CHAPTER 7

COMBINATORIAL ALGORITHMS FOR CONSTRUCTING GENERALIZED-HOMOGENEOUS POLYNOMIALS. SOME CLASSES OF NON-ORTHOGONAL POLYNOMIALS

In this chapter, based on some new concepts and definitions - in particular the generalized power of a variable, and of a monomial - we introduce the so-called generalized-homogeneous polynomial. We will establish a formula for the maximal number of terms of the generalized-homogeneous polynomial and find for these polynomials the analog of the Euler formula. We also introduce the notion of the factorial power, and use it to construct and study factorial polynomials. For these, we establish differentiation and integration formulas, and other relations.

We will introduce also the so-called exponent matrix and coefficient matrix, from which we may develop combinatorial algorithms for the construction of basis systems of polynomial solutions for generalized polyharmonic equations, and equations with mixed derivatives; various forms of the Pascal triangle are used in these algorithms. We will consider polynomial solutions for equations with third-order partial derivatives; more detailed studies may be found in [5-10, 13, 14] and in [82-87].

7.1 GENERALIZED-HOMOGENEOUS POLYNOMIALS. EULER'S FORMULA

As is known, a polynomial is said to be homogeneous if every term is of the same degree. We generalize this basic notion, which will allow the possibility of constructing basis systems of polynomial solutions for a wide class of differential equations, including those with mixed derivatives; we use the methods of combinatorial analysis in carrying out this development. For greater clarity, we will consider the case of three variables; the extension to any number of variables is presented in [14].

Before we consider any new concepts, let us consider the vectors p, q, r:

$$p = (p_1,p_2,p_3), \quad q = (q_1,q_2,q_3), \quad r = (r_1,r_2,r_3),$$

where p and q are nonnegative whole numbers, and r is a positive whole number. We will assume that $r_1 \le r_2 \le r_3$, and introduce the transformation

$$\left.\begin{aligned} q_1 &= r_3\left[\frac{p_1}{r_1}\right] + r_1\left\{\frac{p_1}{r_1}\right\}, \\ q_2 &= r_3\left[\frac{p_2}{r_2}\right] + r_2\left\{\frac{p_2}{r_2}\right\}, \\ q_3 &= r_3\left[\frac{p_3}{r_3}\right] + r_3\left\{\frac{p_3}{r_3}\right\} = p_3, \end{aligned}\right\} \tag{7.1}$$

where [A] denotes the integer part of A, and {A} the fractional part. It is not hard to see that (7.1) is unique, and takes nonnegative whole numbers into nonnegative whole numbers. That is, suppose that two distinct values p_k' and p_k'' correspond to the same value q_k. We can then put $p_k' = n$, and $p_k'' = n+m$, with n,m nonnegative whole numbers, and we will have

$$r_3\left[\frac{n+m}{r_k}\right] + r_k\left\{\frac{n+m}{r_k}\right\} = r_3\left[\frac{n}{r_k}\right] + r_k\left\{\frac{n}{r_k}\right\}, \quad k = 1, 2, 3.$$

We write this in the form

$$(r_3 - r_k)\left[\frac{n+m}{r_k}\right] + r_k\left[\frac{n+m}{r_k}\right] + r_k\left\{\frac{n+m}{r_k}\right\}$$

$$= (r_3 - r_k)\left[\frac{n}{r_k}\right] + r_k\left[\frac{n}{r_k}\right] + r_k\left\{\frac{n}{r_k}\right\},$$

and note that for any nonnegative whole number a and natural number b it is true that

$$b\left[\frac{a}{b}\right] + b\left\{\frac{a}{b}\right\} = a;$$

then

$$(r_3 - r_k)\left(\left[\frac{n+m}{r_k}\right] - \left[\frac{n}{r_k}\right]\right) + m = 0,$$

and since $r_3 - r_k \geq 0$, $\left[\frac{n+m}{r_k}\right] - \left[\frac{n}{r_k}\right] \geq 0$, it follows that the left hand side reduces to zero only for $m=0$, and so $p_k' = p_k''$ and (7.1) is unique.

Consider also the transformation

$$\left.\begin{aligned} p_1 &= r_1\left[\frac{q_1}{r_3}\right] + r_3\left\{\frac{q_1}{r_3}\right\}, \\ p_2 &= r_2\left[\frac{q_2}{r_3}\right] + r_3\left\{\frac{q_2}{r_3}\right\}, \\ p_3 &= r_3\left[\frac{q_3}{r_3}\right] + r_3\left\{\frac{q_3}{r_3}\right\} = q_3. \end{aligned}\right\} \quad (7.2)$$

Let us represent $q_k(k=1,2)$ in the form $q_k=m_kr_3+s_k$, where $s_k=0,1,\ldots,r_3-1$. It is not hard to see that (7.2) is unique for $s_k=0,1,\ldots,r_k-1$, but not for $s_k=r_k,r_{k+1},\ldots,r_3-1$. Also, let $x=(x_1,x_2,x_3)$, with p_k the power of x_k.

Definition 7.1. The number q_k, obtained from the transformation (7.1), is said to be the generalized power (or degree) of the variable x_k relative to the pair (r_k,r_3).

If $r_1=r_2=r_3$, then the generalized power reduces to the ordinary power.

Consider now the monomial $x^p=x_1^{p_1}x_2^{p_2}x_3^{p_3}$.

Definition 7.2. The number $n=|q|=q_1+q_2+q_3$, where q_k is the generalized power of the variable x_k relative to (r_k,r_3), is said to be the generalized power (or degree) of the monomial relative to $r=(r_1,r_2,r_3)$.

Monomials having equal generalized powers relative to the same r are said to be generalized-homogeneous.

Definition 7.3. A polynomial whose terms have equal generalized powers relative to some $r=(r_1,r_2,r_3)$ is said to be generalized-homogeneous, and the number $n=|q|=q_1+q_2+q_3$ is its generalized degree.

As we know, the maximal number of distinct homogeneous monomials in three variables with total degree n is $N_n=(n+1)(n+2)/2$. Noting this, and using the method of combinatorial analysis as in [14, 15, 41], we find that the number of generalized-homogeneous monomials with degree n relative to $r=(r_1,r_2,r_3)$ is given by

$$N_n(r_1,r_2,r_3) = \frac{1}{2}m(m+1)r_1r_2 + (m+1)f_s(r_1,r_2,r_3) \quad (n = mr_3 + s), \tag{7.3}$$

where

$$f_s(r) = \begin{cases} \frac{1}{2}(s+1)(s+2), \; s = 0,1,\ldots,r_1-1, \\ \frac{1}{2}r_1(r_1+1) + r_1(s-r_1+1), \; s = r_1, r_1+1,\ldots,r_2-1, \\ \frac{1}{2}r_1(r_2-r_1+1) + \frac{1}{2}(2r_1+r_2-s+2)(s-r_1+1), \; s = r_2,\ldots,r_1+r_2-2, \\ \frac{1}{2}r_1 r_2, \; s = r_1+r_2-1, r_1+r_2,\ldots,r_3-1. \end{cases}$$

If $r_1=r_2=r_3=1$, then $m=n$, $s=0$, $f_0(1,1,1)=1$, and $N_n(1,1,1)=(n+1)(n+2)/2$, so that in this case the generalized-homogeneous monomial of degree n reduces to an ordinary monomial of degree n.

It can be shown from (7.3) that for $n\to\infty$,

$$N_n(r_1,r_2,r_3) \cong r_1 r_2 (n+1)(n+2) / 2r_3^2. \tag{7.4}$$

This asymptotic formula allows us to determine (approximate) the number of distinct generalized-homogeneous monomials for large n and arbitrary r_1,r_2,r_3.

As is known, a homogeneous function of degree λ, continuously differentiable in its domain of definition, satisfies

$$f(x_1,x_2,x_3) = \frac{1}{\lambda} \sum_{k=1}^{3} x_k \frac{\partial}{\partial x_k} f(x_1,x_2,x_3),$$

which is Euler's formula, and is widely used in various parts of mathematics. If $f(x_1,x_2,x_3)$ is a polynomial $P^n(x_1,x_2,x_3)$, the formula takes the form

$$P^n(x_1,x_2,x_3) = \frac{1}{n}\left(x_1\frac{\partial}{\partial x_1}P^n + x_2\frac{\partial}{\partial x_2}P^n + x_3\frac{\partial}{\partial x_3}P^n\right). \tag{7.5}$$

Let $P_r^n(x_1,x_2,x_3)$ be a generalized-homogeneous polynomial of degree n relative to $r=(r_1,r_2,r_3)$, with generalized degree $n=|q|=q_1+q_2+q_3$. It may be shown [10, 14] that any generalized-homogeneous polynomial may be written in the form

$$P_r^n(x_1,x_2,x_3) = \sum_{\alpha_1=0}^{r_1-1}\sum_{\alpha_2=0}^{r_2-1} P_{r;\alpha}^n(x_1,x_2,x_3),$$

where $\alpha=(\alpha_1,\alpha_2)$. As the result of a transformation [10, 14] we obtain the analog of Euler's formula for generalized-homogeneous polynomials:

$$P^n(x_1,x_2,x_3) = \sum_{k=1}^{3}\frac{r_3}{r_k}\sum_{\alpha_1=0}^{r_1-1}\sum_{\alpha_2=0}^{r_2-1}\frac{x_k}{n+\delta(r;\alpha)}\frac{\partial}{\partial x_k}P_{r;\alpha}^n(x_1,x_2,x_3), \tag{7.6}$$

where

$$\delta(r,\alpha) = \sum_{j=1}^{2}(r_3-r_j)\frac{\alpha_i}{r_j}.$$

If $r_1=r_2=r_3$, then $\alpha_1=\alpha_2=0$ and $\delta(r,\alpha)=0$, and (7.6) reduces to the ordinary Euler formula. The generalized formula (7.6) is used to derive recurrence and other relations for generalized-homogeneous polynomials.

7.2 FACTORIAL POLYNOMIALS. ALGORITHMS FOR CONSTRUCTING "EXPONENT MATRICES"

Consider the simple case of an arbitrary polynomial in one variable

$$P_n(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1}x + a_n, \tag{7.7}$$

where the a_i are real numbers. Various operations performed on such polynomials - differentiation, integration, arithmetic operations - may be considerably simplified by introducing the factorial powers. We transform the polynomial $P_n(x)$ to a factorial polynomial by using the factorial power of the variable x, i.e., we set

$$x^{m,!} = \frac{x^m}{m!} = x(m).$$

The result, which we indicate by $P_n(x) = P_{n,!}(x)$, is that

$$P_{n,!}(x) = b_0 x^{n,!} + b_1 x^{n-1,!} + \dots + b_{n-1} x^{1,!} + b_n, \tag{7.8}$$

where $b_k = (n-k)!\, a_k$. With any polynomial (7.8), which is said to be the basic factorial polynomial associated with the polynomial (7.7), we may relate two systems of so-called associated polynomials

$$P_{n-m,!}(x) \quad (m=1,2,\dots,n) \text{ and } P_{n+m,!}(x) \quad (m=1,2,\dots),$$

where these are defined by:

$$P_{n-m,!}(x) = b_0 x^{n-m,!} + b_1 x^{n-m-1,!} + \dots + b_{n-m+1} x^{1,!} + b_{n-m}, \tag{7.9}$$

$$P_{n+m,!}(x) = b_0 x^{n+m,!} + b_1 x^{n+m-1,!} + \dots + b_{n-1} x^{m+1,!} + b_n x^{m,!}$$
$$+ c_{n+1} x^{m-1,!} + c_{n+2} x^{m-2,!} + \dots + c_{n+m-1} x^{1,!} + c_{n+m}, \quad (7.10)$$

and $c_{n+1}, c_{n+2}, \dots, c_{n+m}$ are arbitrary constants.

We introduce the associated polynomial to simplify differentiation and integration; thus,

$$\frac{d^k}{dx^k} P_{n,!}(x) = P_{n-k,!}(x), \quad (7.11)$$

$$\int \cdots \int P_{n,!}(x)\, dx \dots dx = P_{n+k,!}(x). \quad (7.12)$$

After carrying out such operations on $P_{n,!}(x)$, the result may again be transformed to an ordinary polynomial, with the coefficients being related by $a_k = b_k/(n-k)!$. The operations of addition and subtraction of factorial polynomials are as for ordinary polynomials; for finding products and quotients we use the formulas

$$x^{n,!} x^{m,!} = \binom{n+m}{m} x^{n+m,!}, \quad (7.13)$$

$$\frac{x^{n,!}}{x^{m,!}} = \binom{n}{m}^{-1} x^{n-m,!}, \quad m<n. \quad (7.14)$$

It is not hard to extend the idea of the factorial polynomial to the case of more than one variable and obtain the relations corresponding to (7.7)-(7.14). The factorial powers, monomials, and polynomials, as established in [10, 14] and other works, are introduced not only to simplify forms; they also are important in obtaining solutions, invariant with respect to the order, of partial differential equations of high order.

We turn now to the construction of generalized-homogeneous monomials for a given n and $r=(r_1,r_2,r_3)$. First, we construct the "exponent matrix" for the degree n of the homogeneous monomial; then, from these elements we exclude any with at least one variable which has a non-unique inverse transformation. Finally, from the remaining elements we produce the ordinary powers by using the inverse transformation. An example should clarify these matters, and we choose the values $n=8$, $r_1=2$, $r_2=3$, $r_3=4$. The first step is to write the "exponent matrix", whose elements are the possible triples $\{q_1q_2q_3\}$ of generalized powers of the variables x_1,x_2,x_3:

$$A_8 = \begin{bmatrix} 008 \\ 107 & 017 \\ 206 & 116 & 026 \\ 305 & 215 & 125 & 035 \\ 404 & 314 & 224 & 134 & 044 \\ 503 & 413 & 323 & 233 & 143 & 053 \\ 602 & 512 & 422 & 332 & 242 & 152 & 062 \\ 701 & 611 & 521 & 431 & 341 & 251 & 161 & 071 \\ 800 & 710 & 620 & 530 & 440 & 350 & 260 & 170 & 080 \end{bmatrix}$$

From the elements of A_8 we exclude those for which there is at least one $q_k=m_kr_3+s_k$, with $s_k \geq r_k$. It is not hard to see that these will be the elements with $q_1=2,3,6,7$ or $q_2=3,7$. Denoting these by {xxx}, we obtain the matrix

$$B_8 = \left[\begin{array}{ccccccccc}
008 \\
107 & 017 \\
xxx & 116 & 026 \\
xxx & xxx & 125 & xxx \\
404 & xxx & xxx & xxx & 044 \\
503 & 413 & xxx & xxx & 143 & 053 \\
xxx & 512 & 422 & xxx & xxx & 152 & 062 \\
xxx & xxx & 521 & xxx & xxx & xxx & 161 & xxx \\
800 & xxx & xxx & xxx & 440 & xxx & xxx & xxx & 080
\end{array}\right]$$

Then, using the inverse transformation we find the matrix with elements $\{p_1p_2p_3\}$

$$C_8 = \left[\begin{array}{ccccccccc}
008 \\
107 & 017 \\
xxx & 116 & 026 \\
xxx & xxx & 125 & xxx \\
204 & xxx & xxx & xxx & 034 \\
303 & 213 & xxx & xxx & 133 & 043 \\
xxx & 312 & 222 & xxx & xxx & 142 & 052 \\
xxx & xxx & 321 & xxx & xxx & xxx & 151 & xxx \\
400 & xxx & xxx & xxx & 230 & xxx & xxx & xxx & 060
\end{array}\right]$$

Except for the excluded elements, each element of C_8 corresponds to a generalized-homogeneous monomial of (total) degree eight relative to r=(2,3,4); that is, the monomial $x_1^{p_1}x_2^{p_2}x_3^{p_3}$. The number of such terms is given by formula (7.3), which in this case produces the value 21. (In the special case when the polynomial is homogeneous, the matrices A_8, B_8, C_8 will all be the same.)

For practical applications, we may also need certain submatrices of C_n, also called "exponent matrices" and generated from "generating elements" (which are elements of C_n) as follows. For $\alpha_1=0,1,\ldots,r_1$, and $\alpha_2=0,1,\ldots,r_2$, the generating elements are the triples $\{\alpha_1,\alpha_2,n-\alpha_1-\alpha_2\}$ (these would occur in C_n). For each generating element $\{\alpha_1,\alpha_2,n-\alpha_1-\alpha_2\}$ construct the left triangular "exponent matrix" $C_n(\alpha_1,\alpha_2)$ as follows:

(a) $(\alpha_1,\alpha_2,n-\alpha_1-\alpha_2)$ is the upper left corner element;

(b) in succeeding rows and columns,

  (1) the first exponent of the triple increases by r_1 from row to row and decreases by r_1 from column to column,

  (2) the second exponent of the triple increases by r_2 from column to column,

  (3) the third exponent of the triple deceases by r_3 from row to row;

(c) the decreases in (b_1) and (b_3) determine the size of the matrix, since all exponents must be nonnegative.

Thus, for the current example $n=8$, $r=(2,3,4)$, the six generating elements are {008}, {017}, {026}, {107}, {116}, {125}, and the corresponding matrices $C_8(\alpha_1,\alpha_2)$ are

$$C_8(0,\,0) = \begin{bmatrix} 008 & & \\ 204 & 034 & \\ 400 & 230 & 060 \end{bmatrix} \qquad C_8(1,\,0) = \begin{bmatrix} 107 & \\ 303 & 133 \end{bmatrix}$$

$$C_8(0,\,1) = \begin{bmatrix} 017 & \\ 213 & 043 \end{bmatrix} \qquad C_8(1,\,1) = \begin{bmatrix} 116 & \\ 312 & 142 \end{bmatrix}$$

$$C_8(0,\,2) = \begin{bmatrix} 026 & \\ 222 & 052 \end{bmatrix} \qquad C_8(1,\,2) = \begin{bmatrix} 125 & \\ 321 & 151 \end{bmatrix} .$$

Along with the left triangular matrices A_n, B_n, C_n, and $C_n(\alpha_1, \alpha_2)$, we may also construct the right triangular matrices A_n', B_n', C_n', and $C_n'(\alpha_1, \alpha_2)$. Thus, C_8' has the form

$$C_8' = \begin{bmatrix} & & & & & & & & 008 \\ & & & & & & & 107 & 017 \\ & & & & & & xxx & 116 & 026 \\ & & & & & xxx & xxx & 125 & xxx \\ & & & & 204 & xxx & xxx & xxx & 034 \\ & & & 303 & 213 & xxx & xxx & 133 & 043 \\ & & xxx & 312 & 222 & xxx & xxx & 142 & 052 \\ & xxx & xxx & 321 & xxx & xxx & xxx & 151 & xxx \\ 400 & xxx & xxx & xxx & 230 & xxx & xxx & xxx & 060 \end{bmatrix}$$

the generating elements are as before, and (with the construction appropriately modified) the $C_8(\alpha_1, \alpha_2)$ are

$$C_8^{l}(0,\ 0) = \left[\begin{array}{ccc} & & 008 \\ & 204 & 034 \\ 400 & 230 & 060 \end{array}\right] \qquad C_8^{l}(1,\ 0) = \left[\begin{array}{cc} & 107 \\ 303 & 133 \end{array}\right]$$

$$C_8^{l}(0,\ 1) = \left[\begin{array}{cc} & 017 \\ 213 & 043 \end{array}\right] \qquad C_8^{l}(1,\ 1) = \left[\begin{array}{cc} & 116 \\ 312 & 142 \end{array}\right]$$

$$C_8^{l}(0,\ 2) = \left[\begin{array}{cc} & 026 \\ 222 & 052 \end{array}\right] \qquad C_8^{l}(1,\ 2) = \left[\begin{array}{cc} & 125 \\ 321 & 151 \end{array}\right].$$

The elements of these matrices may represent not only the ordinary but also the factorial powers of the variables, as when, e.g., {abc} corresponds to $x^{a,!}y^{b,!}z^{c,!}$. The introduction of factorial powers makes it possible to develop algorithms for the construction of basis systems of polynomial solutions, invariant relative to r_1,r_2,r_3, of partial differential equations for any n. Both the left and right "exponent matrices" will be used in Section 7.3 for the construction of basis systems of polynomial solutions by means of algorithms in which the Pascal triangle is used.

7.3 COMBINATORIAL ALGORITHMS FOR THE CONSTRUCTION OF "COEFFICIENT MATRICES" AND BASIS SYSTEMS OF POLYNOMIAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS.

Algorithms for constructing basis systems of polynomial solutions are often applied to various classes of partial differential equations, among which are the polyharmonic, polywave, and polycaloric equations of Euler-Poisson-Darboux and Beltrami. Here we present combinatorial algorithms for constructing basis systems of polynomial solutions for more complicated equations, in particular:

$$\Delta_r u(x) \equiv \left(D_1^{r_1} + D_2^{r_2} + D_3^{r_3}\right) u(x) = 0, \tag{7.15}$$

the generalized Laplace equation, and

$$\nabla_r v(x) \equiv \left(D_1^{r_1} D_2^{r_2} + D_1^{r_1} D_3^{r_3} + D_2^{r_2} D_3^{r_3}\right) v(x) = 0, \tag{7.16}$$

the generalized equation of Mangeron type [82-87]. In (7.15), (7.16), $x=(x_1,x_2,x_3)$, $D_k=\frac{\partial}{\partial x_k}$, $r=(r_1,r_2,r_3)$, where $r_1 \le r_2 \le r_3$ are natural numbers.

The author in [14] constructed and studied basis systems of polynomial solutions of (7.15), the general representation of these solutions being of the form

$$H^{n,m,0}_{r;\alpha_1,\alpha_2}(x) = n! \sum_{i=0}^{l_\alpha - m} \sum_{k=0}^{m} (-1)^{i+k} \binom{i+k}{k} \times \frac{x_1^{r_1 i + r_1 k + \alpha_1}}{(r_1 i + r_1 k + \alpha_1)!} \frac{x_2^{r_2 m - r_2 k + \alpha_2}}{(r_2 m - r_2 k + \alpha_2)!} \frac{x_3^{n - r_3 m - r_3 i - \alpha_1 - \alpha_2}}{(n - r_3 m - r_3 i - \alpha_1 - \alpha_2)!}, \tag{7.17}$$

where n,m,α_1,α_2 are nonnegative numbers, and $l_\alpha = \left[\frac{n-\alpha_1-\alpha_2}{r_3}\right]$. In [14] the following theorems are proved.

Theorem 7.1. For $\alpha_1=0,1,\ldots,r_1-1, \alpha_2=0,1,\ldots,r_2-1$, and $m=0,1,\ldots,l_\alpha$, the generalized harmonic polynomials (7.17) form a basis system of

$$N_1(r,n) = \sum_{\alpha_1=0}^{r_1-1} \sum_{\alpha_2=0}^{r_2-1} (l_\alpha + 1) \tag{7.18}$$

linearly independent polynomials satisfying (7.15).

A basis system of polynomial solutions of (7.16) may be constructed from two general representations [14] of the form

$$P^{n,m,0}_{r;\alpha_1,\alpha_2}(x) = n! \sum_{i=0}^{l_\alpha - m} (-1)^i \sum_{k=0}^{i} \binom{i}{k} \times x_1^{r_1 i - r_1 k + r_1 m + \alpha_1, !} x_2^{r_2 k + \alpha_2, !} x_3^{n - r_3 m - r_3 i - \alpha_1 - \alpha_2, !}, \tag{7.19}$$

$$Q^{n,m,0}_{r;\alpha_1,\alpha_2}(x) = n! \sum_{i=0}^{l_\alpha - m} (-1)^i \sum_{k=0}^{i} \binom{i}{k} \times x_1^{r_1 k + \alpha_1, !} x_2^{r_2 i - r_2 k + r_2 m + \alpha_2, !} x_3^{n - r_3 m - r_3 i - \alpha_1 - \alpha_2, !}. \tag{7.20}$$

It may be shown that for m=0, $P^{0,0,0}_{r;\alpha_1,\alpha_2} \equiv Q^{0,0,0}_{r;\alpha_1,\alpha_2}$, and for m=1,2,...,$l_\alpha$, (7.19) and (7.20) are linearly independent.

Theorem 7.2. With α_1, α_2,m as in Theorem 7.1, the polynomials (7.19) and (7.20) form a basis system of

$$N_{1,2}(r,n) = \sum_{\alpha_1=0}^{r_1-1} \sum_{\alpha_2=0}^{r_2-1} (2l_\alpha+1) \tag{7.21}$$

linearly independent solutions of (7.16).

We note also that in [14] the iterated versions of (7.15), (7.16) are discussed, i.e., $\Delta_r^N(x)=0$, $\nabla_r^N(x)=0$. By introducing the normalizing factor $\binom{i}{p}$ and taking account of p in the exponent, the formulas (7.17), (7.19), (7.20) can be generalized to give polynomial representations of the solutions of $\Delta_r^N(x)=0$, $\nabla_r^N(x)=0$, of the form $H^{n,m,p}_{r;\alpha_1,\alpha_2}(x)$; $P^{n,m,p}_{r;\alpha_1,\alpha_2}(x)$ and $Q^{n,m,p}_{r;\alpha_1,\alpha_2}(x)$, in which p=0,1,...,N−1.

The formulas (7.17), (7.19), (7.20) could be used directly to construct polynomial solutions, but in the forms given they entail a good deal of calculation. In this connection, we suggest below a combinatorial algorithm for the construction of basis systems of polynomial solutions to (7.15) and (7.16), based on the introduction of the triangular matrices of factorial exponents and "coefficient matrices" in the form of the Pascal triangle, modified so that coefficients in odd-numbered rows are negative.

First, we construct the left and right triangular matrices of factorial exponents for any $n, r=(r_1,r_2,r_3)$, and $\alpha_1=0,1,\ldots,r_1-1$, $\alpha_2=0,1,\ldots,r_2-1$. Denote the left triangular matrices by $C_n(\alpha_1,\alpha_2)$ and the right by $C_n'(\alpha_1,\alpha_2)$, and their elements by the symbol $(i,k)_\alpha$, where $0\le i,k\le l_\alpha$. These matrices then appear as

$$C_n(\alpha_1,\alpha_2) = \begin{bmatrix} (0,0)_\alpha & & & & \\ (1,0)_\alpha & (0,1)_\alpha & & & \\ (2,0)_\alpha & (1,1)_\alpha & (0,2)_\alpha & & \\ \vdots & \vdots & \vdots & \ddots & \\ (l_\alpha,0)_\alpha & (l_\alpha-1,1)_\alpha & (l_\alpha-2,2)_\alpha & \cdots & (0,l_\alpha)_\alpha \end{bmatrix},$$

$$C_n'(\alpha_1,\alpha_2) = \begin{bmatrix} & & & & (0,0)_\alpha \\ & & & (1,0)_\alpha & (0,1)_\alpha \\ & & (2,0)_\alpha & (1,1)_\alpha & (0,2)_\alpha \\ & \cdots & \vdots & \vdots & \vdots \\ (l_\alpha,0)_\alpha & \cdots & (2,l_\alpha-2)_\alpha & (1,l_\alpha-1)_\alpha & (0,l_\alpha)_\alpha \end{bmatrix}.$$

In these matrices, each of the elements $(i,k)_\alpha$ corresponds to a triple of factorial powers by virtue of the relation

$$(i,\ k)_\alpha \rightarrow \{r_1 i+\alpha_1,\ r_2 k+\alpha_2,\ n-r_3 i-r_3 k-\alpha_1-\alpha_2\}, \tag{7.22}$$

and the matrices consist of $(l_\alpha+1)$ rows, where $l_\alpha = \left[\frac{n-\alpha_1-\alpha_2}{r_3}\right]$.

Thus, for example, let $n=15$, $r=(2,3,4)$, and take $\alpha_1=1, \alpha_2=0$. It is simple to construct $C_{15}(1,0)$ and $C'_{15}(1,0)$, and the resulting matrices are

$$C_{15}(1,0) = \left[\begin{array}{cccccccccccc} 1 & 0 & 14 & & & & & & & & & \\ 3 & 0 & 10 & 1 & 3 & 10 & & & & & & \\ 5 & 0 & 6 & 3 & 3 & 6 & 1 & 6 & 6 & & & \\ 7 & 0 & 2 & 5 & 3 & 2 & 3 & 6 & 2 & 1 & 9 & 2 \end{array}\right],$$

$$C'_{15}(1,0) = \left[\begin{array}{cccccccccccc} & & & & & & & & & 1 & 0 & 14 \\ & & & & & & 3 & 0 & 10 & 1 & 3 & 10 \\ & & & 5 & 0 & 6 & 3 & 3 & 6 & 1 & 6 & 6 \\ 7 & 0 & 2 & 5 & 3 & 2 & 3 & 6 & 2 & 1 & 9 & 2 \end{array}\right].$$

The other matrices are likewise easily constructed, and there will be a total of $r_1 r_2 = 2\cdot 3 = 6$ distinct matrices of either kind.

We describe now a scheme for constructing a basis system of polynomial solutions for arbitrary n, first for (7.15) and then for (7.16). For (7.15), only the left triangular matrices $C_n(\alpha_1,\alpha_2)$ are needed, along with the Pascal triangle whose odd rows have negative elements.

To shorten the description, we discuss the case in which $C_n(\alpha_1,\alpha_2)$ has five rows. Since the algorithm is invariant with respect to the values of the factorial exponents, all elements of this triangle will be denoted simply by {xxx}. We write this symbolic array and the Pascal triangle in the form

$$
\begin{array}{cccccccccc}
1 & \underline{1} & 1 & \underline{1} & 1 & xxx & & & & \\
 & 4 & \underline{3} & 2 & \underline{1} & xxx & xxx & & & \\
 & & 6 & \underline{3} & 1 & xxx & xxx & xxx & & \\
 & & & 4 & \underline{1} & xxx & xxx & xxx & xxx & \\
 & & & & 1 & xxx & xxx & xxx & xxx & xxx,
\end{array}
$$

where the negative elements are underlined. Whatever the values of the elements in the matrix of factorial exponents, the coefficients of the monomials are determined by successive superpositions of the Pascal triangle. We show the scheme for the (in this case) five positions of the triangles relative to one another.

$$
\begin{array}{ccccccccc}
1 & \underline{1} & 1 & \underline{1} & 1\cdot xxx & & & & \\
 & 4 & \underline{3} & 2 & \underline{1}\cdot xxx & xxx & & & \\
 & & 6 & \underline{3} & 1\cdot xxx & xxx & xxx & & \\
 & & & 4 & \underline{1}\cdot xxx & xxx & xxx & xxx & \\
 & & & & 1\cdot xxx & xxx & xxx & xxx & xxx
\end{array}
$$

$$
\begin{array}{cccccccc}
 & & & xxx & & & & \\
1 & \underline{1} & 1 & \underline{1}\cdot xxx & 1\cdot xxx & & & \\
 & 4 & \underline{3} & 2\cdot xxx & \underline{1}\cdot xxx & xxx & & \\
 & & 6 & \underline{3}\cdot xxx & 1\cdot xxx & xxx & xxx & \\
 & & & 4\cdot xxx & \underline{1}\cdot xxx & xxx & xxx & xxx \\
 & & & & 1 & & &
\end{array}
$$

		xxx				
		xxx	xxx			
1	1	1·xxx	1·xxx	1·xxx		
	4	3·xxx	2·xxx	1·xxx	xxx	
		6·xxx	3·xxx	1·xxx	xxx	xxx
			4	1		
				1		

	xxx				
	xxx	xxx			
	xxx	xxx	xxx		
1	1·xxx	1·xxx	1·xxx	1·xxx	
	4·xxx	3·xxx	2·xxx	1·xxx	xxx
		6	3	1	
			4	1	
				1	

xxx				
xxx	xxx			
xxx	xxx	xxx		
xxx	xxx	xxx	xxx	
1·xxx	1·xxx	1·xxx	1·xxx	1·xxx
	4	3	2	1
		6	3	1
			4	1
				1

If $C_n(\alpha_1,\alpha_2)$ contains a greater, or lesser, number of rows, the corresponding Pascal triangle must consist of the same number of rows. And, carrying out this superposition scheme with all of the $r_1 r_2$ matrices $C_n(\alpha_1,\alpha_2)$, we will form a total of

$$N_1(r,n) = \sum_{\alpha_1=0}^{r_1-1} \sum_{\alpha_2=0}^{r_2-1} (l_\alpha+1)$$

linearly independent polynomial solutions of (7.15).

To illustrate the procedure described, we form the basis system of generalized-homogeneous polynomial solutions of (7.15) of degree n=8, with $r_1=2, r_2=3, r_3=4$. The six matrices $C_8(\alpha_1,\alpha_2)$ were given in the example of Section 7.2, and we will need the Pascal triangle in the form

$$\begin{bmatrix} 1 & 1 & 1 \\ & 2 & 1 \\ & & 1 \end{bmatrix}$$

for those matrices having three rows. As a result of applying the algorithm and converting to factorial form, we will have (by Theorem 7.1) 13 linearly independent polynomial solutions of (7.15):

$$H^{8,0,0}_{4;0,0}(x) = 8!\left(x_3^{8,!} - x_1^{2,!}x_3^{4,!} + x_1^{4,!}\right),$$

$$H^{8,1,0}_{4;0,0}(x) = 8!\left(-x_1^{2,!}x_3^{4,!} + 2x_1^{4,!} + x_2^{3,!}x_3^{4,!} - x_1^{2,!}x_2^{3,!}\right),$$

$$H^{8,2,0}_{4;0,0}(x) = 8!\left(x_1^{4,!} - x_1^{2,!}x_2^{3,!} + x_2^{6,!}\right),$$

$$H^{8,0,0}_{4;1,0}(x) = 8!\left(x_1^{1,!}x_2^{7,!} - x_1^{3,!}x_3^{3,!}\right),$$

$$H^{8,1,0}_{4;1,0}(x) = 8!\left(-x_1^{3,!}x_3^{3,!} + x_1^{1,!}x_2^{3,!}x_3^{3,!}\right),$$

$$H^{8,0,0}_{4;0,1}(x) = 8!\left(x_2^{1,!}x_3^{7,!} - x_1^{2,!}x_3^{1,!}x_4^{3,!}\right),$$

$$H^{8,1,0}_{4;0,1}(x) = 8!\left(-x_1^{2,!}x_2^{1,!}x_3^{3,!} + x_2^{4,!}x_3^{3,!}\right),$$

$$H^{8,0,0}_{4;1,1}(x) = 8!\left(x_1^{1,!}x_2^{1,!}x_3^{6,!} - x_1^{3,!}x_2^{1,!}x_3^{2,!}\right),$$

$$H^{8,1,0}_{4;1,1}(x) = 8!\left(-x_1^{3,!}x_2^{1,!}x_3^{2,!} + x_1^{1,!}x_2^{4,!}x_3^{2,!}\right),$$

$$H^{8,0,0}_{4;0,2}(x) = 8!\left(x_2^{2,!}x_3^{6,!} - x_1^{2,!}x_2^{2,!}x_3^{2,!}\right),$$

$$H^{8,1,0}_{4;0,2}(x) = 8!\left(-x_1^{2,!}x_2^{2,!}x_3^{2,!} + x_2^{5,!}x_3^{2,!}\right),$$

$$H^{8,0,0}_{4;1,2}(x) = 8!\left(x_1^{1,!}x_2^{2,!}x_3^{5,!} - x_1^{3,!}x_2^{2,!}x_3^{1,!}\right),$$

$$H^{8,1,0}_{4;1,2}(x) = 8!\left(-x_1^{3,!}x_2^{2,!}x_3^{1,!} + x_1^{1,!}x_2^{5,!}x_3^{1,!}\right).$$

If necessary, any or all of the 13 polynomials may be written in ordinary form; thus, e.g.,

$$H^{8,0,0}_{4;0,0}(x) = x_3^8 - 840x_1^2x_3^4 + 1680x_1^4.$$

We turn now to the algorithm for constructing a basis system of polynomial solutions of degree n for (7.16). In this case we will need both the left and right triangular matrices $C_n(\alpha_1,\alpha_2)$ and $C_n'(\alpha_1,\alpha_2)$, and also modified (odd rows have negative coefficients) left and right triangular forms of the Pascal triangle.

For brevity, we again discuss the case for matrices with five rows, first for the part of the algorithm which uses the left triangular matrices $C_n(\alpha_1,\alpha_2)$. Denoting, as before, the elements of the matrix by {xxx}, we write the symbolic and Pascal triangles in the array

1					xxx				
1	1				xxx	xxx			
1	2	1			xxx	xxx	xxx		
1	3	3	1		xxx	xxx	xxx	xxx	
1	4	6	4	1	xxx	xxx	xxx	xxx	xxx .

The algorithm for this case is similar to, but differs in detail from, the algorithm for the previous case. We show the scheme for the five superpositions of the triangles relative to one another, and which give the constructions for the polynomials $P^{n,m,0}_{r;\alpha_1,\alpha_2}(x)$:

1·xxx				
1·xxx	1·xxx			
1·xxx	2·xxx	1·xxx		
1·xxx	3·xxx	3·xxx	1·xxx	
1·xxx	4·xxx	6·xxx	4·xxx	1·xxx

xxx				
1·xxx	xxx			
1·xxx	1·xxx	xxx		
1·xxx	2·xxx	1·xxx	xxx	
1·xxx	3·xxx	3·xxx	1·xxx	xxx
1	4	6	4	1

xxx				
xxx	xxx			
1·xxx	xxx	xxx		
1·xxx	1·xxx	xxx	xxx	
1·xxx	2·xxx	1·xxx	xxx	xxx
1	3	3	1	
1	4	6	4	1

xxx				
xxx	xxx			
xxx	xxx	xxx		
1·xxx	xxx	xxx	xxx	
1·xxx	1·xxx	xxx	xxx	xxx
1	2	1		
1	3	3	1	
1	4	6	4	1

xxx				
xxx	xxx			
xxx	xxx	xxx		
xxx	xxx	xxx	xxx	
1·xxx	xxx	xxx	xxx	xxx
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

The first part of the algorithm gives us the polynomials $P^{n,m,0}_{r;\alpha_1,\alpha_2}(x)$, for $m=0,1,2,\ldots,l_\alpha$. For the construction of the polynomials $Q^{n,m,0}_{r;\alpha_1,\alpha_2}(x)$, $m=1,2,\ldots,l_\alpha$, we write the symbolic triangle and the Pascal triangle in the form

				1					xxx	
			1	1				xxx	xxx	
		1	2	1			xxx	xxx	xxx	
	1	3	3	1		xxx	xxx	xxx	xxx	
1	4	6	4	1	xxx	xxx	xxx	xxx	xxx	.

We show the scheme for the four (m begins at 1) superpositions of the triangles relative to one another, and which give the constructions for the polynomials $Q^{n,m,0}_{r;\alpha_1,\alpha_2}(x)$:

				xxx
			xxx	1·xxx
		xxx	1·xxx	1·xxx
	xxx	1·xxx	2·xxx	1·xxx
xxx	1·xxx	3·xxx	3·xxx	1·xxx
1	4	6	4	1

				xxx
			xxx	xxx
		xxx	xxx	1·xxx
	xxx	xxx	1·xxx	1·xxx
xxx	xxx	1·xxx	2·xxx	1·xxx
	1	3	3	1
1	4	6	4	1

				xxx
			xxx	xxx
		xxx	xxx	xxx
	xxx	xxx	xxx	1·xxx
xxx	xxx	xxx	1·xxx	1·xxx
		1	2	1
	1	3	3	1
1	4	6	4	1

				xxx
			xxx	xxx
		xxx	xxx	xxx
	xxx	xxx	xxx	xxx
xxx	xxx	xxx	xxx	1·xxx
			1	1
		1	2	1
	1	3	3	1
1	4	6	4	1

In practical applications of this Pascal triangle algorithm one must write out the starting form clearly, and carry out carefully these patterns using the elements of $C_n(\alpha_1,\alpha_2)$ and $C_n'(\alpha_1,\alpha_2)$.

Writing out all the matrices $C_n(\alpha_1,\alpha_2)$ and $C_n'(\alpha_1,\alpha_2)$, a total of $2\tau_1\tau_2$, and applying the algorithm, we will, by construction, have a total of

$$N_{1,2}(r,n) = \sum_{\alpha_1=0}^{r_1-1} \sum_{\alpha_2=0}^{r_2-1} (2l_\alpha+1)$$

linearly independent polynomial solutions of (7.16).

As an example, we form the basis system of polynomial solutions of (7.16) of degree n=8, with $r_1=2, r_2=3, r_4=4$. The Pascal triangles needed are

$$\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 2 & 1 \end{bmatrix}, \qquad \begin{bmatrix} & 1 \\ 1 & 1 \end{bmatrix}.$$

The first of these (left triangular) is used in connection with the matrices $C_8(\alpha_1,\alpha_2)$, and the second with the matrices $C_8'(\alpha_1,\alpha_2)$, constructed in Section 7.2:

$$C_8(0,0),\ C_8(1,0),\ C_8(0,1),\ C_8(1,1),\ C_8(0,2),\ C_8(1,2),$$

and

$$C_8'(0,0),\ C_8'(1,0),\ C_8'(0,1),\ C_8'(0,2),\ C_8'(1,2).$$

As a result of the first part of the algorithm, we will have 13 linearly independent polynomials $P^{8,m,0}_{4;\alpha_1,\alpha_2}(x)$:

$$
\begin{aligned}
P_{4;0,0}^{8,0,0}(x) &= 8!\left(x_3^{8,!} - x_1^{2,!}x_3^{4,!} - x_2^{3!}x_3^{4,!} + x_1^{4,!} + 2x_1^{2,!}x_2^{3,!} + x_2^{6,!}\right),\\
P_{4;0,0}^{8,1,0}(x) &= 8!\left(x_1^{2,!}x_3^{4,!} - x_1^{4,!} - x_1^{2,!}x_2^{3,!}\right),\\
P_{4;0,0}^{8,2,0}(x) &= 8!\,x_1^{4,!},\\
P_{4;1,0}^{8,0,0}(x) &= 8!\left(x_1^{1,!}x_3^{7,!} - x_1^{3,!}x_3^{3,!} - x_1^{1,!}x_2^{3,!}x_3^{3,!}\right),\\
P_{4;1,0}^{8,1,0}(x) &= 8!\,x_1^{3,!}x_3^{3,!},\\
P_{4;0,1}^{8,0,0}(x) &= 8!\left(x_2^{1,!}x_3^{7,!} - x_1^{2,!}x_2^{1,!}x_3^{3,!} - x_2^{4,!}x_3^{3,!}\right),\\
P_{4;0,1}^{8,1,0}(x) &= 8!\,x_1^{2,!}x_2^{1,!}x_3^{3,!},\\
P_{4;1,1}^{8,0,0}(x) &= 8!\left(x_1^{1,!}x_2^{1,!}x_3^{6,!} - x_1^{3,!}x_2^{1,!}x_3^{2,!} - x_1^{1,!}x_2^{4,!}x_3^{2,!}\right),\\
P_{4;1,1}^{8,1,0}(x) &= 8!\,x_1^{3,!}x_2^{1,!}x_3^{2,!},\\
P_{4;0,2}^{8,0,0}(x) &= 8!\left(x_2^{2,!}x_3^{6,!} - x_1^{2,!}x_2^{2,!}x_3^{2,!} - x_2^{5,!}x_3^{2,!}\right),\\
P_{4;0,2}^{8,1,0}(x) &= 8!\,x_1^{2,!}x_2^{2,!}x_3^{2,!},\\
P_{4;1,2}^{8,0,0}(x) &= 8!\left(x_1^{1,!}x_2^{2,!}x_3^{5,!} - x_1^{3,!}x_2^{2,!}x_3^{1,!} - x_1^{1,!}x_2^{5,!}x_3^{1,!}\right),\\
P_{4;1,2}^{8,1,0}(x) &= 8!\,x_1^{3,!}x_2^{2,!}x_3^{1,!}.
\end{aligned}
$$

Applying the second part of the algorithm results in seven linearly independent polynomials $Q_{4;\alpha_1,\alpha_2}^{8,m,0}(x)$:

$$Q^{8,1,0}_{4;0,0}(x) = 8!\left(x_2^{3,!}x_3^{4,!} - x_1^{2,!}x_2^{3,!} - x_2^{6,!}\right),$$

$$Q^{8,2,0}_{4;0,0}(x) = 8!\,x_2^{6,!},$$

$$Q^{8,1,0}_{4;1,0}(x) = 8!\,x_1^{1,!}x_2^{3,!}x_3^{3,!},$$

$$Q^{8,1,0}_{4;1,1}(x) = 8!\,x_1^{1,!}x_2^{4,!}x_3^{2,!},$$

$$Q^{8,1,0}_{4;0,1}(x) = 8!\,x_2^{4,!}x_3^{3,!},$$

$$Q^{8,1,0}_{4;0,2}(x) = 8!\,x_2^{5,!}x_3^{2,!},$$

$$Q^{8,1,0}_{4;1,2}(x) = 8!\,x_1^{1,!}x_2^{5,!}x_3^{1,!}.$$

In all, we obtain 13 polynomials $P^{8,m,0}_{4;\alpha_1,\alpha_2}(x)$ and 7 polynomials $Q^{8,m,0}_{4;\alpha_1,\alpha_2}(x)$, for a total of 20 linearly independent generalized-homogeneous polynomial solutions of degree eight for (7.16); this agrees with the number given by

$$N_{1,2}(r,8) = \sum_{\alpha_1=0}^{1} \sum_{\alpha_2=0}^{2} (2l_\alpha + 1),$$

where

$$l_\alpha = \left[\frac{8-\alpha_1-\alpha_2}{4}\right],$$

and a simple calculation gives $N_{1,2}(r,8)=20$. As before, we note that any of these polynomials may also be represented as a polynomial with ordinary powers of the variables.

7.4 POLYNOMIALS OF BINOMIAL TYPE AND RELATED POLYNOMIALS

A sequence of polynomials $\{f_n\}$ is said to be a sequence of binomial type if

$$f_n(x+y) = \sum_{i=0}^{n} \binom{n}{i} f_i(x) f_{n-i}(y), \tag{7.23}$$

where $n=0,1,\ldots$, and $f_0=1$. Classical polynomial sequences of binomial type are, e.g., x^n, $(x)_n=x(x-1)\cdots(x-n+1)$, and $x(x-na)^{n-1}$. The theory of polynomial sequences of binomial type was developed by G.-C. Rota and R. Mullin [331], who gave a complete characterization of these sequences, worked out algorithms for determining their corresponding constants, and discussed some enumeration problems. R.B. Brown [95] gave an example of considerable combinatorial interest in this connection; he also generalized some of the results in [331] and considered the ring structure of sequences of binomial type. L. Brand [91] discussed functions of binomial type for negative factorials, i.e., sequences of the form $(x)_{-n}=[(x+1)(x+2)\cdots(x+n)]^{-1}$. Using polynomial sequences of binomial type, the author [7-10] constructed and studied binomial and trinomial polynomials.

Let r be a natural number, and let $s=0,1,\ldots,r-1$. Then the polynomial of binomial type

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^n y^{n-i}$$

may be written in the form

$$(x+y)^n = \sum_{s=0}^{r-1} \sum_{k=0}^{\left[\frac{n-s}{r}\right]} \binom{n}{rk+s} x^{rk+s} y^{n-rk-s}. \tag{7.24}$$

The right side of (7.24) may be considered as a sum of r polynomials, corresponding to s=0,1,...,r−1.

Using factorial powers, and taking the n! outside the summation, we write out two classes of polynomials:

$$G_{r,s}^{n,0}(x,y) = n! \sum_{k=0}^{\left[\frac{n-s}{r}\right]} x^{rk+s,1} y^{n-rk-s,1},$$

$$H_{r,s}^{n,0}(x,y) = n! \sum_{k=0}^{\left[\frac{n-s}{r}\right]} (-1)^k x^{rk+s,1} y^{n-rk-s,1}.$$

Then, putting in the summation the normalizing factor $\binom{k}{p}$, where $p=0,1,\ldots,\left[\frac{n-s}{r}\right]$, we obtain

$$G_{r,s}^{n,p}(x,y) = n! \sum_{k=p}^{\left[\frac{n-s}{r}\right]} \binom{k}{p} x^{rk+s,1} y^{n-rk-s,1}, \tag{7.25}$$

$$H_{r,s}^{n,p}(x,y) = n! \sum_{k=p}^{\left[\frac{n-s}{r}\right]} (-1)^k \binom{k}{p} x^{rk+s,1} y^{n-rk-s,1}. \tag{7.26}$$

The polynomials (7.26), (7.27) are called p-binomials. Omitting the details of the derivations of the numerous properties of these polynomials, we give their formulas for differentiation, indefinite integration, and some recurrence relations.

Differentiation formulas for the p-binomials with respect to the variables x and y are of the form

$$\left.\begin{aligned}
&\frac{\partial}{\partial x} G_{r,s}^{n,p}(x,y) = nG_{r,s-1}^{n-1,p}(x,y), \quad s\neq 0,\\
&\frac{\partial}{\partial x} G_{r,0}^{n,p}(x,y) = nG_{r,r-1}^{n-1,p}(x,y) + nG_{r,r-1}^{n-1,p-1}(x,y),\\
&\frac{\partial}{\partial y} G_{r,s}^{n,p}(x,y) = nG_{r,s}^{n-1,p}(x,y), \quad s=0,1,\ldots,r-1,
\end{aligned}\right\} \tag{7.27}$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial x} H_{r,s}^{n,p}(x,y) = nH_{r,s-1}^{n-1,p}(x,y), \quad s\neq 0,\\
&\frac{\partial}{\partial x} H_{r,0}^{n,p}(x,y) = -nH_{r,r-1}^{n-1,p}(x,y) - nH_{r,r-1}^{n-1,p-1}(x,y),\\
&\frac{\partial}{\partial y} H_{r,s}^{n,p}(x,y) = nH_{r,s}^{n-1,p}(x,y), \quad s=0,1,\ldots,r-1.
\end{aligned}\right\} \tag{7.28}$$

For p=0 in (7.27), (7.28) we get

$$G_{r,r-1}^{n-1,-1}(x,y) = H_{r,r-1}^{n-1,-1}(x,y) = 0.$$

Replacing in (7.27), (7.28) the power n by (n+1), and the index s by (s+1), and integrating both sides with respect to the differentiation variables, we obtain (to within an arbitrary function independent of the variable of integration) the indefinite integrals of the p-binomials:

$$\left.\begin{aligned}
&\int G_{r,s}^{n,p}(x,y)\,dx = \frac{1}{n+1} G_{r,s+1}^{n+1,p}(x,y),\\
&\int G_{r,s}^{n,p}(x,y)\,dy = \frac{1}{n+1} G_{r,s}^{n+1,p}(x,y),
\end{aligned}\right\} \tag{7.29}$$

$$\left.\begin{aligned} \int H_{r,s}^{n,p}(x,y)\,dx &= \frac{1}{n+1} H_{r,s+1}^{n+1,p}(x,y), \\ \int H_{r,s}^{n,p}(x,y)\,dy &= \frac{1}{n+1} H_{r,s}^{n+1,p}(x,y). \end{aligned}\right\} \tag{7.30}$$

From the differentiation and indefinite integration formulas, it is not difficult to obtain by integration by parts formulas for the indefinite integral of a product of two p-binomials; the values of definite integrals may also be calculated.

Using Euler's formula for a homogeneous function of degree n,

$$F(x,y) = \frac{x}{n}\frac{\partial}{\partial x}F^{n}(x,y) + \frac{y}{n}\frac{\partial}{\partial y}F^{n}(x,y),$$

and (7.27), (7.28), we can derive the recurrence relations

$$\left.\begin{aligned} G_{r,s}^{n,p}(x,y) &= xG_{r,s-1}^{n-1,p}(x,y) + yG_{r,s}^{n-1,p}(x,y), \quad s\neq 0, \\ G_{r,0}^{n,p}(x,y) &= xG_{r,r-1}^{n-1,p}(x,y) + xG_{r,r-1}^{n-1,p-1}(x,y) + yG_{r,0}^{n-1}(x,y), \end{aligned}\right\} \tag{7.31}$$

$$\left.\begin{aligned} H_{r,s}^{n,p}(x,y) &= xH_{r,s-1}^{n-1,p}(x,y) + yH_{r,s}^{n-1,p}(x,y), \quad s\neq 0, \\ H_{r,0}^{n,p}(x,y) &= -xH_{r,r-1}^{n-1,p}(x,y) - xH_{r,r-1}^{n-1,p-1}(x,y) + yH_{r,0}^{n-1,p}(x,y). \end{aligned}\right\} \tag{7.32}$$

Again from (7.27), (7.28) we may obtain differential equations whose solutions are p-binomials. Using the first two formulas in (7.27) and calculating the r^{th}-order derivative of the function $G_{r,s}^{n,p}(x,y)$ we find

$$\frac{\partial^r}{\partial x^r} G^{n,p}_{r,s}(x,y) = \frac{\partial^{r-s}}{\partial x^{r-s}}\left(\frac{\partial^s}{\partial x^s} G^{n,p}_{r,s}(x,y)\right)$$

$$= (n-s+1,1)_s \frac{\partial^{r-s-1}}{\partial x^{r-s-1}}\left(\frac{\partial}{\partial x} G^{n-s,p}_{r,0}(x,y)\right)$$

$$= (n-s,1)_{s+1} \frac{\partial^{r-s-1}}{\partial x^{r-s-1}}\left(G^{n-s-1,p}_{r,r-1}(x,y) + G^{n-s-1,p-1}_{r,r-1}(x,y)\right)$$

$$= (n-r+1,1)_r\left(G^{n-r,p}_{r,s}(x,y) + G^{n-r,p-1}_{r,s}(x,y)\right),$$

from which we have

$$\frac{\partial^r}{\partial x^r} G^{n,p}_{r,s}(x,y) = (n-r+1,1)_r G^{n-r,}_{r,s}(x,y) + (n-r+1,1)_r G^{n-r,p-1}_{r,s}(x,y).$$

And, using the third formula of (7.27), we find

$$\frac{\partial^r}{\partial y^r} G^{n,p}_{r,s}(x,y) = (n-r+1,1)_r G^{n-r,p}_{r,s}(x,y),$$

and subtracting this equation from the previous one we obtain

$$\left(\frac{\partial^r}{\partial x^r} - \frac{\partial^r}{\partial y^r}\right) G^{n,p}_{r,s}(x,y) = (n-r+1,1)_r G^{n-r,p-1}_{r,s}(x,y).$$

Here we assume that $n-r \geq 0$, $p \geq 1$. If $n < r$, then the right hand side is certainly zero. Let $n > pr$; then from this formula we will have

$$\left(\frac{\partial^r}{\partial x^r} - \frac{\partial^r}{\partial y^r}\right)^p G_{r,s}^{n,p}(x,y) = (n-pr+1,1)_{pr} G_{r,s}^{n-pr,0}(x,y), \quad p \neq 0,$$

$$\left(\frac{\partial^r}{\partial x^r} - \frac{\partial^r}{\partial y^r}\right) G_{r,s}^{n,0}(x,y) = 0.$$

If $n<pr$, the right side of the first equation is certainly zero. Thus, the p-binomial $G_{r,s}^{n,p}(x,y)$ is a polynomial solution of

$$\left(\frac{\partial^r}{\partial x^r} - \frac{\partial^r}{\partial y^r}\right)^{p+1} u(x,y) = 0,$$

for $n \geq pr$, $s=0,1,\ldots,r-1$.

A similar argument shows that the p-binomial $H_{r,s}^{n,p}(x,y)$, for $n \geq pr$, $s=0,1,\ldots,r-1$, gives a polynomial solution of

$$\left(\frac{\partial^r}{\partial x^r} + \frac{\partial^r}{\partial y^r}\right)^{r+1} v(x,y) = 0.$$

If $r=2$, $s=0,1$, then we obtain a basis system of $2p+2$ polywave polynomials $G_{2,s}^{n,p}(x,y)$, and the same number of polyharmonic polynomials $H_{2,s}^{n,p}(x,y)$, which for $p=0$ reduce to the corresponding wave and harmonic polynomials $G_{2,s}^{n,0}(x,y)$ and $H_{2,s}^{n,0}(x,y)$.

Note that by taking linear combinations of p-binomial polynomials, we can obtain polynomial solutions of various classes of differential equations. Consider, for example, the differential equation of order 2m

$$\left(\frac{\partial^{2m}}{\partial x^m \partial y^m} + \frac{\partial^{2m}}{\partial x^{2m}} + \frac{\partial^{2m}}{\partial y^{2m}}\right) u(x,y) = 0; \tag{7.33}$$

we will construct a basis system of polynomial solutions using linear combinations of the p-binomial polynomials $G^{n,0}_{r,s}(x,y)$. Omitting the details of the choice of these combinations, it can be shown that the linearly independent polynomials

$$P^{n,0}_{2m,s}(x,y) = G^{n,0}_{3m,s}(x,y) - G^{n,0}_{3m,m+s}(x,y), \quad s = 0,1,2,\ldots,2m-1, \quad n>2m,$$

form a basis system of 2m linearly independent polynomial solutions of (7.33). In fact, let $s \le m-1$; then

$$\frac{\partial^{2m}}{\partial x^m \partial y^m} G^{n,0}_{3m,s}(x,y) = (n-2m+1,\ 1)_{2m} G^{n-2m,0}_{3m,2m+s}(x,y),$$

$$\frac{\partial^{2m}}{\partial x^{2m}} G^{n,0}_{3m,s}(x,y) = (n-2m+1,\ 1)_{2m} G^{n-2m,0}_{3m,m+s}(x,y),$$

$$\frac{\partial^{2m}}{\partial y^{2m}} G^{n,0}_{3m,s}(x,y) = (n-2m+1,\ 1)_{2m} G^{n-2m,0}_{3m,s}(x,y),$$

$$\frac{\partial^{2m}}{\partial x^m \partial y^m} G^{n,0}_{3m,m+s}(x,y) = (n-2m+1,\ 1)_{2m} G^{n-2m,0}_{3m,s}(x,y),$$

$$\frac{\partial^{2m}}{\partial x^{2m}} G^{n,0}_{3m,m+s}(x,y) = (n-2m+1,\ 1)_{2m} G^{n-2m,0}_{3m,2m+s}(x,y),$$

$$\frac{\partial^{2m}}{\partial y^{2m}} G^{n,0}_{3m,m+s}(x,y) = (n-2m+1,\ 1)_{2m} G^{n-2m,0}_{2m,m+s}(x,y).$$

Thus, it follows that the polynomials $P^{n,0}_{2m,s}(x,y)$ satisfy (7.33) for $s=0,1,\ldots,m-1$. In a similar way, it can be shown that these polynomials also satisfy (7.33) for $s=m,\ m+1,\ \ldots,\ 2m-1$.

To construct p-trinomial polynomials we use the expansion of a trinomial in the form

$$(x+y+z)^n = \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k,i-k} x^k y^{i-k} z^{n-i}$$

$$= \sum_{s_1=0}^{r-1} \sum_{s_2=0}^{r-1} \sum_{i=0}^{\left[\frac{n-s_1-s_2}{r}\right]} \sum_{k=0}^{i} \binom{n}{rk+s_1,\ ri-rk+s_2}$$

$$\times\, x^{rk+s_1} y^{ri-rk+s_2} z^{n-ri-s_1-s_2}.$$

Using factorial powers of the variables and inserting the normalization, we distinguish two classes of p-trinomial polynomials:

$$G^{n,p}_{r;s_1,s_2}(x,y,z) = n! \sum_{i=p}^{\left[\frac{n-s_1-s_2}{r}\right]} \sum_{k=0}^{i} \binom{i}{p} \tag{7.34}$$

$$\times\, x^{rk+s_1,1} y^{ri-rk+s_2,1} z^{n-ri-s_1-s_2,1},$$

$$H^{n,p}_{r;s_1,s_2}(x,y,z) = n! \sum_{i=p}^{\left[\frac{n-s_1-s_2}{r}\right]} \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{p} \tag{7.35}$$

$$\times\, x^{rk+s_1,1} y^{ri-rk+s_2,1} z^{n-ri-s_1-s_2,1}.$$

In these expressions, $s_1=0,1,\ldots,r-1$, $s_2=0,1,\ldots,r-1$, and $p=0,1,\ldots,[(n-s_1-s_2)/r]$. Their differentiation formulas are:

$$\frac{\partial}{\partial x} G^{n,p}_{r;s_1,s_2}(x,y,z) = nG^{n-1,p}_{r;s_1-1,s_2}(x,y,z), \quad s_1 \neq 0,$$

$$\frac{\partial}{\partial x} G^{n,p}_{r;0,s_2}(x,y,z) = nG^{n-1,p}_{r;r-1,s_2}(x,y,z) + nG^{n-1,p-1}_{r;r-1,s_2}(x,y,z),$$

$$\frac{\partial}{\partial y} G^{n,p}_{r;s_1,s_2}(x,y,z) = nG^{n-1,p}_{r;s_1,s_2-1}(x,y,z), \quad s_2 \neq 0,$$

$$\frac{\partial}{\partial y} G^{n,p}_{r;s_1,0}(x,y,z) = nG^{n-1,p}_{r;s_1,r-1}(x,y,z) + nG^{n-1,p-1}_{r;s_1,r-1}(x,y,z),$$

$$\frac{\partial}{\partial z} G^{n,p}_{r;s_1,s_2}(x,y,z) = nG^{n-1,p}_{r;s_1,s_2}(x,y,z),$$

$$\frac{\partial}{\partial x} H^{n,p}_{r;s_1,s_2}(x,y,z) = nH^{n-1,p}_{r;s_1-1,s_2}(x,y,z), \quad s_1 \neq 0,$$

$$\frac{\partial}{\partial x} H^{n,p}_{r;0,s_2}(x,y,z) = -nH^{n-1,p}_{r;r-1,s_2}(x,y,z) - nH^{n-1,p-1}_{r;r-1,s_2}(x,y,z),$$

$$\frac{\partial}{\partial y} H^{n,p}_{r;s_1,s_2}(x,y,z) = nH^{n-1,p}_{r;s_1,s_2-1}(x,y,z), \quad s_2 \neq 0,$$

$$\frac{\partial}{\partial y} H^{n,p}_{r;s_1,0}(x,y,z) = -nH^{n-1,p}_{r;s_1,r-1}(x,y,z) - nH^{n-1,p-1}_{r;s_1,r-1}(x,y,z),$$

$$\frac{\partial}{\partial z} H^{n,p}_{r;s_1,s_2}(x,y,z) = nH^{n-1,p}_{r;s_1,s_2}(x,y,z).$$

The indefinite integral formulas for the polynomials $G^{n,p}_{r,s_1,s_2}(x,y,z)$, up to an arbitrary function not dependent on the variable of integration, take the form

$$\int G^{n,p}_{r;s_1,s_2}(x,y,z)\,dx = \frac{1}{n+1} G^{n+1,p}_{r;s_1+1,s_2}(x,y,z),$$

$$\int G^{n,p}_{r;s_1,s_2}(x,y,z)\,dy = \frac{1}{n+1} G^{n+1,p}_{r;s_1,s_2+1}(x,y,z),$$

$$\int G^{n,p}_{r;s_1,s_2}(x,y,z)\,dz = \frac{1}{n+1} G^{n+1,p}_{r;s_1,s_2}(x,y,z).$$

Using the differentiation formulas, and Euler's formula as before, recurrence relations for the polynomials (7.34) take the form

$$G^{n,p}_{r;s_1,s_2}(x,y,z) = xG^{n-1,p}_{r;s_1-1,s_2}(x,y,z) + yG^{n-1,p}_{r;s_1,s_2-1}(x,y,z)$$

$$+ zG^{n-1,p}_{r;s_1,s_2}(x,y,z), \quad s_1 \neq 0, \quad s_2 \neq 0,$$

$$G^{n,p}_{r;0,s_2}(x,y,z) = xG^{n-1,p}_{r;r-1,s_2}(x,y,z) + xG^{n-1,p-1}_{r;r-1,s_2}(x,y,z)$$

$$+ yG^{n-1,p}_{r;0,s_2-1}(x,y,z) + zG^{n-1,p}_{r;0,s_2}(x,y,z), \quad s_2 \neq 0,$$

$$G^{n,p}_{r;s_1,0}(x,y,z) = xG^{n-1,p}_{r;s_1-1,0}(x,y,z) + yG^{n-1,p}_{r;s_1,r-1}(x,y,z)$$

$$+ yG^{n-1,p-1}_{r;s_1,r-1}(x,y,z) + zG^{n-1,p}_{r;s_1,0}(x,y,z), \quad s_1 \neq 0,$$

$$G^{n,p}_{r;0,0}(x,y,z) = xG^{n-1,p}_{r;r-1,0}(x,y,z) + xG^{n-1,p-1}_{r;r-1,0}(x,y,z)$$

$$+ yG^{n-1,p}_{r;0,r-1}(x,y,z) + yG^{n-1,p-1}_{r;0,r-1}(x,y,z) + zG^{n-1,p}_{r;0,0}(x,y,z).$$

Corresponding integration and recurrence formulas for the polynomials $H^{n,p}_{r,s_1,s_2}(x,y,z)$ may be derived in similar fashion.

Various methods of constructing polynomial solutions for partial differential equations, including the Laplace, polywave, polyvibration, and polyharmonic equations, may be found in [82-87, 215, 290, 317, 374, 400, 406].

7.5 OTHER CLASSES OF NONORTHOGONAL POLYNOMIALS

As is well known, orthogonal polynomials in one variable have been studied in great detail, and widely applied in problems in mathematics, mechanics, and physics; two excellent summaries are the books [48, 370]. An extensive bibliography, containing a survey of over 500 works, is given in the detailed book of P. Nevai [293], dedicated to the memory of Geza Freud.

We consider here nonorthogonal polynomials in one independent variable, whose coefficients in expansions in powers of x are binomial coefficients, Fibonacci and Lucas numbers, and other special quantities. We first discuss two classes of polynomials suggested in [10], and of interest from the point of view of constructing solutions to differential equations of Sobolev type, which arise in various mechanics contexts.

We define the polynomials by

$$P_{r,s}^{m,n}(x) = \sum_{k=0}^{n} \binom{n+m}{k+m} x^{rk+s,!}, \tag{7.36}$$

$$Q_{r,s}^{m,n}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+m}{k+m} x^{rk+s,!}, \tag{7.37}$$

where m,n are any nonnegative integers, and r=1,2,..., s=0,1,2,....

We will give differentiation, indefinite integration, and recurrence formulas for the polynomials (7.36); those for the polynomials (7.37) are similar.

It is easy to establish the formulas

$$\left.\begin{aligned} \frac{d}{dx} P_{r,s}^{m,n}(x) &= P_{r,s-1}^{m,n}(x), \quad s \neq 0, \\ \frac{d}{dx} P_{r,0}^{m,n}(x) &= P_{r,r-1}^{m+1,n-1}(x), \end{aligned}\right\} \tag{7.38}$$

$$\int P_{r,s}^{m,n}(x)\,dx = P_{r,s+1}^{m,n}(x), \tag{7.39}$$

$$P_{r,s}^{m+1,n}(x) = P_{r,s}^{m+1,n-1}(x) + P_{r,s}^{m,n}(x). \tag{7.40}$$

From (7.38)-(7.40) we may derive other relations, including integrals of products of two polynomials for values of m,n,s the same or differing. We calculate, for example, the r[th] derivative, which is needed in applications; we have

$$\frac{d^r}{dx^r}P_{r,s}^{m,n}(x) = \frac{d^{r-s}}{dx^{r-s}}\frac{d^s}{dx^s}P_{r,s}^{m,n}(x) = \frac{d^{r-s}}{dx^{r-s}}P_{r,0}^{m,n}(x)$$

$$= \frac{d^{r-s-1}}{dx^{r-s-1}}\frac{d}{dx}P_{r,0}^{m,n}(x) = \frac{d^{r-s-1}}{dx^{r-s-1}}P_{r,r-1}^{m+1,n-1}(x) = P_{r,s}^{m+1,n-1}(x),$$

that is,

$$\frac{d^r}{dx^r}P_{r,s}^{m,n}(x) = P_{r,s}^{m+1,n-1}(x).$$

In a similar way, we could consider the extension of these polynomials, as in

$$P_{r,s}^{m,n;p,q}(x) = \sum_{k=p}^{n} \binom{k}{p}\binom{n+m}{k+m} x^{rk+rq+s,1},$$

$$Q_{r,s}^{m,n;p,q}(x) = \sum_{k=p}^{n} (-1)^k \binom{k}{p}\binom{n+m}{k+m} x^{rk+rq+s,1},$$

which also have convenient analytic and computational properties.

We include now a short review of references dealing with the construction and study of Fibonacci, Lucas, and other generalized polynomials.

V.E. Hoggatt and M. Bicknell [202] considered Fibonacci, tribonacci, and more general r-bonacci polynomials, defined by the relations

$$R_{-(r-2)}(x) = R_{-(r-1)}(x) = \cdots = R_{-1}(x) = R_0(x) = 0,$$

$$R_1(x) = 1,\ R_2(x) = x^{r-1},$$

$$R_{n+r}(x) = x^{r-1}R_{n+r-1}(x) + x^{r-2}R_{n+r-2}(x) + \cdots + R_n(x).$$

For r=2, the polynomial $R_n(x)$ reduces to the Fibonacci polynomial $F_n(x)$, and for r=3 to the tribonacci polynomial $T_n(x)$. They showed that the r-bonacci polynomial $R_n(x)$, written with descending powers of x, has as coefficients the coefficients in the n^{th} ascending diagonal of the generalized Pascal triangle of order r, i.e., the coefficients in the expansion

$$(1+x+x^2+\cdots+x^{r-1})^n,\ n=0,1,2,\ldots.$$

The general representation of $R_n(x)$ has the form

$$R_n(x) = \sum_{j=0}^{[(r-1)(n-1)/r]} \binom{n-j-1}{j}_r x^{(r-1)(n-1)-rj},$$

where $\binom{n}{j}_r = 0$ for $j>n$. For r=2,3, this gives

$$F_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1},$$

$$T_n(x) = \sum_{j=0}^{[2(n-1)/3]} \binom{n-j-1}{j}_3 x^{2n-3j-2}.$$

They also studied Q-matrices, generating r-bonacci polynomials.

V.E. Hoggatt, M. Bicknell, and E.L. King [207] defined four sequences of polynomials: the Fibonacci polynomials $f_n(x)$, the Lucas polynomials $l_n(x)$, and the polynomials $g_n(x)$, $h_n(x)$ satisfying

$$g_0(x) = 0,\ g_1(x) = 1,\ g_{n+2}(x) = xg_{n+1}(x) - g_n(x),$$
$$h_0(x) = 2,\ h_1(x) = x,\ h_{n+2}(x) = xh_{n+1}(x) - h_n(x),$$

and established various identities and relations among these.

In [212] V.E. Hoggatt and J.W. Phillips used the generalized binomial coefficients $C_n(p,r)$, $r \geq 2$, $p \geq 0$, and the Fibonacci and Lucas polynomials

$$f_1(x) = 1,\ f_2(x) = x,\ f_n(x) = xf_{n-1} + f_{n-2}(x),$$
$$l_1(x) = x,\ l_2(x) = x^2+2,\ l_n(x) = xl_{n-1}(x) + l_{n-2}(x),$$

to define new classes of polynomials in the form of the sums

$$\sum_{n=0}^{p(r-1)} (\pm 1)^n C_n(p,r) f_{bn+j}^m(x), \quad \sum_{n=0}^{p(r-1)} (\pm 1)^n C_n(p,r)\, l_{bn+j}^m(x), \quad C_n(p,r) = \binom{p}{n}_r .$$

V.E. Hoggatt and C.T. Long [211] introduced the so-called generalized Fibonacci polynomials

$$u_0(x,y) = 0,\ u_1(x,y) = 1,\ u_{n+2}(x,y) = xu_{n+1}(x,y) + yu_n(x,y),\ n = 0,1,2,\ldots .$$

They showed that these polynomials have a number of properties analogous to the Fibonacci sequence, and studied some of their divisibility properties.

A. Krishnaswami [244] discussed a class of functions which he called Pascal functions, the coefficients of which are formed from the Pascal triangle diagonals; he showed that the Fibonacci polynomials are a special case of the Pascal functions.

H. Hosoya [217, 219] studied some interesting analytic, combinatorial, and graph properties of orthogonal polynomials, including those of Chebyshev, Hermite, and Laguerre.

In a cycle of papers: A.N. Phillipou [310]; A.N. Phillipou, C. Georghiou, G.N. Phillipou [311, 312]; A.N. Phillipou, F.S. Makri [313]; and G.N. Phillipou and C. Georghiou [314] considered generalized Fibonacci polynomials, and Fibonacci polynomials of order k. A review of these works and a bibliography is given in the detailed article of A.N. Phillipou [310]. A sequence of polynomials $\left\{f_n^{(k)}(x)\right\}_{n=0}^{\infty}$ is said to be a sequence of Fibonacci polynomials of order k if $f_0^{(k)}(x)=0$, $f_1^{(k)}(x)=1$, and

$$f_n^{(k)}(x) = \begin{cases} \sum_{i=1}^{n} x^{k-i} f_{n-i}^{(k)}(x), & 2 \le n \le k+1, \\ \sum_{i=1}^{k} x^{k-i} f_{n-i}^{(k)}(x), & n \ge k+2. \end{cases}$$

If we set $f_n^{(r)}(x)=0$ for $-(r-2) \le n \le -1$, then $f_n^{(r)}(x)=R_n(x)$, $n \ge -(r-2)$, i.e., reduces to the r-bonacci polynomial mentioned earlier. In [311], the explicit representation of these polynomials is obtained in a form involving multinomial coefficients,

$$f_{n+1}^{(k)}(x) = \sum_{n_1,\dots,n_k} \binom{n_1+\dots+n_k}{n_1,\dots,n_k} x^{k(n_1+\dots+n_k)-n}, \quad n \ge 0.$$

In [312, 313] other properties are studied, and an application to probability theory is given, and [314] is a further consideration of their properties, including their connection with the generalized Pascal triangle of order k, whose coefficients are defined by

$$A_{0,0}^{(k)} = 1,\ A_{0,m}^{(k)} = 0, \quad m \geq 1,$$

$$A_{n,m}^{(k)} = \begin{cases} \sum_{i=0}^{m} A_{n-1,m-i}^{(k)}, & 0 \leq m < k, \\ \sum_{i=0}^{k-1} A_{n-1,m-i}^{(k)}, & m \geq k, \end{cases}$$

where n≥1. They showed that

$$F_{n+1}^{k}(x) = \sum_{i=0}^{[n-n/k]} A_{n-i,i}^{(k)} x^{n-i}, \quad n \geq 0,$$

and that

$$A_{n,m}^{(k)} = \sum_{i=0}^{\left[\frac{m}{k}\right]} (-1)^i \binom{n}{i} \binom{m+n-1-ki}{n-1}.$$

V.E. Hoggatt and D.A. Lind [210] discussed the question of the so-called height of the Fibonacci polynomial

$$f_n(x) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n-j-1}{j} x^{n-2j-1},$$

and its connection with these functions, where the height is taken to be the largest coefficient of the polynomial. They proved that the heights of two successive polynomials lie either in the same column, or in neighboring columns, of the Pascal triangle. In the latter case, $m(n)/m(n+1)=k/h(k)$, where $m(n)$ is the height of $f_n(x)$, and $h(k)=\frac{1}{2}\left(k+1+\sqrt{5k^2-2k+1}\right)$ if $m(n+1)$ lies in the k^{th} column. They also showed that F_n/F_{n+1} for $n\geq 2$, and L_n/L_{n+1} for $n\geq 4$, may be represented as the quotient of the heights of two successive polynomials whose heights do not lie in the same column.

Other discussions related to some classes of non-orthogonal polynomials may be found in the references [124, 198, 273, 283].