CHAPTER 2

DIVISIBILITY AND THE DISTRIBUTION WITH RESPECT TO THE MODULUS p, AND ITS POWERS, OF BINOMIAL, TRINOMIAL, AND MULTINOMIAL COEFFICIENTS

In this chapter we discuss questions of the divisibility of binomial, trinomial, and multinomial coefficients by a prime p and its powers for the Pascal triangle, pyramid, and hyperpyramid. We also consider the number and distribution of these coefficients with respect to the modulus p and its powers in a row, triangle, or cross section of a pyramid.

A great number of works have been devoted to the study of the divisibility of these coefficients. Fundamental in these investigations are the theorems of Legendre, Lucas, and Kummer, and other important results are those of L. Carlitz [99-105], P. Erdös [131-133], N.J. Fine [139], H. Harborth [172-181], F.T. Howard [221-225], D. Singmaster [346-352], M. Sved [366-369], and the present author [11, 12, 14-16]. A survey of early divisibility results may be found in L.E. Dickson [122], and work from more recent decades is reviewed in the detailed article of D. Singmaster [352].

2.1 DIVISIBILITY OF BINOMIAL COEFFICIENTS

References containing material on the divisibility of binomial coefficients by a prime p and its powers are [11, 103, 109, 139, 148, 149, 176, 221-223, 233, 238, 256, 265, 297, 325, 365, 369, 401]. In dealing with the arithmetic properties of binomial coefficients and other coefficients containing factorials, it is convenient to have Legendre's Theorem:

Theorem 2.1. Let p be a prime, and s the highest power of p such that p^s divides n!. Then

$$s = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots = \frac{n-a}{p-1}, \tag{2.1}$$

where the p-ary representation of n is $n=(a_r a_{r-1} \cdots a_1 a_0)_p$, and $a=a_0+a_1+\cdots+a_r$.

To obtain the residue mod p of the binomial coefficients we have Lucas's Theorem:

Theorem 2.2. Let p be a prime, n and m nonnegative integers ($m=0,1,2,\ldots,n$), and let the p-ary representations of these be $n=(a_r a_{r-1} \cdots a_0)$, $m=(b_r b_{r-1} \cdots b_0)$, where $a_r \neq 0$, and $0 \leq a_k < p$, $0 \leq b_k < p$. Then

$$\binom{n}{m} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots\binom{a_r}{b_r} \pmod{p}, \tag{2.2}$$

where $\binom{a_k}{b_k} = 0$ if $b_k > a_k$.

Using Theorem 2.1, E. Kummer [246] obtained a formula for determining the highest power s of the prime p for which $\binom{n}{m}$ is exactly divisible by p^s (and not by p^{s+1}):

Theorem 2.3. Let p, m, n and the p-ary representations be as in Theorem 2.2, and let $n-m = (c_r c_{r-1} \cdots c_0)_p$. Then $\binom{n}{m}$ is exactly divisible by p^s if and only if

$$s = \frac{1}{p-1} \sum_{k=0}^{r} (b_k + c_k - a_k). \tag{2.3}$$

Let us denote by $h(n,p)$ the number of binomial coefficients in the n^{th} row of the Pascal triangle which are divisible by p, and $g(n,p)$ the number of these coefficients not divisible by p. Also, denote by $g_j(n,p)$ the number of these coefficients which when divided by p have the remainder $j \le p-1$, and by $h_s(n,p)$ the number of these coefficients exactly divisible by p^s. Then we have

$$g(n,p) = g_1(n,p) + g_2(n,p) + \cdots + g_{p-1}(n,p)$$
$$h(n,p) = h_1(n,p) + h_2(n,p) + \cdots + h_{q_n}(n,p),$$

where $q_n = \max\{s\}$ in the n^{th} row. Since row n has $n+1$ entries, we have

$$h(n,p) = (n+1) - g(n,p).$$

Theorem 2.4. Let p be a prime, and n a row number of the Pascal triangle, with $n=(a_r a_{r-1} \cdots a_0)_p$. Then

$$g(n,p) = (a_r+1)(a_{r-1}+1) \cdots (a_1+1)(a_0+1).$$

The proof of this theorem based on Lucas's Theorem 2.2 was first given in [139].

For the calculation of $h_s(n,p)$, L. Carlitz [103, 104] introduced the functions $\theta_s(n,p)$ and $\phi_s(n,p)$, where the first is the number of binomial coefficients $\binom{n}{m}$ exactly divisible by p^s, and the second is the number of products $(n+1)\binom{n}{m}$ divisible by p^s, with p a prime and $m=0,1,\ldots,n$. For these functions he derived a system of recurrence relations and found the generating functions. He proved that for $s=1$,

$$h_1(n,p) = \sum_{k=0}^{r-1} (a_0+1)(a_1+1) \cdots (a_{k-1}+1)(p-a_k-1)a_{k+1}(a_{k+2}+1) \cdots (a_r+1) \tag{2.5}$$

and for $s>2$ established a formula for $h_s(n,p)$ when n has the form(s)

$$n = ap^r + bp^{r+1}, \quad 0 \le a < p,\ 0 \le b < p;$$

$$n = a(1 + p + \cdots + p^{r+s}), \qquad 0 < a < p$$

$$n = a(1 + p + \cdots + p^{r+s}) - 1.$$

F.T. Howard [221, 222] found a formula for $h_s(n,p)$ when $s=0,1,\ldots,4$; in the case of $s>4$, the formula requires further conditions. In [223] he found an exact formula for $h_2(n,p)$ and for $s>2$ values of $h_s(n,p)$ valid when n is of the form(s):

$$n = ap^k + bp^r,\ 0<a<p,\ 0<b<p,\ k<r,$$

$$n = c_1 p^{k_1} + \cdots + c_m p^{k_m},\ 0<c_i<p,\ k_1 \ge s,\ k_{i+1} - k_i > s.$$

The extension of the divisibility results noted here to the trinomial case will be discussed in 2.3 and 2.4. We mention two examples of the application of Theorem 2.4 to the enumeration of the number of binomial coefficients not divisible by p. Let $p=2$ and $n=13$; we write the binary representation $13=(1101)_2$ and find $g(13,2) = (1+1)(1+1)(0+1)(1+1) = 8$. And if $p=3$, $n=14$ we write the ternary representation $14=(112)_3$ and find $g(14,3) = (1+1)(1+1)(2+1) = 12$.

We now turn to the Pascal triangle whose base is the row numbered n. Denote by $H(n,p)$ the (total) number of coefficients divisible by p in this triangle, and by $G(n,p)$ the number not divisible by p. Also, let $G_j(n,p)$ denote the number of coefficients in this triangle whose remainder after division by p is $j \le p-1$, and let $H_s(n,p)$ be the number exactly divisible by p^s. Then

$$G(n,p) = G_1(n,p) + G_2(n,p) + \cdots + G_{p-1}(n,p),$$

$$H(n,p) = H_1(n,p) + H_2(n,p) + \cdots + H_{q_n}(n,p),$$

where $q_n = \max\{s\}$ over the triangle. We note also that the triangle contains $N(n)=\frac{1}{2}(n+1)(n+2)$ entries; thus $H(n,p)=N(n)-G(n,p)$.

Theorem 2.5. Let n be the row number of the base of the Pascal triangle, and let p be a prime. Then

$$G(n,p) = \frac{1}{2} \sum_{i=0}^{r} b_{r-i} \binom{p+1}{2}^{r-i} \prod_{j=0}^{i} (b_{r-j}+1), \tag{2.6}$$

where $n+1 = (b_r b_{r-1} \cdots b_0)_p$.

Proof: For any n, we have

$$G(n+1,\ p) = G(n,p) + g(n+1,\ p), \tag{2.7}$$

and so

$$G(n,p) = G(n+1,\ p) - g(n+1,\ p). \tag{2.8}$$

From (2.4), we can write

$$G(n+1,\ p) = \sum_{k=0}^{n+1} \prod_{j=0}^{r} (\beta_{r-j}+1), \tag{2.9}$$

where $k = (\beta_r \beta_{r-1} \cdots \beta_0)_p$. If we now pass from the single sum with index k to a multiple sum with indices $\beta_r, \cdots, \beta_0$ and take into account the implied limits of summation we have

$$G(n+1,\, p) = \sum_{\beta_r=0}^{b_r-1} \sum_{\beta_{r-1}=0}^{p-1} \cdots \sum_{\beta_1=0}^{p-1} \sum_{\beta_0=0}^{p-1} \prod_{j=0}^{r} (\beta_{r-j}+1)$$

$$+ (b_r+1) \sum_{\beta_{r-1}=0}^{b_{r-1}-1} \sum_{\beta_{r-2}=0}^{p-1} \cdots \sum_{\beta_1=0}^{p-1} \sum_{\beta_0=0}^{p-1} \prod_{j=1}^{r} (\beta_{r-j}+1)$$

$$+ \left[(b_r+1)(b_{r-1}+1)\right] \sum_{\beta_{r-2}=0}^{b_{r-2}-1} \sum_{\beta_{r-3}=0}^{p-1} \cdots \sum_{\beta_1=0}^{p-1} \sum_{\beta_0=0}^{p-1} \prod_{j=2}^{r} (\beta_{r-j}+1)$$

$$+ \cdots + \left[(b_r+1)(b_{r-1}+1) \cdots (b_2+1)\right] \sum_{\beta_1=0}^{b_1-1} \sum_{\beta_0=0}^{p-1} \prod_{j=r-1}^{r} (\beta_{r-j}+1)$$

$$+ \left[(b_r+1)(b_{r-1}+1) \cdots (b_1+1)\right] \sum_{\beta_0=0}^{b_0-1} (\beta_0+1)$$

$$+ (b_r+1)(b_{r-1}+1) \cdots (b_1+1)(b_0+1) . \qquad (2.10)$$

Each of the sums in (2.10) is an elementary calculation. It follows that

$$G(n+1,\, p) = \frac{b_r(b_r+1)}{2} \binom{p+1}{2}^{r} + (b_r+1)\, \frac{b_{r-1}(b_{r-1}+1)}{2} \binom{p+1}{2}^{r-1}$$

$$+ (b_r+1)(b_{r-1}+1)\, \frac{b_{r-2}(b_{r-2}+1)}{2} \binom{p+1}{2}^{r-2} + \cdots$$

$$+ \left[(b_r+1)(b_{r-1}+1) \cdots (b_2+1)\right] \frac{b_1(b_1+1)}{2} \binom{p+1}{2}^{1}$$

$$+ \left[(b_r+1)(b_{r-1}+1) \cdots (b_1+1)\right] \frac{b_0(b_0+1)}{2} \binom{p+1}{2}^{0}$$

$$+ (b_r+1)(b_{r-1}+1) \cdots (b_1+1)(b_0+1)$$

$$= \frac{1}{2} \sum_{i=0}^{r} b_{r-i} \prod_{j=0}^{i} (b_{r-j}+1) \binom{p+1}{2}^{r-i}$$

$$+ (b_r+1)(b_{r-1}+1) \cdots (b_1+1)(b_0+1) = G(n,p) + g(n+1,\, p) ,$$

which proves the theorem. From Theorem 2.5 it follows that if $n=p^r-1$, then

$$\left.\begin{aligned} &G(p^r-1,\ p) = \binom{p+1}{2}^r,\ G(ap^r-1,\ p) = \binom{a+1}{2}\binom{p+1}{2}^r, \\ &G(ap^r+b-1,\ p) = G(ap^r-1,\ p) + (a+1)G(b-1,\ p)\ , \end{aligned}\right\} \tag{2.11}$$

where $0 \le a < p-1$, $1 \le b \le p^r$. If $c \ge p$, then

$$G(cp^r-1,\ p) = \binom{p+1}{2}^r G(c-1,\ p)\ .$$

Let $p=2$. Then $n+1$ may be written in the form

$$n+1 = b_{r_1}2^{r_1} + b_{r_2}2^{r_2} + \dots + b_{r_q}2^{r_q}\ ,$$

and it follows from Theorem 2.5 that

$$G(n,2) = \sum_{i=1}^{q} 2^{i-1}3^{r_i}\ . \tag{2.12}$$

If $n=2^r-1$, then $G(2^r-1,2) = 3^r$, $G(2^r+b-1,2) = G(2^r-1,2) + 2G(b-1,2)$, and $G(c2^r-1,2) = 3^rG(c-1,2)$.

It also follows that if we subtract $G(n,p)$ from the total number of elements, we have

$$H(n,p) = N(n) - G(n,p) = \binom{n+2}{2} - G(n,p)\ . \tag{2.13}$$

For each p, from some n onward $H(n,p) >> G(n,p)$. Thus, for $p=3$ we have $G(26,3) = 216$, $H(26,3) = 162$; $G(80,3) = 1296$, $H(80,3) = 2025$; $G(242,3) = 7776$, $H(242,3) = 21868$; $G(728,3) = 46656$, $H(728,3) = 485514$. We need to clarify this order of increase of H(n,p) and G(n,p). For this, it is sufficient to consider, rather than $\{n\}$, the subsequence $\{p^r-1\}$ for $r\to\infty$.

Theorem 2.6. For $p\geq 2$, $\lim_{n\to\infty} G(n,p)/H(n,p)=0$.

Proof: Since G and H are nondecreasing functions of n, then for $p^r-1\leq n<p^{r+1}-1$, using the first equation in (2.11) and equation (2.13), we have

$$G(n,p)/H(n,p) \leq G(p^{r+1}-1,\ p)/H(p^r-1,\ p)$$
$$= \binom{p+1}{2}^{r+1} \Bigg/ \left[\binom{p^r+1}{2} - \binom{p+1}{2}^r\right]$$
$$= p(p+1) \Bigg/ \left[\left(\frac{2p}{p+1}\right)^r + \left(\frac{2}{p+1}\right)^r - 2\right].$$

Since for $p\geq 2$, $\frac{2p}{p+1}>1$, $\frac{2}{p+1}<1$, then as $r\to\infty$ $\left(\frac{2p}{p+1}\right)^r\to\infty$, $\left(\frac{2}{p+1}\right)^r\to 0$, and it follows that

$$\lim_{n\to\infty} G(n,p)/H(n,p) = 0, \tag{2.14}$$

which proves the theorem.

We give below a short survey of some basic works on the divisibility of the binomial coefficients.

J.W. Glaisher [148, 149] discussed questions of exact divisibility of the binomial coefficients by powers of a prime and established a formula for the numbers of entries not divisible by p in the rows of the Pascal triangle.

N.J. Fine [139] obtained a formula for the number of binomial coefficients $\binom{n}{m}$, $0\leq m\leq n$, not divisible by p, and gave necessary and sufficient conditions for divisibility by p and for non-divisibility by p. He also proved that as $n\to\infty$ almost all binomial coefficients are divisible by p.

J.B. Roberts [325] discussed the problem of obtaining the number $\theta_j(n)$ of binomial coefficients in the Pascal triangle congruent to j, $0\le j\le p-1$, modulo the prime p. He reduced the problem to the solution of a linear difference equation with constant coefficients, and gave a formula for $\theta_j(n)$ for $p=2$ and any n, and also for $p=3,5$ and $n=p^k-1$, $k\ge0$.

In [177], H. Harborth studied the problem of the number A(n) of binomial coefficients $\binom{n}{m}$ in the Pascal triangle which are divisible by their row number n, as $n\to\infty$. He proved that almost all binomial coefficients are divisible by their row number; the distribution of divisibility by the row number is also considered in section 3.2.

N. Robbins [323, 324] looked at the connection between the function A(n) mentioned above and Euler's function $\varphi(n)$. In [323] he proved that $A(n)\ge\varphi(n)$ for all n, and $A(n)=\varphi(n)$ if $n=p^s(s\ge1)$ or if n is twice a prime Mersenne number. In [324] he found necessary and sufficient conditions for the equality $A(n)=\varphi(n)$, when n is square-free, and also discussed the case when n is a product of three distinct primes.

Consider the number of binomial coefficients $\binom{n}{m}$, for $0\le m\le n\le N$, not divisible by the product $(n)(n-1)\cdots(n-s+1)$, $s\ge1$. H. Harborth [180] proved that for fixed $s\ge1$ and $N\to\infty$, almost all binomial coefficients are divisible by this product. From this, he concludes that almost all binomial coefficients are divisible by $\binom{n}{s}$, and for $s=1$ this is the row number.

L. Carlitz [99] proved that if $n=(a_ra_{r-1}\cdots a_0)_p$, $\sigma(n)=a_r+a_{r-1}+\cdots+a_0$, and $(p-1)$ is divisible by $k\ge\sigma(n)$, then all binomial coefficients $\binom{n}{km}$, $0<km<n$, are divisible by p. He also considered in [100] the number of binomial coefficients $\binom{n}{m}$ satisfying the conditions:

$$\binom{n}{m} \equiv \binom{n}{m-1} \not\equiv 0 \pmod{p}, \; m \not\equiv 0 \pmod{p},$$

$$\binom{n}{m} \equiv \binom{n}{m-1} \equiv 0 \pmod{p}, \; m = 1,2,\ldots,n.$$

In [360, 362, 363] K.B. Stolarsky studied various problems connected with the function B(n) defined as the number of ones in the binary representation of n. In [360] he discussed the recurrence relation $y_{n+1}=y_n+B(n)$, n=1,2,..., and established the asymptotic behavior $y_m \sim (m \log m)/2 \log 2$. In [362], he studied the function $r_h=B(m^h)/B(m)$, where h is positive. He showed that the maximal order of magnitude of $r_h(m)$ is $c(h)\ (\log m)^{\frac{h-1}{h}}$, where $c(h)>0$ depends only on h; the minimal order of magnitude of $r_2(m)$ is not greater than $c(\log \log m)^2/\log m$, where $c>0$ is an absolute constant. In [363], he compared the behavior of the functions B(kn) and B(n), and called n "strong" if $B(kn)>B(n)$; he also studied the question of the number of solutions of $B(3n)-B(n)=a$ for $2^r \le n < 2^{r+1}$. If we denote by F(n) the number of odd binomial coefficients in the first n rows of the Pascal triangle, Stolarsky also studied [361] the asymptotic behavior of F(n), using the expressions

$$\alpha = \limsup_{n\to\infty} F(n)/n^{\theta}, \quad \beta = \liminf_{n\to\infty} F(n)n^{\theta},$$

where $\theta = \log 3/\log 2 = 1.58496\ldots$. He established that α and ß satisfy the conditions $1 \le \alpha \le 1.052$, $0.72 \le \beta \le 0.815$, and that $n^{\theta}/3 < F(n) < 3n^{\theta}$.

These results were sharpened by H. Harborth [176], who showed that $\alpha=1$, $\beta=0.812556\ldots$.

In a series of papers [346-352], D. Singmaster studied various properties of the binomial and multinomial coefficients. In [346] he discussed the problem of the number of ways a whole number a may be represented as a binomial coefficient, and showed that $N(a)=O(\log a)$. In [347] he introduced the functions E(n) and F(n), where $E(n)=s$ if p^s/n and $F(n)\equiv n/p^s \pmod p$; on the basis of the properties of these functions, he determined E(n!), F(n!), $E\left(\binom{n}{k}\right)$, $F\left(\binom{n}{k}\right)$, and so generalized the results of Lucas, Legendre, and Kummer. In [348], he obtained the least n for which the multinomial coefficient $(n; n_1,n_2,\ldots,n_r)$, $n_1,n_2,\ldots,n_r$ given, is divisible by p^s. In [349] he showed that "almost all" binomial coefficients are divisible by any positive whole number d. The notion of "almost all" appears in four versions, using the definitions: $A(\alpha,m)$ is the number of pairs (j,k) for which $0\le j$, $k<m$, $p^s \mid \binom{j+k}{k}$; $B(\alpha,m)$ is the number of pairs (j,k) for which $0\le j+k<m$, $p^s \mid \binom{j+k}{k}$; $C(\alpha,n)$ is the number of values of k for which $0\le k\le n$, $p^s \mid \binom{n}{k}$; $D(\alpha,k)$ is the density of j's for which $p^s \mid \binom{j+k}{k}$; $s=\alpha$. He showed, then, that

$$\lim_{m\to\infty} A(\alpha,m)/m^2 = 0, \quad \lim_{m\to\infty} B(\alpha,m)/(m(m+1)/2) = 0,$$

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} C(\alpha,i)/(i+1) = 0, \quad \lim_{k\to\infty} \frac{1}{k+1}\sum_{i=0}^{k} D(\alpha,i) = 0.$$

In [350] he discussed the greatest common divisor of corresponding triples of binomial coefficients in the Pascal triangle, and in [351] he considered the equation $\binom{n+1}{k+1} = \binom{n}{k+2}$ and showed there are infinitely many solutions of the form $n=F_{2\ell+2}F_{2\ell+3}-1$, $k=F_{2i}F_{2i+3}-1$, where F_k is a Fibonacci number. Finally, [352] is a systematic review of more than seventy papers by various authors, and also contains some new results on divisibility of binomial and multinomial coefficients by a prime p and its powers.

R. Fray [143] posed the problem of determining the least positive number *a* for which

$$\binom{n+a}{m} \equiv \binom{n}{m} \pmod{p^r} \qquad (2.15)$$

for all m=0,1,...,n, r=1,2,.... He showed that if $p^s \leq n < p^{s+1}$, the least a satisfying (2.15) for all m is $a=p^{r+s}$, and if $p^s \leq m < p^{s+1}$, the solution is again $a=p^{r+s}$.

H. Gupta [167] solved the problem of determining the smallest positive n so that for a given positive m the binomial coefficient $\binom{n}{m}$ will have at least m prime divisors.

In [234, 235] G.S. Kazandzidis worked on a method for obtaining the highest power of a prime p which will divide $\binom{n}{m}$ and $\binom{np}{mp}$.

H.B. Mann and D. Shanks [274], using the Pascal triangle, established a criterion that a natural number m be prime: m is a prime if and only if for $\frac{m}{3} \leq n \leq \frac{m}{2}$, n divides $\binom{n}{m-2n}$.

In [175, 178] H. Harborth, with the help of the Pascal triangle, generalized the criterion of [174], showing that m is a prime if and only if for $\frac{m}{c+1} \leq n \leq \frac{m}{c}$, n divides $\binom{n}{m-cn}$; here, for fixed $c \leq 2$, $m \geq 2$, n is not a multiple of a prime less than or equal to c^2-c-1. The details for c=3 are given in [175], and for c=4 in [178].

J. Bernard and G. Letac [67] proved that if a and b are whole numbers satisfying $|a| < p$, $|b| < p$, where p is prime, $(a+b) \geq 0$, and $\binom{n+m}{m}$ is divisible by p^s, then

$$\binom{pn+pm+a+b}{pm+b} \equiv 0 \pmod{p^s}. \qquad (2.16)$$

Lastly, E.F. Ecklund [127] proved that $\binom{n}{m}$ has a prime divisor $p \leq \max\left\{\frac{n}{m}, \frac{n}{2}\right\}$, with the exception of $\binom{7}{3}$ for $n \geq 2m$.

The question of the divisibility of the binomial coefficients by powers of a prime p is discussed in the following section.

2.2 THE DISTRIBUTION OF THE BINOMIAL COEFFICIENTS IN THE PASCAL TRIANGLE MODULO p AND ITS POWERS.

Here we consider problems associated with the number and distribution of the binomial coefficients in the Pascal triangle with respect to their remainder, or residue, after division by the prime p, and also the distribution of these coefficients strictly divisible by a power of p. These problems are also connected with the quantities $G_j(n,p)$, $H_s(n,p)$ introduced in section 2.1.

C.T. Long [261, 262] introduced two triangular forms composed of binomial coefficients:

$$\begin{array}{ccc} & \binom{np^k}{mp^k} & \\ & \cdot \quad\quad \cdot & \\ \cdot & & \cdot \\ \binom{np^k+p^k-1}{mp^k} & \cdots & \binom{np^k+p^k-1}{mp^k+p^k-1} \end{array}, \quad 0 \le m \le n, \tag{2.17}$$

and

$$\begin{array}{ccc} \binom{np^k}{mp^k+1} & \cdots & \binom{np^k}{mp^k+p^k-1} \\ \cdot & & \cdot \\ & \cdot \quad\quad \cdot & \\ & \binom{np^k+p^k-2}{mp^k+p^k-1} & \end{array}, \quad n \ge 1,\ 0 \le m \le n-1, \tag{2.18}$$

where m, n, k are whole numbers, and $k \geq 1$. If we replace the elements in (2.17) by their residues mod p, and denote the result by $\Delta_{n,m}$, the author discusses the triangle of triangles

$$\begin{array}{ccccc} & & \Delta_{0,0} & & \\ & \Delta_{1,0} & & \Delta_{1,1} & \\ \Delta_{2,0} & & \Delta_{2,1} & & \Delta_{2,2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{2.19}$$

"isomorphic" to the usual Pascal triangle. He shows that for practical purposes the triangle $\Delta_{n,m}$ may be taken to be the triangle

$$\begin{array}{ccccc} & & \binom{n}{m}\binom{0}{0} & & \\ & \binom{n}{m}\binom{1}{0} & & \binom{n}{m}\binom{1}{1} & \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \binom{n}{m}\binom{p^k-1}{0} & & \cdots & & \binom{n}{m}\binom{p^k-1}{p^k-1} \end{array} \tag{2.20}$$

consisting of the residues mod p, which by Lucas's Theorem are congruent mod p to the corresponding elements of the triangle (2.17). It is also shown that the triangle $\Delta_{n,m}$ satisfies the recurrence relation

$$\Delta_{n+1,m+1} = \Delta_{n,m} + \Delta_{n,m+1} \tag{2.21}$$

"isomorphic" to the ordinary one, $\binom{n+1}{m+1} = \binom{n}{m} + \binom{n}{m+1}$. On the right side of (2.21), addition is carried out mod p.

Between each pair of adjacent triangles $\Delta_{n,m}$ and $\Delta_{n,m+1}$, $m=0,1,\ldots,n-1$, in (2.19), we may place (the elements of) a triangle $\nabla_{n,m}$ consisting of the residues mod p of the binomial coefficients in (2.18). By Lucas's Theorem the residues of each of the binomial coefficients in (2.18) are all zeros, and so all the elements of $\nabla_{n,m}$ are null.

<u>First Problem</u>. Let p be a prime, $k\geq 0$, $n=0,1,\ldots,p-1$, and $m=0,1,\ldots,n$. Then among the $p(p+1)/2$ triangles $\Delta_{n,m}$ constituting p^k rows, as in (2.19), there will be only $p-1$ distinct types. Their general form, which we denote by $\Delta_k^{(i)}$, $k\geq 0$, $i=1,2,\ldots,p-1$, is

$$\begin{array}{ccccccccc} & & & & i & & & & \\ & & & i & & i & & & \\ & & i & & & & i & & \\ & \cdot & & & & & & \cdot & \\ & \cdot & & & & & & \cdot & \\ & \cdot & & & & & & \cdot & \\ i & & & & & & & & i \\ i & p-i & & & \cdots & & & p-i & i \end{array} \tag{2.22}$$

The interior elements of the triangle are calculated by the usual rule as the sums of two elements in the preceding row, except here mod p. Also, the elements of the triangle $\nabla_{n,m}$ are zeros independently of n and m, and for a given k compose p^k-1 rows, these triangles are all the same and we denote them by $\nabla_k^{(0)}$.

Using the "geometric" (Pascal rule) interpretation, we can establish that each triangle $\Delta_k^{(i)}$ is itself a "geometric" sum of $p(p+1)/2$ triangles $\Delta_{k-1}^{(j)}$ and $p(p-1)/2$ triangles $\nabla_{k-1}^{(0)}$. Thus, we can write out the "geometric" equations

$$\left.\begin{aligned}
\Delta_k^{(1)} &= a_{1,1}\Delta_{k-1}^{(1)} + a_{1,2}\Delta_{k-1}^{(2)} + \cdots + a_{1,p-1}\Delta_{k-1}^{(p-1)} + \frac{p(p-1)}{2}\nabla_{k-1}^{(0)},\\
\Delta_k^{(2)} &= a_{2,1}\Delta_{k-1}^{(1)} + a_{2,2}\Delta_{k-1}^{(2)} + \cdots + a_{2,p-1}\Delta_{k-1}^{(p-1)} + \frac{p(p-1)}{2}\nabla_{k-1}^{(0)},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\Delta_k^{(p-1)} &= a_{p-1,1}\Delta_{k-1}^{(1)} + a_{p-1,2}\Delta_{k-1}^{(2)} + \cdots\\
&\quad + a_{p-1,p-1}\Delta_{k-1}^{(p-1)} + \frac{p(p-1)}{2}\nabla_{k-1}^{(0)},
\end{aligned}\right\} \tag{2.23}$$

where $a_{i,j}$ is the number of triangles $\Delta_{k-1}^{(j)}$ occurring in $\Delta_k^{(i)}$. These coefficients $a_{i,j} \equiv a_{i,j}^{(p)}$ depend on p but not on k, and their values coincide with the numbers of ones, twos, ..., up to $(p-1)$ inclusive, contained in the triangles $\Delta_1^{(1)}$, $\Delta_1^{(2)}$, ..., $\Delta_1^{(p-1)}$, whose general form is that of (2.22), and which have p rows.

Let $P_k^{(i)}(r)$ be the number of occurrences of r in the triangle $\Delta_k^{(i)}$. Thus, $P_k^{(1)}(1)$ is the number of ones in $\Delta_k^{(1)}$, $P_k^{(2)}(3)$ is the number of threes in $\Delta_k^{(2)}$, and so on. Using (2.23) and taking into account that the $\nabla_{k-1}^{(0)}$ contain only zeros and may be neglected, we can form the system of first order recurrence relations

$$\left.\begin{aligned}
P_k^{(1)}(r) &= a_{1,1}P_{k-1}^{(1)}(r) + a_{1,2}P_{k-1}^{(2)}(r) + \cdots + a_{1,p-1}P_{k-1}^{(p-1)}(r),\\
P_k^{(2)}(r) &= a_{2,1}P_{k-1}^{(1)}(r) + a_{2,2}P_{k-1}^{(2)}(r) + \cdots + a_{2,p-1}P_{k-1}^{(p-1)}(r),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
P_k^{(p-1)}(r) &= a_{p-1,1}P_{k-1}^{(1)}(r) + a_{p-1,2}P_{k-1}^{(2)}(r) + \cdots + a_{p-1,p-1}P_{k-1}^{(p-1)}(r),
\end{aligned}\right\} \tag{2.24}$$

where $r=1,2,\ldots,p-1$, $a_{i,j} \equiv a_{i,j}^{(p)}$.

The solution of the system depends on the initial data, which are determined by the occurrences of r in the triangles $\Delta_0^{(j)}$, $1 \le j \le p-1$. Since each of the $\Delta_0^{(j)}$ contains but one element equal to j, and does not contain $r \ne j$, the data take the form of (p−1) groups:

$$\left.\begin{array}{l} P_0^{(1)}(1) = 1,\ P_0^{(2)}(1) = 0,\ P_0^{(3)}(1) = 0, \cdots,\ P_0^{(p-1)}(1) = 0, \\ P_0^{(1)}(2) = 0,\ P_0^{(2)}(2) = 1,\ P_0^{(3)}(2) = 0, \cdots,\ P_0^{(p-1)}(2) = 0, \\ \cdot \\ P_0^{(1)}(p-1) = 0,\ P_0^{(2)}(p-1) = 0,\ P_0^{(3)}(p-1) = 0, \cdots,\ P_0^{(p-1)}(p-1) = 1. \end{array}\right\} \qquad (2.25)$$

From (2.24) and (2.25) we may determine the number of occurrences of r in any triangle $\Delta_k^{(i)}$, $1 \le i \le p-1$. The triangle $\Delta_k^{(1)}$ itself represents the Pascal triangle mod p, and so $P_k^{(1)}(r)$ gives the value of $G_r(p^k-1,p)$; the values of $P_k^{(j)}(r)$, $2 \le j \le p-1$, are the number of occurrences of r in $\Delta_k^{(j)}$, as shown schematically in (2.22).

The matrix of the system (2.24) is

$$A_p = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,p-1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,p-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p-1,1} & a_{p-1,2} & \cdots & a_{p-1,p-1} \end{bmatrix}, \quad a_{i,j} \equiv a_{i,j}^{(p)}$$

which, except for p=3, is not symmetric. The element $a_{i,j}$, as indicated before, is the number of occurrences of i in $\Delta_1^{(j)}$ for a given value of p. We give below some examples.

Let p=3. Then

$$\Delta_1^{(1)} = \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array} \qquad \Delta_1^{(2)} = \begin{array}{ccccc} & & 2 & & \\ & 2 & & 2 & \\ 2 & & 1 & & 2 \end{array},$$

and so

$$a_{1,1}^{(3)} = 5,\ a_{1,2}^{(3)} = 1,\ a_{2,1}^{(3)} = 1,\ a_{2,2}^{(3)} = 5.$$

The system corresponding to (2.24) is

$$\left.\begin{aligned} P_k^{(1)}(r) &= 5P_{k-1}^{(1)}(r) + P_{k-1}^{(2)}(r), \\ P_k^{(2)}(r) &= P_{k-1}^{(1)}(r) + 5P_{k-1}^{(2)}(r), \end{aligned}\right\} \qquad (2.26)$$

where r=1,2, and the matrix is

$$A_3 = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}.$$

The initial data are obtained from $\Delta_0^{(1)}$ and $\Delta_0^{(2)}$, and we have

$$P_0^{(1)}(1) = 1,\ P_0^{(2)}(1) = 0,\ P_0^{(1)}(2) = 0,\ P_0^{(2)}(2) = 1.$$

The solution of (2.26), with these initial values is

$$\left.\begin{aligned} P_k^{(1)}(1) &= \tfrac{1}{2}(6^k+4^k), \quad & P_k^{(1)}(2) &= \tfrac{1}{2}(6^k-4^k) \\ P_k^{(2)}(1) &= \tfrac{1}{2}(6^k-4^k), \quad & P_k^{(2)}(2) &= \tfrac{1}{2}(6^k+4^k). \end{aligned}\right\} \qquad (2.27)$$

It follows from (2.27) that for $p=3$ the number of ones in the Pascal triangle whose base is row 3^k-1 is

$$G_1(3^k-1,3) = P_k^{(1)}(1) = \tfrac{1}{2}(6^k+4^k),$$

and the number of twos is

$$G_2(3^k-1,3) = P_k^{(1)}(2) = \tfrac{1}{2}(6^k-4^k).$$

It also follows that the total number of coefficients in this triangle which are not divisible by 3 is

$$G(3^k-1,3) = G_1 + G_2 = 6^k,$$

which agrees with Theorem 2.5. If $3^k<n<3^{k+1}$, then for the enumerations $G_1(n,3)$, $G_2(n,3)$ we need to use the appropriate "geometric" equations and the formulas for $P_\ell^{(1)}(1)$, $P_\ell^{(1)}(2)$, where $\ell<k$ and k and ℓ must be specified.

Let $p=5$. Then

$$\Delta_1^{(1)} = \begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&1&&4&&1 \end{array} \qquad \Delta_1^{(2)} = \begin{array}{ccccccccc} &&&&2&&&& \\ &&&2&&2&&& \\ &&2&&4&&2&& \\ &2&&1&&1&&2& \\ 2&&3&&2&&3&&2 \end{array}$$

$$\Delta_1^{(3)} = \begin{array}{ccccccccc} &&&&3&&&& \\ &&&3&&3&&& \\ &&3&&1&&3&& \\ &3&&4&&4&&3& \\ 3&&2&&3&&2&&3 \end{array} \qquad \Delta_1^{(4)} = \begin{array}{ccccccccc} &&&&4&&&& \\ &&&4&&4&&& \\ &&4&&3&&4&& \\ &4&&2&&2&&4& \\ 4&&1&&4&&1&&4 \end{array},$$

and so we have

$$a_{1,1}^{(5)} = 10, \quad a_{1,2}^{(5)} = 1, \quad a_{1,3}^{(5)} = 2, \quad a_{1,4}^{(5)} = 2;$$

$$a_{2,1}^{(5)} = 2, \quad a_{2,2}^{(5)} = 10, \quad a_{2,3}^{(5)} = 2, \quad a_{2,4}^{(5)} = 1;$$

$$a_{3,1}^{(5)} = 1, \quad a_{3,2}^{(5)} = 2, \quad a_{3,3}^{(5)} = 10, \quad a_{3,4}^{(5)} = 2;$$

$$a_{4,1}^{(5)} = 2, \quad a_{4,2}^{(5)} = 2, \quad a_{4,3}^{(5)} = 1, \quad a_{4,4}^{(5)} = 10,$$

and the matrix A_5

$$A_5 = \begin{bmatrix} 10 & 1 & 2 & 2 \\ 2 & 10 & 2 & 1 \\ 1 & 2 & 10 & 2 \\ 2 & 2 & 1 & 10 \end{bmatrix}.$$

In vector form the system is

$$\begin{bmatrix} P_k^{(1)}(r) \\ P_k^{(2)}(r) \\ P_k^{(3)}(r) \\ P_k^{(4)}(r) \end{bmatrix} = \begin{bmatrix} 10 & 1 & 2 & 2 \\ 2 & 10 & 2 & 1 \\ 1 & 2 & 10 & 2 \\ 2 & 2 & 1 & 10 \end{bmatrix} \begin{bmatrix} P_{k-1}^{(1)}(r) \\ P_{k-1}^{(2)}(r) \\ P_{k-1}^{(3)}(r) \\ P_{k-1}^{(4)}(r) \end{bmatrix}$$

for r=1,2,3,4, and the initial conditions are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The solution of this system with the given initial data has the form

$$P_k^{(1)}(1) = P_k^{(2)}(2) = P_k^{(3)}(3) = P_k^{(4)}(4)$$

$$= \frac{1}{4}(15^k + 9^k) + \frac{1}{2}H_0^k(1,8),$$

$$P_k^{(1)}(2) = P_k^{(2)}(4) = P_k^{(3)}(1) = P_k^{(4)}(3)$$

$$= \frac{1}{4}(15^k - 9^k) - \frac{1}{2}H_1^k(1,8),$$

$$P_k^{(1)}(3) = P_k^{(2)}(1) = P_k^{(3)}(4) = P_k^{(4)}(2)$$

$$= \frac{1}{4}(15^k - 9^k) + \frac{1}{2}H_1^k(1,8),$$

$$P_k^{(1)}(4) = P_k^{(2)}(3) = P_k^{(3)}(2) = P_k^{(4)}(1)$$

$$= \frac{1}{4}(15^k + 9^k) - \frac{1}{2}H_0^k(1,8),$$

where

$$H_\alpha^k(x, y) = \sum_{i=0}^{\left[\frac{k-\alpha}{2}\right]} (-1)^i \binom{k}{2i+\alpha} x^{2i+\alpha} y^{k-2i-\alpha}, \quad \alpha = 0, 1,$$

is itself known to be a harmonic polynomial in two variables [6]. With this information we can find the distribution of the residues 1,2,3,4 mod 5 in the Pascal triangle whose base is the row numbered $5^k - 1$:

$$\left.\begin{aligned} G_1(5^k - 1,5) &= \frac{1}{4}(15^k + 9^k) + \frac{1}{2}H_0^k(1,8), \\ G_2(5^k - 1,5) &= \frac{1}{4}(15^k - 9^k) - \frac{1}{2}H_1^k(1,8), \\ G_3(5^k - 1,5) &= \frac{1}{4}(15^k - 9^k) + \frac{1}{2}H_1^k(1,8), \\ G_4(5^k - 1,5) &= \frac{1}{4}(15^k + 9^k) - \frac{1}{2}H_0^k(1,8). \end{aligned}\right\}$$

In like fashion for p=7,11, the matrices will be

$$A_7 = \begin{bmatrix} 15 & 2 & 2 & 1 & 4 & 4 \\ 1 & 15 & 4 & 2 & 4 & 2 \\ 4 & 2 & 15 & 4 & 1 & 2 \\ 2 & 1 & 4 & 15 & 2 & 4 \\ 2 & 4 & 2 & 4 & 15 & 1 \\ 4 & 4 & 1 & 2 & 2 & 15 \end{bmatrix} \qquad A_{11} = \begin{bmatrix} 27 & 5 & 5 & 4 & 3 & 3 & 2 & 4 & 4 & 9 \\ 3 & 27 & 2 & 5 & 4 & 5 & 4 & 4 & 9 & 3 \\ 4 & 4 & 27 & 3 & 4 & 5 & 3 & 9 & 5 & 2 \\ 5 & 3 & 4 & 27 & 4 & 2 & 9 & 5 & 3 & 4 \\ 4 & 2 & 3 & 5 & 27 & 9 & 4 & 3 & 4 & 5 \\ 5 & 4 & 3 & 4 & 9 & 27 & 5 & 3 & 2 & 4 \\ 4 & 3 & 5 & 9 & 2 & 4 & 27 & 4 & 3 & 5 \\ 2 & 5 & 9 & 3 & 5 & 4 & 3 & 27 & 4 & 4 \\ 3 & 9 & 4 & 4 & 5 & 4 & 5 & 2 & 27 & 3 \\ 9 & 4 & 4 & 2 & 3 & 3 & 4 & 5 & 5 & 27 \end{bmatrix}$$

These matrices for any prime p, as here for p=3,5,7,11, have a property which we might call quasi-symmetry, in which the elements satisfy three types of conditions:

$$a_{1,1} = a_{2,2} = a_{3,3} = \cdots = a_{p-1,p-1},$$

$$a_{1,p-1} = a_{2,p-2} = a_{3,p-3} = \cdots = a_{p-1,1},$$

$$a_{i,j} = a_{p-i,p-j}, \; i \neq j, \; i+j \neq p.$$

The first and second of these require the equality of the elements on the main diagonal, and of those on the counterdiagonal, respectively. The third requires, in effect, the equality of elements in positions above and below the main diagonal, related to one another by a 180° rotation about an axis perpendicular to the center of the array. Without going into the matter here, we mention that these quasi-symmetric matrices which arise in connection with the Pascal triangle mod p have a number of interesting properties.

Questions connected with the distribution in the Pascal triangle of the binomial coefficients mod p are discussed in papers by A. Fadini [137], J.B. Roberts [325], M. Sved and J. Pitman [369], among others. In particular, [137] uses a triangle of triangles like that of Long [262]; [325] gives the distribution of the binomial coefficients mod 3 and mod 5; in [369] are tables of the distribution mod 3 up to the 50^{th} row, and also for the composite modulus $9=3^2$ up to the 60^{th} row. In the last, there are also tables of values α, β for the expression of the binomial coefficients in the forms $\alpha \cdot 3^2 + \beta \cdot 3$ and $\alpha \cdot 7^2 + \beta \cdot 7$, and other tables.

As examples of the properties mentioned in this section, we show the distribution of the binomial coefficients mod 2 in Figure 18, and mod 3 in Figure 19 (the dots stand for zeros).

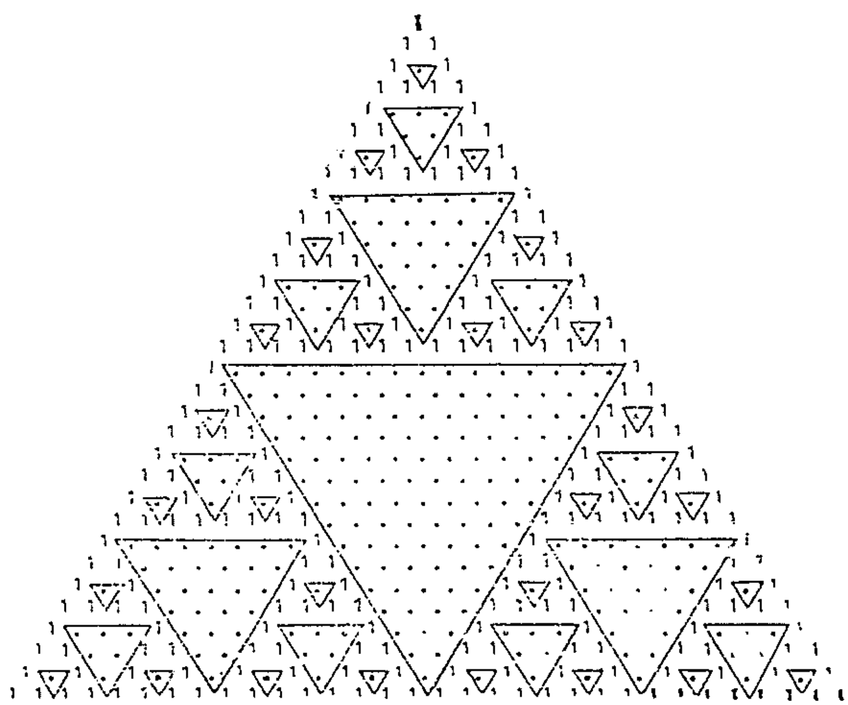


Figure 18

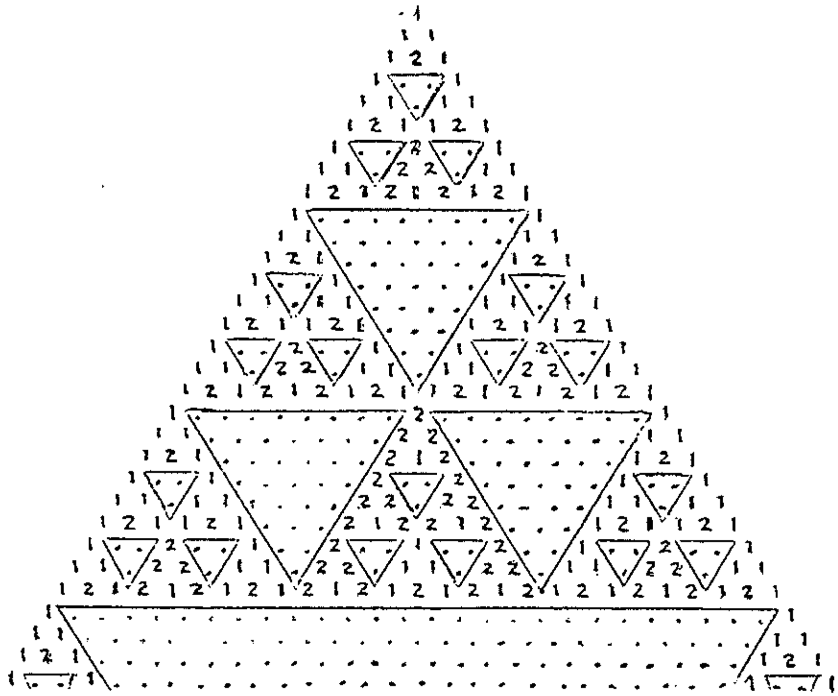


Figure 19

Second Problem. Here again we are interested in the distribution of the binomial coefficients in the Pascal triangle, but now the criterion (for forming the distribution) is that of strict divisibility by a power of the prime p. That is, using the notation introduced by Long [261], we denote by $\begin{bmatrix} n \\ m \end{bmatrix}$ the exponent of the highest power of p which divides $\binom{n}{m}$, and consider the triangle whose elements are the values $\begin{bmatrix} n \\ m \end{bmatrix}$, $0 \le m \le n$, called the p-index Pascal triangle. This triangle has a number of interesting properties, and was first discussed by K.R. McLean [277].

We list some of these properties, formulated and proved in [261]. Let p be a prime, and N and n natural numbers; then

$$\begin{bmatrix} np^k-1 \\ m \end{bmatrix} = 0, \text{ if } 1 \le n < p,\ 0 \le m < np^k-1;$$

$$\begin{bmatrix} p^k \\ m \end{bmatrix} \ge 1, \text{ if } 1 \le m < p^k,\ \begin{bmatrix} p^k \\ 0 \end{bmatrix} = \begin{bmatrix} p^k \\ p^k \end{bmatrix} = 0;$$

$$\begin{bmatrix} Np^k+np^{k-1}-1 \\ m \end{bmatrix} = \begin{cases} 0 \text{ for } rp^k \le m \le rp^k+np^{k-1}-1, & 0 \le r \le N, \\ 1 \text{ for } rp^k+np^{k-1} \le m \le (r+1)p^k, & 0 \le r < N, \end{cases}$$

where $1 \le n < p$, $1 \le N < p$.

Let p be a prime, k a natural number, and n,m integers, $0 \le m \le n$. Denote by $T_{n,m}^{(k)}$ the p-index triangle formed from the triangle (2.17), and which will be of the form

$$\begin{matrix} & \begin{bmatrix} np^k \\ mp^k \end{bmatrix} & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \begin{bmatrix} np^k+p^k-1 \\ mp^k \end{bmatrix} & \cdots & \begin{bmatrix} np^k+p^k-1 \\ mp^k+p^k-1 \end{bmatrix} \end{matrix} \qquad (2.28)$$

In [261] it is shown that $T_{n,m}^{(k)} = \begin{bmatrix} n \\ m \end{bmatrix} + T_{0,0}^{(k)}$, where the symbolic addition on the right-hand side indicates that the element $\begin{bmatrix} n \\ m \end{bmatrix}$ is added to each entry of $T_{0,0}^{(k)}$. Thus, it follows that if we know the distribution of elements in the triangle $T_{0,0}^{(k)}$, we can easily find the distribution in the triangle $T_{n,m}^{(k)}$, and moreover we obtain the distribution of the binomial coefficients according to the criterion of the strict divisibility by a power of the prime p in the triangle (2.17). From the triangles $T_{n,m}^{(k)}$ we can form an indefinitely increasing triangle.

We introduce the p-index triangle $R_{n,m}^{(k)}$ corresponding to the triangle (2.18):

$$\begin{matrix} \begin{bmatrix} np^k \\ mp^k+1 \end{bmatrix} & \cdots & \begin{bmatrix} np^k \\ mp^k+p^k-1 \end{bmatrix} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \begin{bmatrix} np^k+p^k-2 \\ mp^k+p^k-1 \end{bmatrix} & \end{matrix} \tag{2.29}$$

where $n \geq 1$, $0 \leq m \leq n-1$.

It is not difficult to show that the correct equation for this triangle is $R_{n,m}^{(k)} = \begin{bmatrix} n \\ m \end{bmatrix} + R_{1,0}^{(k)}$. It follows that if we know the distribution of the entries in the triangle $R_{1,0}^{(k)}$, we can find the distribution of the elements in $R_{n,m}^{(k)}$ for any n,m, and moreover we will obtain the distribution of the binomial coefficients strictly divisible by a power of p in the triangle (2.18) for any n,m.

Consider the Pascal triangle whose base is the row numbered $N=p^k-1$. The corresponding p-index triangle $T_{0,0}^{(k)}$ consists of $p(p+1)/2$ p-index triangles $T_{n,m}^{(k-1)}$, where $n=0,1,\ldots,p-1$, and $0 \leq m \leq n$. The distributions of the elements in all the triangles $T_{n,m}^{(k-1)}$ are identical, and so we may replace each of them by $T_{0,0}^{(k-1)}$. Besides the triangles $T_{n,m}^{(k-1)}$, $T_{0,0}^{(k)}$ also contains $p(p-1)/2$ p-index triangles $R_{n,m}^{(k-1)}$, where $n=1,2,\ldots,p-1$, $0 \leq m \leq n-1$. The distributions in these are also identical, and so we may replace each of them by the triangle $R_{1,0}^{(k-1)}$.

Consider now the p-index triangle (2.29) for $n=1$, $m=0$, i.e., $R_{1,0}^{(k)}$. It may be shown that $R_{1,0}^{(k)}$ itself consists of $p(p-1)/2$ triangles $\overline{T}_{n,m}^{(k-1)}$, $1 \leq n \leq p-1$, $0 \leq m \leq p-n-1$, and $p(p+1)/2$ triangles $\overline{R}_{n,m}^{(k-1)}$, $1 \leq n \leq p$, $0 \leq m \leq p-n$. As before, each of the $\overline{T}_{n,m}^{(k-1)}$ may be replaced by

$\overline{T}^{(k-1)}_{1,1}$, and each of the $\overline{R}^{(k-1)}_{n,m}$ may be replaced by $\overline{R}^{(k-1)}_{1,0}$. Figure 20 shows $T^{(k)}_{0,0}$ and $R^{(k)}_{1,0}$, denoted by T_k and R_k, and the arrangement of the triangles $T^{(k-1)}_{0,0}$, $R^{(k-1)}_{1,0}$, $\overline{T}^{(k-1)}_{1,1}$, $\overline{R}^{(k-1)}_{1,0}$, denoted by T_{k-1}, R_{k-1}, $T^{(1)}_{k-1}$, $R^{(1)}_{k-1}$.

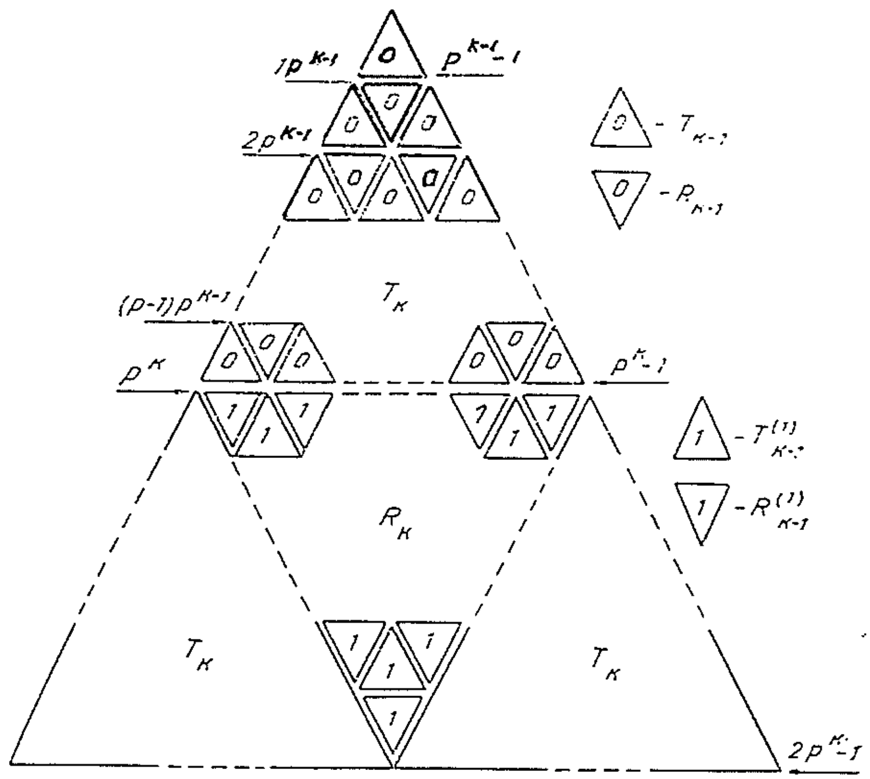


Figure 20

Using the equation given earlier and taking in account that $n<p$, we find that

$$\overline{T}^{(k-1)}_{1,1} = 1 + T^{(k-1)}_{0,0}, \qquad \overline{R}^{(k-1)}_{1,0} = 1 + R^{(k-1)}_{1,0}.$$

It follows that we can form the "geometric" equations

$$T^{(k)}_{0,0} = \tfrac{1}{2}p(p+1)T^{(k-1)}_{0,0} + \tfrac{1}{2}p(p-1)R^{(k-1)}_{1,0},$$

$$R^{(k)}_{1,0} = \tfrac{1}{2}p(p-1)\overline{T}^{(k-1)}_{1,1} + \tfrac{1}{2}p(p+1)\overline{R}^{(k-1)}_{1,0}.$$

Denote now by $P_k(s)$ and $Q_k(s)$ the number of occurrences of the value s in the respective triangles $T_{0,0}^{(k)}$ and $R_{1,0}^{(k)}$, where s is the greatest exponent of p such that p^s divides the corresponding binomial coefficient. From the geometric equations above, we can write the recurrence relations

$$\left.\begin{aligned} P_k(s) &= \tfrac{1}{2}p(p+1)P_{k-1}(s) + \tfrac{1}{2}p(p-1)Q_{k-1}(s), \\ Q_k(s) &= \tfrac{1}{2}p(p-1)P_{k-1}(s-1) + \tfrac{1}{2}p(p+1)Q_{k-1}(s-1), \end{aligned}\right\} \tag{2.30}$$

where k=2,3,..., and s=0,1,...,k−1 in the first equation and s=1,2,...,k in the second. For the initial conditions ($P_1(s)$, $Q_1(s)$ for s=0,1), we enumerate the numbers of zeros and ones in $T_{0,0}^{(1)}$, $R_{1,0}^{(1)}$, and find that

$$\begin{aligned} P_1(0) &= p(p+1)/2, \quad & P_1(1) &= 0 \\ Q_1(0) &= 0, & Q_1(1) &= p(p-1)/2. \end{aligned}$$

The system (2.30) is a special case of the system

$$\left.\begin{aligned} X_{k,s} &= aX_{k-1,s} + bY_{k-1,s}, \\ Y_{k,s} &= bX_{k-1,s-1} + aY_{k-1,s-1}, \end{aligned}\right\} \tag{2.31}$$

which we solve by the method discussed in [6]. If we further choose

$$X_{1,0} = a, \quad X_{1,1} = 0, \quad Y_{1,0} = 0, \quad Y_{1,1} = b,$$

then it may be shown by complete induction that the solution takes the form

$$X_{k,s} = a^k \sum_{i=0}^{\mu(k,s)} \left(\frac{b}{a}\right)^{2i+2} \binom{s-1}{i} \binom{k-s}{i+1}, \; s=0,\, 1,\ldots,\, k-1,$$

$$Y_{k,s} = a^k \sum_{i=0}^{\nu(k,s)} \left(\frac{b}{a}\right)^{2i+1} \binom{s-1}{i} \binom{k-s}{i}, \; s=1,\, 2,\ldots,\, k,$$

where $k=2,3,\ldots$; $\mu(k,s)=\min\{s-1,\, k-s-1\}$; $\nu(k,s)=\min\{s-1,k-s\}$.

It follows then that

$$\left.\begin{aligned} P_k(s) &= \binom{p+1}{2}^k \sum_{i=0}^{\mu(k,s)} \left(\frac{p-1}{p+1}\right)^{2i+2} \binom{s-1}{i} \binom{k-s}{i+1}, \\ Q_k(s) &= \binom{p+1}{2}^k \sum_{i=0}^{\nu(k,s)} \left(\frac{p-1}{p+1}\right)^{2i+1} \binom{s-1}{i} \binom{k-s}{i}. \end{aligned}\right\} \tag{2.32}$$

In particular,

$$P_k(0) = \binom{p+1}{2}^k, \; P_k(1) = (k-1)\, \frac{p-1}{p+1} \binom{p}{2} \binom{p+1}{2}^{k-1},$$

$$Q_k(0) = 0, \; Q_k(1) = \binom{p}{2} \binom{p+1}{2}^{k-1}.$$

Here $P_k(0)$ is the number of binomial coefficients not divisible by p in the Pascal triangle up through row p^k-1, i.e., the quantity $G(p^k\text{-}1,p)$ discussed previously in 2.1. The value of $P_k(s)$ is the number of coefficients exactly divisible by p^s, i.e,, the quantity $H_s(p^k-1,p)$.

Consider the Pascal triangle up through row N, where $p^k \le N \le p^{k+1}-2$. Then, putting $N=np^k+\ell$ for $0\le n \le p-1$, $0 \le \ell \le p^k-2$, if we know the distribution of the p-index triangle for the given n,ℓ in the part of the triangle above row p^k-1, we can find the number of binomial coefficients exactly divisible by p^s, $1\le s \le k$.

The determination of the number of coefficients exactly divisible by p^j in the Pascal triangle up through row p^r-1 is discussed in [103, 349, 369]. Let

$$S_j(r) = \sum_{a=0}^{p^r-1} \theta_j(a), \qquad S_j'(r) = \sum_{a=0}^{p^r-1} \psi_j(a),$$

where $\theta_j(a)$ is the number of binomial coefficients $\binom{a}{b}$ exactly divisible by p^j, and $\psi_j(a)$ is the number of products $(a+1)\binom{a}{b}$ exactly divisible by p^j. L. Carlitz [103] has shown, using generating functions, that

$$S_j(r) = \sum_{0<2k\le r} \binom{j-1}{k-1}\binom{r-j}{k}\binom{p}{2}^{2k}\binom{p+1}{2}^{r-2k}, \quad (0<j\le r),$$

$$S_j'(r) = \sum_{0\le 2k<r} \binom{j}{k}\binom{r-j-1}{k}\binom{p}{2}^{2k+1}\binom{p+1}{2}^{r-2k-1}, \quad (0\le j<r).$$

By a simple transformation, $S_j(r)$ and $S_j'(r)$ may be written in the form (2.32).

Analogous expressions for the number exactly divisible by p^j were introduced by D. Singmaster [349], who used the notations $A(\alpha,m)$, $B(\alpha,m)$; it can be shown, for example, that $B(j,p^r) = S_j(r)$. The problem was also studied by M. Sved and J. Pitman [369], who obtained the formulas

$$D(\alpha,m) = \sum_{h=1}^{m-\alpha}\sum_{j=0}^{\alpha-1} \binom{\alpha-1}{j}\binom{m-h-\alpha}{j}\binom{p}{2}^{2j+1}\binom{p+1}{2}^{m-h-(2j+1)}\binom{p^k}{2},$$

$$E(\alpha,m) = \sum_{j=0}^{\alpha-1} \binom{\alpha-1}{j}\binom{m-\alpha}{j+1}\binom{p}{2}^{2j+2}\binom{p+1}{2}^{m-2j-2},$$

where, in the Pascal triangle up through row p^m-1, $D(\alpha,m)$ is the number of binomial coefficients divisible by p^α, and $E(\alpha,m)$ is the number exactly divisible by p^α. These formulas can be transformed into the form of $S_j(r)$ or $P_k(s)$.

The distribution of binomial coefficients exactly divisible by 2^s is shown in Figure 21, and that for 3^s in Figure 22. It is interesting to represent the p-index Pascal triangles by colors of various shades for $s=0,1,2,\ldots$; a fragment of a colored p-index triangle for $p=2$ appears in C.K. Abachiev [1,2].

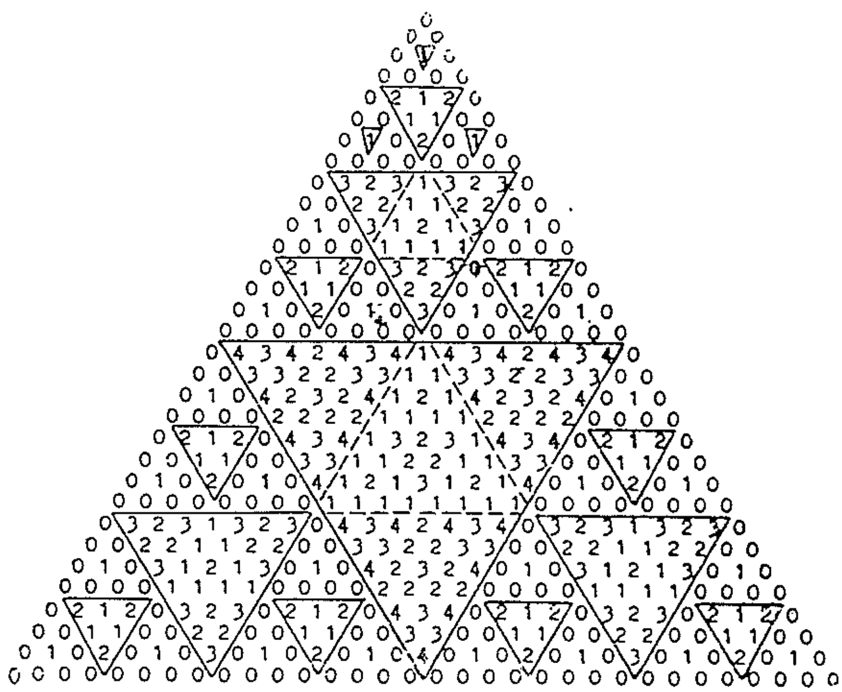


Figure 21

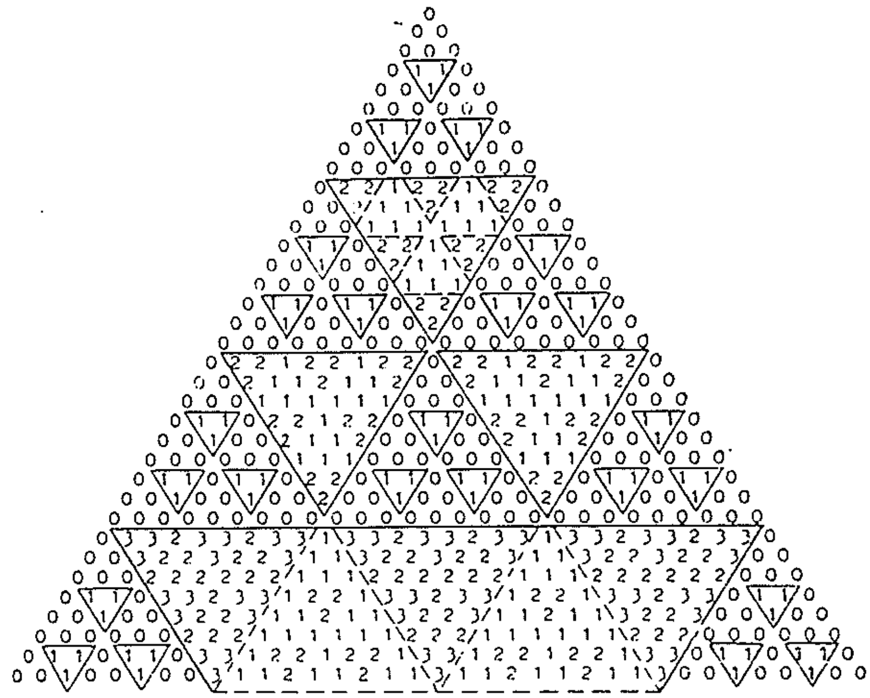


Figure 22

2.3 DIVISIBILITY OF TRINOMIAL COEFFICIENTS AND THEIR DISTRIBUTION MODULO THE PRIME p, AND ITS POWERS, IN THE PASCAL PYRAMID

To study divisibility questions for the trinomial coefficients $(n; m_1,m_2)$, discussed in 1.5, we need to extend some theorems established for the binomial coefficients in 2.1. We first note the analog of Lucas's Theorem, the generalization of which to the multi-dimensional case is given in [121].

<u>Theorem 2.7</u>. Let p be a prime, n,m_1,m_2 nonnegative whole numbers, $m_1 \leq n$, $m_2 \leq m_1$, and let the p-ary representations of these be $n=(a_r a_{r-1} \cdots a_0)_p$, $m_1=(b_r^1 b_{r-1}^1 \cdots b_0^1)_p$, $m_2=(b_r^2 b_{r-1}^2 \cdots b_0^2)_p$, where $a_r \neq 0$, $0 \leq a_k < p$, $0 \leq b_k^i < p$. Then

$$(n;\ m_1,m_2) \equiv \left(a_0;\ b_0^1,b_0^2\right)\left(a_1;\ b_1^1 b_1^2\right) \cdots \left(a_r;\ b_r^1 b_r^2\right) \pmod p, \tag{2.33}$$

where $(a_k;\ b_k^1 b_k^2) = 0$ if $b_k^1 > a_k$ or $b_k^2 > b_k^1$, $0 \le k \le r$.

Consider now the Pascal pyramid (cf. Fig. 15). We denote by g(n,p,3) the number of trinomial coefficients not divisible by p in the n^{th} cross section, and by h(n,p,3) the number divisible by p. Also let $g_j(n,p,3)$ denote the number of these coefficients for which $(n;\ m_1,m_2) \equiv j \pmod p$, $1 \le j \le p-1$, and let $h_v(n,p,3)$ denote the number of coefficients exactly divisible by p^v; again, these are for the n^{th} cross section. When the whole pyramid down to the n^{th} cross section, inclusive, is considered, the total numbers of coefficients satisfying the corresponding divisibility conditions will be denoted by G(n,p,3), H(n,p,3) $G_j(n,p,3)$, and $H_v(n,p,3)$.

Theorem 2.8. Let $n = (a_r a_{r-1} \cdots a_0)_p$ be the number of a cross section in the Pascal pyramid, and p a prime. Then

$$g(n,p,3) = \prod_{k=1}^{p-1} \binom{k+2}{2}^{f_k}, \tag{2.34}$$

where f_k is the number of digits k, $1 \le k \le p-1$, among $a_0, a_1, \ldots, a_r$.

The proof of Theorem 2.8 follows from Theorem 2.7. Note that if the cross section number $n = p^r$, $r = 1,2,\ldots$, then it follows from Theorem 2.8 that in this cross section only the three coefficients (n; 0,0), (n; n,0), (n; n,n) are ones, and not divisible by p.

Theorem 2.9. With the same hypothesis as Theorem 2.8, we have

$$G(n,p,3) = \frac{1}{3} \sum_{i=0}^{r} b_{r-i} \binom{p+2}{3}^{r-i} \prod_{j=0}^{i} \binom{b_{r-j}+2}{2}, \tag{2.35}$$

where $n+1 = (b_r b_{r-1} \cdots b_0)_p$.

This theorem is proved in the same way as Theorem 2.5. From Theorem 2.9, if $n=p^r-1$, we note that

$$G(p^r-1,p,3) = \binom{p+2}{3}^r. \qquad (2.36)$$

Since the total number of coefficients in the n^{th} cross section is $\binom{n+2}{2}$, we have that

$$h(n,p,3) = \binom{n+2}{2} - g(n,p,3),$$

and, since the total number of coefficients in the pyramid down through the n^{th} cross section is $\binom{n+3}{3}$, by the same token,

$$H(n,p,3) = \binom{n+3}{3} - G(n,p,3).$$

Theorem 2.10. Let p be a prime. Then for $n\rightarrow\infty$

$$\lim[G(n,p,3)/H(n,p,3)] = 0.$$

The proof of this theorem uses (2.36), and is like the proof of Theorem 2.6. As for the binomial coefficients, we may formulate two principal problems for the trinomial coefficients. The first is to obtain the value of $g_j(n,p,3)$, the number of trinomial coefficients in the n^{th} cross section with residue j (mod p), and the value of $G_j(n,p,3)$, the total number of coefficients with residue j (mod p) in the whole pyramid down through the n^{th} cross section. The second problem is that of obtaining the distributions of the coefficients with respect to strict divisibility by p^v, for both the cross section and the pyramid, as above.

The solution of the first problem we may think of as depending on determining the residues of the three elements in the corners of the triangular elements in the $(n-1)^{st}$ cross section. Obtaining $g_j(n,p,3)$ and $G_j(n,p,3)$ then reduces to the formulation of the corresponding recurrence relations and their solutions. As examples, we show the distributions of the trinomials mod 2 in Figure 23, and mod 3 in Figure 24, through the 12^{th} cross section.

An algorithm for constructing the distributions of the trinomial coefficients with respect to strict divisibility by p^v for any cross section is as follows. Let n be the cross section number, and construct the Pascal triangle for $\binom{n}{m_1}$, $0 \le m_1 \le n$. Using the algorithm of section 2.2, construct the "triangular distribution" of the binomial coefficients (for strict divisibility by p^v) for this triangle. Then, based on the equation $(n; m_1, m_2) = \binom{n}{m_1}\binom{n}{m_2}$, add to each of the elements of the rows of the "triangular distribution" the elements of the base row rotated counterclockwise by 90°. The result is the desired distribution.

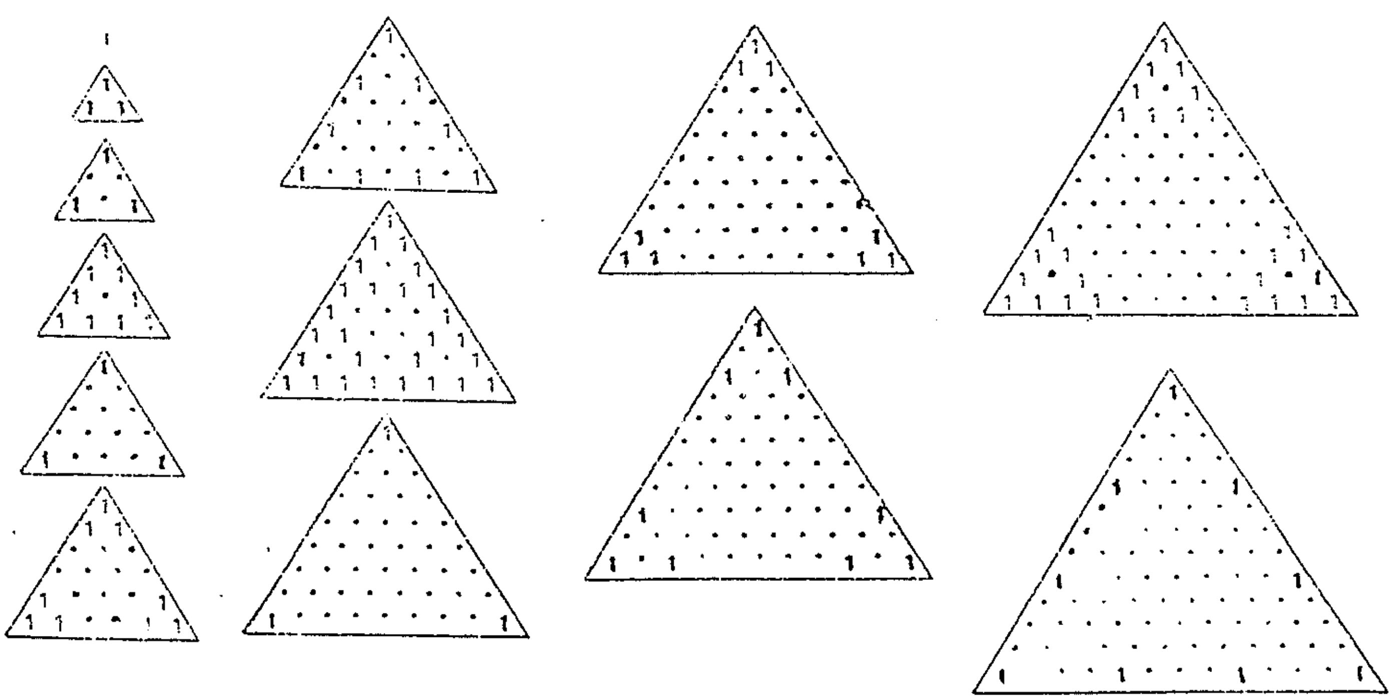


Figure 23

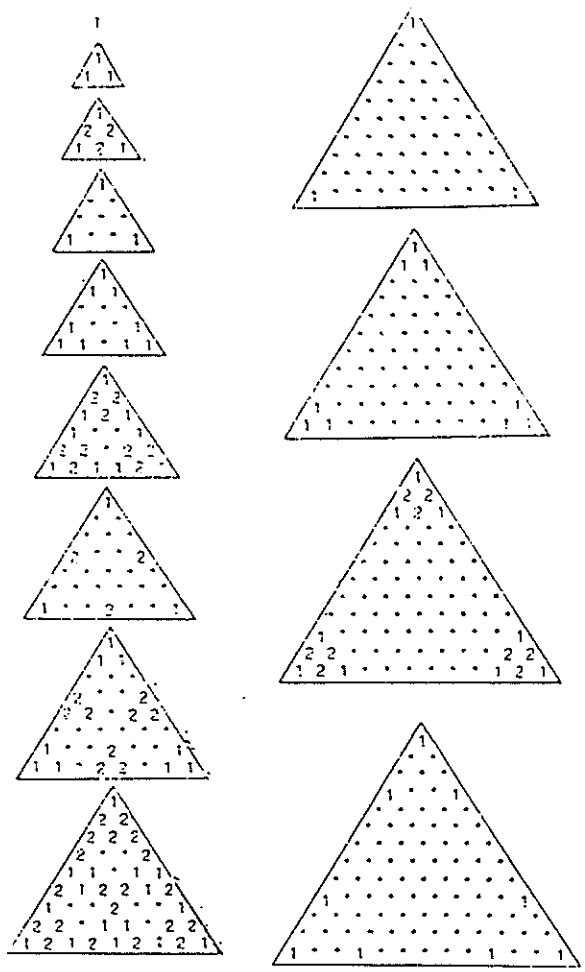
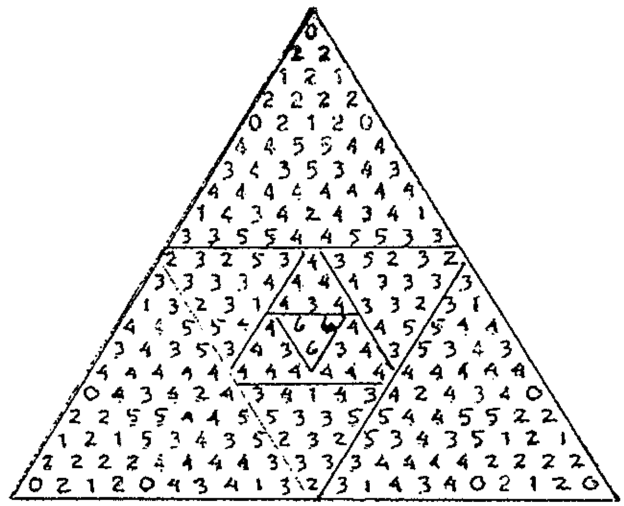


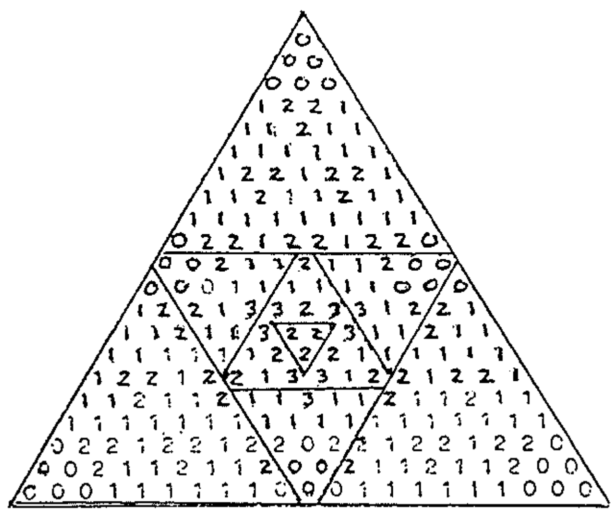
Figure 24

We note that the distribution of the trinomial coefficients with respect to divisibility by p^v in the cross section $n=p^r-1$ coincides with the corresponding distribution of the binomial coefficients in the Pascal triangle whose base is row $n=p^r-1$, for any r.

As examples we show the distributions of the trinomial coefficients with respect to strict divisibility by 2^v in Figure 25a, and by 3^v in Figure 25b, for the 20[th] cross section of the Pascal pyramid.



a



b

Figure 25

2.4 DIVISIBILITY OF THE MULTINOMIAL COEFFICIENTS BY THE PRIME p AND ITS POWERS

Questions of divisibility specifically for the multinomial coefficients, the determination of the number divisible, or not divisible, by a prime or power of a prime, in a cross section of the hyperpyramid or the whole hyperpyramid, and problems related to these topics are treated in [14-16, 18, 67, 121, 225, 275, 348]. The discussion of these results again begins with the extension of Lucas's Theorem [266] to the multinomial case, which is given in L.E. Dickson [121]; it will also be useful to represent the multinomial coefficients in the form (1.41), and denote them by $(n; n_1, n_2, \ldots, n_s)$.

We write the p-ary representations

$$n = (a_r a_{r-1} \cdots a_0)_p, \quad n_i = (b_r^i b_{r-1}^i \cdots b_0^i)_p, \tag{2.37}$$

where $a_r \neq 0$, $0 \leq a_k < p$, $0 \leq b_k^i < p$, $0 \leq k \leq r$, $1 \leq i \leq s$.

<u>Theorem 2.11</u>. Let p be a prime, n and n_i nonnegative whole numbers with p-ary representations (2.37). Then

$$(n; n_1, n_2, \ldots, n_s) \equiv \prod_{k=0}^{r} \left(a_k;\ b_k^1 b_k^2 \cdots b_k^s\right) \pmod{p}, \tag{2.38}$$

in which $(a_k;\ b_k^1 b_k^2 \cdots b_k^s) = 0$ if $b_k^1 + b_k^2 + \cdots + b_k^s \neq a_k$.

It follows from Theorem 2.11 that $(n; n_1, \ldots, n_s) \not\equiv 0 \pmod{p}$ if and only if $b_k^1 + b_k^2 + \cdots + b_k^s = a_k$ for all values of k.

Known and new methods for determining the highest power of the prime p which divides a multinomial coefficient are presented in detail in the article of R.J. Martin and G.L. Mullen [275].

Theorem 2.12. If p is a prime, the multinomial coefficient $(n; n_1,\ldots,n_s)$ is divisible by p^ν if and only if

$$E(p,n!) - \sum_{i=1}^{s} E(p,n_i!) \geq \nu,$$

where E(p,n!) denotes the highest power of p which divides n!.

Let n and n_i be written in p-ary form (2.37), and form the system of equations:

$$\left.\begin{array}{l} b_0^1 + \cdots + b_0^s = \varepsilon_0 p + a_0, \\ \varepsilon_0 + b_1^1 + \cdots + b_1^s = \varepsilon_1 p + a_1, \\ \dots\dots\dots\dots\dots\dots \\ \varepsilon_{r-2} + b_{r-1}^1 + \cdots + b_{r-1}^s = \varepsilon_{r-1} p + a_{r-1}, \\ \varepsilon_{r-1} + b_r^1 + \cdots + b_r^s = a_r. \end{array}\right\} \quad (2.39)$$

Then R.D. Fray in [143] proved the following result.

Theorem 2.13. The highest power of the prime p which divides $(n; n_1,n_2,\ldots,n_s)$ is given by

$$\nu = \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_{r-1},$$

where the values $\varepsilon_0,\ldots,\varepsilon_{r-1}$ are chosen from among $0,1,\ldots,s-1$ so as to satisfy (2.39).

If we put $S(n) = a_0+a_1+\cdots+a_r$ and $S(n_i) = b_0^i+b_1^i+\cdots+b_r^i$, then it may be shown [225] that

$$\nu = \frac{1}{p-1}\left[S(n_1)+S(n_2)+\cdots+S(n_s)-S(n)\right]. \tag{2.40}$$

In [275] Martin and Mullen worked out a new, more effective method for calculating ν, based on obtaining the residues of $n_1,n_2,\ldots,n_s$ modulo distinct powers of p. Denote by n_i^j the residue of n_i (mod p^j), for $1\le i\le s$, $1\le j\le h$, where $p^h\le n<p^{h+1}$, $h=[\log n/\log p]$. Then the following theorem is from [275].

Theorem 2.14. The multinomial coefficient $(n;\, n_1,n_2,\ldots,n_s)\equiv 0 \pmod{p^\nu}$ if and only if

$$\sum_{j=1}^{h} \frac{1}{p^j}\left(n_1^j+n_2^j+\cdots+n_s^j\right) \ge \nu.$$

It follows from Theorem 2.14 that for $\nu=1$, $(n;n_1,\ldots,n_s)\equiv 0 \pmod p$ if and only if for some value of j, $(n_1^j+n_2^j+\cdots+n_s^j)\ge p^j$.

D. Singmaster [348] discussed the question of the least value of n for which $(n;\, n_1\ldots,n_s)$ is divisible by p^ν, and obtained the following theorem.

Theorem 2.15. Let the power ν of the prime $p\ge s$ be represented in the form $\nu=a(s-1)+b$, where $0<b\le s-1$. Then the least value of n for which $(n;\, n_1,\ldots,n_s)$ is divisible by p^ν is $n=bp^{a+1}$.

He also considered [352] various properties of the multinomials and proved the following result.

Theorem 2.16. The multinomial coefficient $(n; n_1,\ldots,n_s)$ in which n is strictly divisible by p^v, and n_i is strictly divisible by p^{t_i}, is divisible by p^{s-t} if $t \leq v$, where $t=\min\{t_i\}$.

The problems of determining the number of multinomial coefficients not divisible by p, or divisible by p^v, are discussed in [15, 16, 18, 225]; in these problems it is sometimes convenient to use the form (1.43) for the multinomial coefficients.

Let g(n,p,s) be the number of multinomial coefficients $(n; n_1,\ldots,n_s)$ not divisible by p in the n^{th} cross section of the Pascal hyperpyramid, and h(n,p,s) the number divisible by p. Also, let $g_j(n,p,s)$ be the number congruent to j (mod p); then we have

$$g(n,p,s) = g_1(n,p,s) + g_2(n,p,s) + \cdots + g_{p-1}(n,p,s).$$

Likewise

$$h(n,p,s) = h_1(n,p,s) + h_2(n,p,s) + \cdots + h_q(n,p,s),$$

where $h_v(n,p,s)$ denotes the number of multinomial coefficients $(n; m_1,m_2,\ldots,m_{s-1})$ in the n^{th} cross section divisible by p^v, and $q=\max\{v\}$. For the total numbers of coefficients in the hyperpyramid satisfying the corresponding conditions we use the notations G(n,p,s), H(n,p,s), $G_j(n,p,s)$, and $H_v(n,p,s)$. Then

$$G(n,p,s) = G_1(n,p,s) + \cdots + G_{p-1}(n,p,s),$$

$$H(n,p,s) = H_1(n,p,s) + \cdots + H_q(n,p,s).$$

Theorem 2.17. Let $n = (a_r a_{r-1}\ldots a_0)_p$ be a cross section number in the hyperpyramid, and p a prime. Then

$$g(n,p,s) = \prod_{k=1}^{p-1} \binom{k+s-1}{s-1}^{f_k}, \qquad (2.41)$$

where f_k is the number of digits k among $a_r,\dots,a_0$. In the proof of Theorem 2.17 we use Theorem 2.11 and a corresponding transformation; as a result, (2.41) differs from the representation obtained in [18, 225].

In [225], F.T. Howard extended the results in [100, 103, 222, 223] and obtained formulas for the quantities $\theta_0(s,n)$, $\theta_1(s,n)$, $\theta_2(s,n)$, where these are the numbers of multinomial coefficients strictly divisible by p^0, p^1, p^2. For $\theta_\nu(s,n)$, $\nu>2$, he constructed the corresponding generating function and found explicit expressions for $\theta_\nu(s,n)$ for certain values of n.

Denote by C(i) the coefficients in the expansion of $(1+x+x^2+\cdots+x^{p-1})^s$ in powers of x, where p is a given prime, and s is the "dimension" of the multinomial $(n;\, n_1,\dots,n_s)$. Howard proved that

$$C(a+bp) = \sum_{k=0}^{b} (-1)^k \binom{s}{k} \binom{s+a+bp-kp-1}{s-1}. \tag{2.42}$$

In (2.42), which is analogous to (1.16) for generalized binomial coefficients, a and b satisfy $0\le a<p$, $0\le b$. Using only the coefficients C(i) and the p-ary representation $n=(a_r a_{r-1}\cdots a_0)_p$, Howard [225] proved a theorem containing the following formulas:

$$\theta_0(s,n) = C(a_0)C(a_1)\ldots C(a_r),$$

$$\theta_1(s,n) = \sum_{i=0}^{r-1} C(a_0)\ldots C(a_{i-1})C(a_i+p)C(a_{i+1}-1)C(a_{i+2})\ldots C(a_r),$$

$$\theta_2(s,n) = \sum_{i=0}^{r-2} C(p+a_i)C(p+a_{i+1}-1)C(a_{i+2}-1)A_i + \sum_{i=0}^{r-1} C(2p+a_i)C(a_{i+1}-2)B_i$$

$$+ \sum_{i=0}^{r-3} \sum_{k=i+2}^{r-1} C(p+a_i)C(a_{i+1}-1)C(p+a_k)C(a_{k+1}-1)H_{i,k}.$$

The values A_i, B_i, $H_{i,k}$, mentioned in [225], may be written in the form

$$A_i = P_n/Q_iC(a_{i+2}), \quad B_i = P_n/Q_i, \quad H_{i,k} = P_n/Q_iQ_k,$$

where

$$P_n = \prod_{j=0}^{r} C(a_j), \quad Q_i = C(a_i)C(a_{i+1}), \quad Q_k = C(a_k)C(a_{k+1}).$$

Also given is $\theta_v(n)$ for the values $n=a+bp$, $n=a+p^2$, $n=a+2p^2$, $n=a+bp+p^2$.

N.A. Volodin [18] developed a formula for the number of multinomial coefficients not divisible by p, and for the number divisible by p, in the form of a sum of products of binomial coefficients. Methods for obtaining the number of multinomial coefficients not divisible by p in the Pascal hyperpyramid, are given in [15, 16, 18].

Theorem 2.18. Let the base of the Pascal hyperpyramid of dimension s be the n^{th} cross section, and p a prime. Then

$$G(n,p,s) = \frac{1}{s}\sum_{i=0}^{r} b_{r-i}\binom{p+s-1}{s}^{r-i}\prod_{j=0}^{i}\binom{b_{r-j}+s-1}{s-1}, \tag{2.43}$$

where $n+1=(b_rb_{r-1}\cdots b_0)_p$.

The proof of Theorem 2.18 is analogous to the proofs of Theorem 2.5 and Theorem 2.9. If $n=p^r-1$, $r\geq 1$, from (2,43) we find that

$$G(p^r-1,\ p,\ s) = \binom{p+s-1}{s}^r. \tag{2.44}$$

The total number of multinomial coefficients in the n^{th} cross section of the Pascal hyperpyramid of dimension s is $\binom{n+s-1}{s-1}$. Thus,

$$h(n,p,s) = \binom{n+s-1}{s-1} - g(n,p,s).$$

Likewise,

$$H(n,p,s) = \binom{n+s}{s} - G(n,p,s),$$

since $\binom{n+s}{s}$ is the total number of coefficients in the Pascal hyperpyramid of dimension s and whose base is numbered n.

<u>Theorem 2.19</u>. If p is a prime, then for $n\to\infty$,

$$\lim[G(n,p,s)/H(n,p,s)] = 0.$$

The proof of Theorem 2.19 is like those of Theorem 2.6 and Theorem 2.10, and uses (2.44).

The problems of determining $G_j(n,p,s)$ and $H_v(n,p,s)$ (using the usual notation) may also be formulated for the multinomial coefficients.

2.5 GREATEST COMMON DIVISORS AND LEAST COMMON MULTIPLES OF BINOMIAL COEFFICIENTS. FACTORIZATION

We consider here some questions associated with the greatest common divisor (GCD) and the least common multiple (LCM) of those binomial coefficients which are arranged in some definite pattern in the Pascal triangle.

On the basis of the known equation

$$\binom{n-1}{m}\binom{n}{m-1}\binom{n+1}{m+1} = \binom{n-1}{m-1}\binom{n}{m+1}\binom{n+1}{m}$$

H.W. Gould [159] stated the conjecture that

$$GCD\left\{\binom{n-1}{m}, \binom{n}{m-1}, \binom{n+1}{m+1}\right\} = GCD\left\{\binom{n-1}{m-1}, \binom{n}{m+1}, \binom{n+1}{m}\right\},$$

which was proved in various ways by A.P. Hillman and V.E. Hoggatt [187], D. Singmaster [350], and E.G. Straus [364].

C.T. Long [262] discussed the triangle ∇_n of binomial coefficients of the form

$$\begin{matrix} \binom{n}{1} & \cdots & \binom{n}{n-1} \\ \cdot & & \cdot \\ & \cdot \quad \cdot & \\ & \cdot & \\ & \binom{2n-2}{n-1} & \end{matrix} .$$

If we denote the GCD of all elements in this triangle by d, and the GCD of the three corner elements by D, he proved that: d=D=p if n=p; d=p, $D=p^s$ if $n=p^s$ (s>1); d=1, D=n for all $n \neq p^s$, where p is a prime and s is a whole number.

Let $n = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$ and k be whole numbers satisfying $1 \le k \le \min\{p_i^{s_i}\}$, $1 \le i \le r$, and denote by m the product of all divisors of n of the form p^s, where $p^s \le k \le p^{s+1}$. T. Tonkov [49] proved that

$$GCD\left\{\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{m}\right\} = \frac{n}{m}.$$

In [345] G.J. Simmons showed that there are infinitely many values of m for which m! is a divisor of $\binom{n}{m}$, but m!p for $p \le m$ does not divide this coefficient. He further proved that for given N,m, there exist infinitely many n such that $GCD\left\{\binom{n}{m}, N\right\} = 1$.

J. Albree [56] proved that for $1 \le m \le n-1$, if GCD{m,p}=1, then $GCD\left\{\binom{n}{m}\right\} = p^s$, where s is the highest power for which p^s divides n.

Let

$$a = \binom{n-1}{m-1}\binom{n+1}{m}, \quad b = \binom{n+1}{m}\binom{n}{m+1}, \quad c = \binom{n}{m+1}\binom{n-1}{m-1},$$
$$d = \binom{n}{m-1}\binom{n+1}{m+1}, \quad e = \binom{n+1}{m+1}\binom{n-1}{m}, \quad f = \binom{n-1}{m}\binom{n}{m-1}.$$

Then H.M. Edgar [128] proved that

$$LCM\{a,b,c\} = LCM\{d,e,f\}.$$

In [393], I.S. Williams showed that for powers of primes $p_i^{s_i}$, where $p_i^{s_i} \leq n+1 \leq p_i^{s_i+1}$,

$$LCM \left\{ \binom{n}{m} \right\} = \left(\prod_i p_i^{s_i} \right) / n+1,$$

where the product is taken over all primes $p_i \leq n+1$.

R. Meynieux [280] discussed questions connected with the LCM of binomial coefficients, and with determining the powers of primes which occur in the factorizations into prime factors of binomial coefficients. A typical result is as follows. Let λ_p be the power of the prime p in the factorization of $\binom{n}{m}$; let $\mu_p(n) = \sup_m \lambda_p$; for $m \leq n/2$ let $\rho_p(n)$ be the largest prime such that $\lambda_p = \mu_p$; and let $\rho(n) = \inf \rho_p(n)$ for p belonging to the set of primes occurring in the factorization. Then $\rho(n) \geq (n-1)/3$ and $\lim[\rho(n)/n] = 1/3$ for $n \to \infty$.

Problems connected with the factorization into prime factors of the binomial coefficients, asymptotic estimates, and other topics are treated in the works of P. Erdos [132, 133], P. Erdös and R. Graham [134], P. Erdös, H. Gupta, and S.P. Khare [135], H. Gupta and S.P. Khare [168], and S.P. Khare [237], among others. Omitting details, we summarize three of these papers. Khare [135] proves a theorem on the factorization of binomial coefficients and gives tables of such factorizations for special conditions imposed on n and m. Included also is a discussion of the case where $\binom{n}{m}$ has m prime factors, e.g., $\binom{4}{2} = 2 \cdot 3$, $\binom{10}{4} = 2 \cdot 3 \cdot 5 \cdot 7$, and so on. In [168] it is shown that $\binom{n^2}{n}$ is greater than the product of the first n primes for $2 < n < 1794$, and less than this product for $n \geq 1794$. And [237] gives tables of factors of n! for $n \leq 1000$.

The factorizations of the binomial coefficients $\binom{n}{m}$ up through n=54 are given in the book of T.M. Green and C.L. Hamberg [162]. Matters related one way or another to this topic are also discussed in [108, 130, 140, 166, 169, 279].