

GOLDEN TRIANGLES, RECTANGLES, AND CUBOIDS

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1. INTRODUCTION

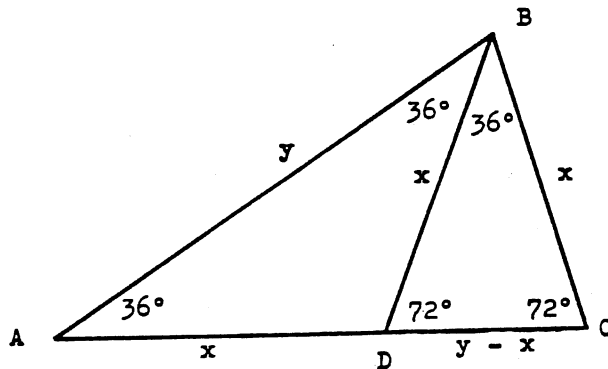
One of the most famous of all geometric figures is the Golden Rectangle, which has the ratio of length to width equal to the Golden Section,

$$\phi = (1 + \sqrt{5})/2 .$$

The proportions of the Golden Rectangle appear consistently throughout classical Greek art and architecture. As the German psychologists Fechner and Wundt have shown in a series of psychological experiments, most people do unconsciously favor "golden dimensions" when selecting pictures, cards, mirrors, wrapped parcels, and other rectangular objects. For some reason not fully known by either artists or psychologists, the Golden Rectangle holds great aesthetic appeal. Surprisingly enough, the best integral lengths to use for sides of an approximation to the Golden Rectangle are adjacent members of the Fibonacci series: 1, 1, 2, 3, 5, 8, 13, ..., and we find 3 x 5 and 5 x 8 filing cards, for instance.

Suppose that, instead of a Golden Rectangle, we study a golden section triangle. If the ratio of a side to the base is $\phi = (1 + \sqrt{5})/2$, then we will call the triangle a Golden Triangle. (See [2], [3].)

Now, consider the isosceles triangle with a vertex angle of 36° . On bisecting the base angle of 72° , two isosceles triangles are formed, and $\triangle BDC$ is similar to $\triangle ABC$ as indicated in the figure:



Since $\triangle ABC$ is similar to $\triangle BDC$,

$$\frac{AB}{BD} = \frac{BC}{DC}, \quad \text{or} \quad \frac{y}{x} = \frac{x}{y-x},$$

so that

$$y^2 - yx - x^2 = 0.$$

Dividing through by $x^2 \neq 0$,

$$\frac{y^2}{x^2} - \frac{y}{x} - 1 = 0.$$

The quadratic equation gives

$$\frac{y}{x} = (1 + \sqrt{5})/2 = \phi$$

as the positive root, so that $\triangle ABC$ is a Golden Triangle. Notice also, that, using the common altitude from B, the ratio of the area of $\triangle ABC$ to $\triangle ADB$ is ϕ .

Since the central angle of a regular decagon is 36° , $\triangle ABC$ above shows that the ratio of the radius y to the side x of an inscribed decagon is ϕ . Also, in a regular pentagon, the angle at a vertex between two adjacent diagonals is 36° . By reference to the figure above, the ratio of a diagonal to a side of a regular pentagon is also ϕ .

2. A TRIGONOMETRIC PROPERTY OF THE ISOSCELES GOLDEN TRIANGLE

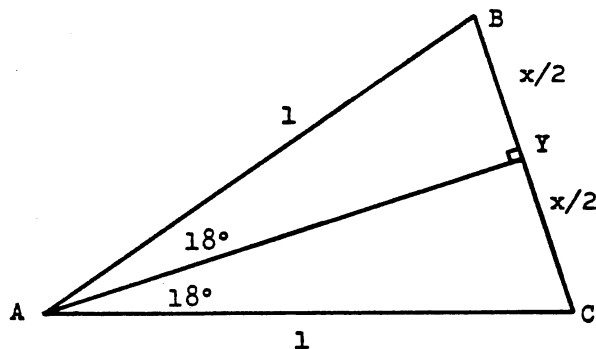
The Golden Triangle with vertex angle 36° can be used for a surprising trigonometric application. Few of the trigonometric functions of an acute angle have values which can be expressed exactly. Usually, a method of approximation is used; most values in trigonometric tables cannot be expressed exactly as terminating decimals, repeating decimals, or even square roots, since they are approximations to transcendental numbers, which are numbers so irrational that they are not the root of any polynomial over the integers.

The smallest positive integral number of degrees for which the trigonometric functions of the angle can be expressed exactly is three degrees. Then, all multiples of 3° can also be expressed exactly by repeatedly using formulas such as $\sin(A + B)$. Strangely enough, the Golden Triangle can be used to derive the value of $\sin 3^\circ$.

In our Golden Triangle, the ratio of the side to the base was

$$y/x = (1 + \sqrt{5})/2.$$

Suppose we let $AB = y = 1$. Then $1/x = (1 + \sqrt{5})/2$, or, $x = (\sqrt{5} - 1)/2$.
Redrawing the figure and bisecting the 36° angle,



we form a right triangle, $\triangle AYC$, with $YC = x/2$. Then,

$$\sin 18^\circ = \frac{YC}{AC} = \frac{x}{2} = \frac{\sqrt{5} - 1}{4} = \frac{1}{2\phi}.$$

Since $\sin^2 A + \cos^2 A = 1$,

$$\cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4} = \frac{\sqrt{\sqrt{5}\phi}}{2}.$$

Since $\sin(A - B) = \sin A \cos B - \sin B \cos A$,

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

Similarly, using $\cos(A - B) = \cos A \cos B + \sin A \sin B$,

$$\cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

Again using the formula for $\sin(A - B)$,

$$\begin{aligned} \sin 3^\circ = \sin(18^\circ - 15^\circ) &= \frac{\sqrt{5} - 1}{4} \cdot \frac{\sqrt{6} + \sqrt{2}}{4} - \frac{\sqrt{6} - \sqrt{2}}{4} \cdot \frac{\sqrt{10 + 2\sqrt{5}}}{4} \\ &= \frac{1}{16} [(\sqrt{5} - 1)(\sqrt{6} + \sqrt{2}) - 2(\sqrt{3} - 1)(\sqrt{5 + \sqrt{5}})] \end{aligned}$$

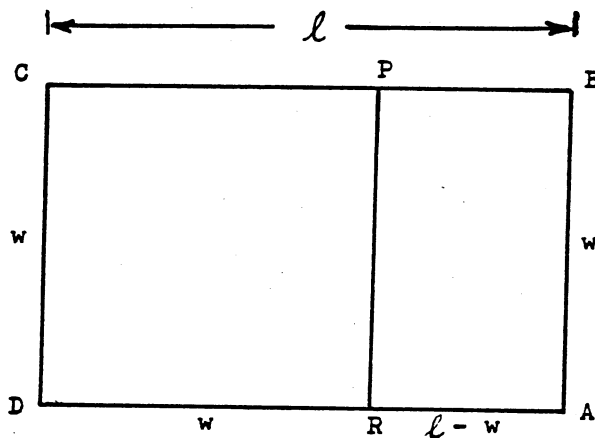
as given by Ransom in [1].

3. GOLDEN RECTANGLE AND GOLDEN TRIANGLE THEOREMS

While a common way to describe the Golden Rectangle is to give the ratio of length to width as $\phi = (1 + \sqrt{5})/2$, this ratio is a consequence of the geometric properties of the Golden Rectangle which are discussed in this section.

Theorem. Given that the ratio of length to width of a rectangle is $k > 1$. A square with side equal to the width can be removed to leave a rectangle similar to the original rectangle if and only if $k = (1 + \sqrt{5})/2$.

Proof. Let the square PCDR be removed from rectangle ABCD, leaving rectangle BPRA.



If rectangles ABCD and BPRA have the same ratio of length to width, then

$$k = \frac{w}{l - w} = \frac{l}{w} .$$

Cross-multiplying and dividing by $w^2 \neq 0$ gives a quadratic equation in $\frac{l}{w}$ which has $(1 + \sqrt{5})/2$ as its positive root. If

$$\frac{l}{w} = (1 + \sqrt{5})/2 = \phi ,$$

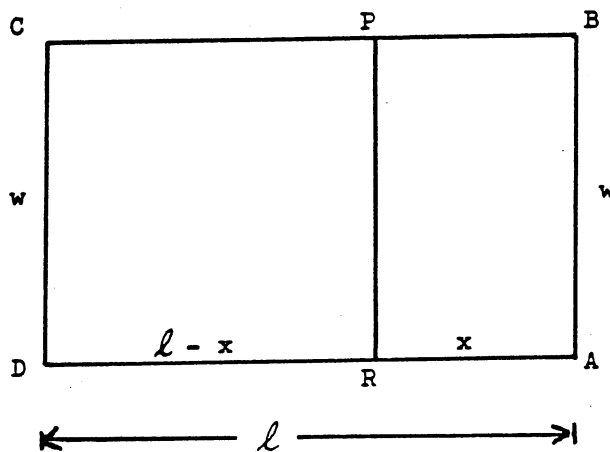
then

$$\frac{w}{l - w} = \frac{1}{\frac{l}{w} - 1} = \frac{1}{\phi - 1} = \phi$$

so that both rectangles have the same ratio of length to width.

Theorem. Given that the ratio of length to width of a rectangle is $k > 1$. A rectangle similar to the first can be removed to leave a rectangle such that the ratio of the areas of the original rectangle and the rectangle remaining is k , if and only if $k = (1 + \sqrt{5})/2$. Further, the rectangle remaining is a square.

Proof. Remove rectangle BPRA from rectangle ABCD as in the figure:



Then

$$\frac{\text{area } ABCD}{\text{area } PCDR} = \frac{lw}{w(l-x)} .$$

But,

$$\frac{lw}{w(l-x)} = \frac{l}{l-x} = k \quad \text{if and only if} \quad \frac{w}{l-x} = 1 ,$$

or $w = l - x$ or PCDR is a square. Thus, our second theorem is a consequence of the first theorem.

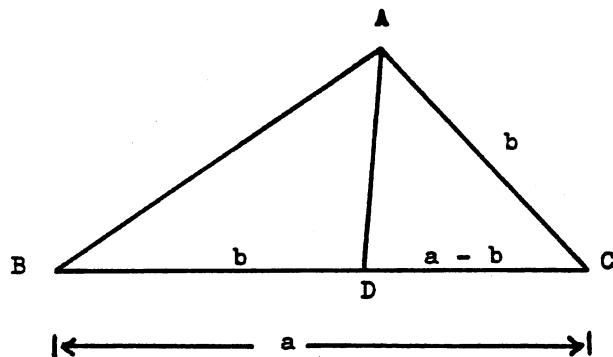
Analogous theorems hold for Golden Triangles.

Theorem. Given that the ratio of two sides a and b of a triangle is $a/b = k > 1$. A triangle with side equal to b can be removed to leave a triangle similar to the first if and only if $k = (1 + \sqrt{5})/2$.

Proof. Remove $\triangle ABD$ from $\triangle ABC$. If $\triangle ADC$ is similar to $\triangle BAC$, then

$$\frac{AC}{BC} = \frac{DC}{AC} \quad \text{or} \quad \frac{b}{a} = \frac{a-b}{b} .$$

Cross-multiply, divide by $b^2 \neq 0$, and solve the quadratic in a/b to give $a/b = (1 + \sqrt{5})/2$ as the only positive root.



If $a/b = (1 + \sqrt{5})/2$, then

$$DC/AC = (a - b)/b = a/b - 1 = (\sqrt{5} - 1)/2$$

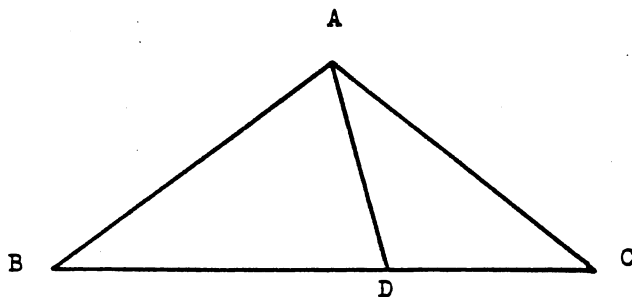
and

$$AC/BC = b/a = 2/(1 + \sqrt{5}) = (\sqrt{5} - 1)/2 = DC/AC .$$

Since $\angle C$ is in both triangles, $\triangle ADC$ is similar to $\triangle BAC$.

Theorem. Given that the ratio of two sides of a triangle is $k > 1$. A triangle similar to the first can be removed to leave a triangle such that the ratio of the areas of the original triangle and the triangle remaining is k , if and only if $k = (1 + \sqrt{5})/2$.

Proof. Let $\triangle ADC$ be similar to $\triangle BAC$, such that $BC/AC = AC/DC = k$.



If the ratio of areas of the original triangle and the one remaining is k , since there is a common altitude from A ,

$$k = \frac{\text{area } \triangle BAC}{\text{area } \triangle BDA} = \frac{(BC)(h/2)}{(BC - DC)(h/2)} = \frac{BC/AC}{BC/AC - DC/AC} = \frac{k}{k - 1/k} .$$

Again cross-multiplying and solving the quadratic in k gives $k = (1 + \sqrt{5})/2$.

If $k = (1 + \sqrt{5})/2$, then

$$BC/AC = AC/DC = (1 + \sqrt{5})/2,$$

and the ratio of areas $BC/(BC - DC)$ becomes $(1 + \sqrt{5})/2$ upon dividing through by AC and then simply substituting the values of BC/AC and DC/AC .

If $k = (1 + \sqrt{5})/2 = BC/AC$, and the ratio of areas of $\triangle BAC$ and $\triangle BDA$ is also k , then

$$k = \frac{BC/AC}{BC/AC - DC/AC} = \frac{k}{k - x} ,$$

which leads to

$$x = k - 1 \quad \text{or} \quad DC/AC = (1 + \sqrt{5})/2 - 1 = 2/(1 + \sqrt{5})$$

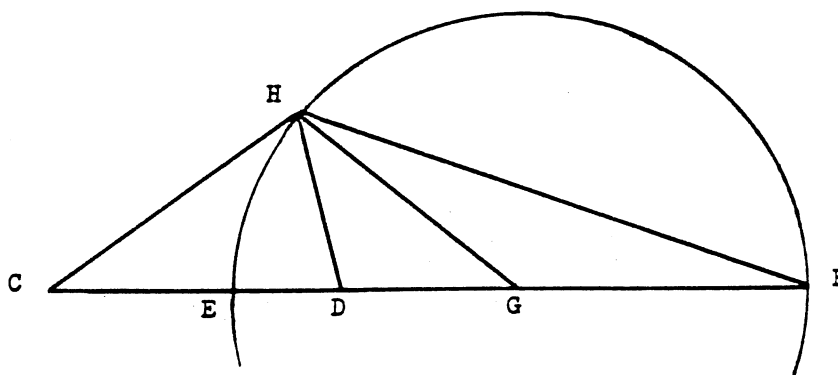
so that

$$AC/DC = (1 + \sqrt{5})/2$$

and $\triangle BAC$ is similar to $\triangle ADC$.

4. THE GENERAL GOLDEN TRIANGLE

Unlike the Golden Rectangle, the Golden Triangle does not have a unique shape. Consider a line segment \overline{CD} of length $\phi = (1 + \sqrt{5})/2$. Place points E , G , and F on line \overline{CD} such that $CE = 1$, $EG = GF = \phi$ as in the diagram.



Then, $ED = \phi - 1$, and

$$CE/ED = 1/(\phi - 1) = \phi,$$

$$CF/DF = (2\phi + 1)/(\phi + 1) = \phi^3/\phi^2 = \phi,$$

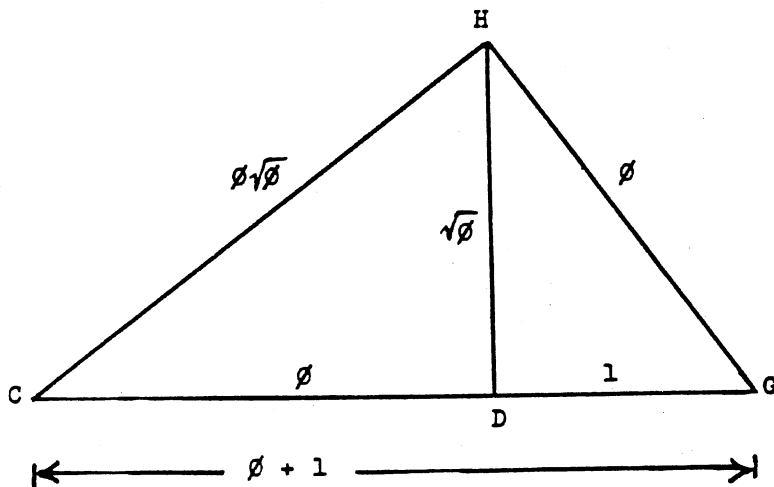
so that E and F divide segment \overline{CD} internally and externally in the ratio ϕ . Then the circle with center G is the circle of Apollonius for \overline{CD} with ratio ϕ . Incidentally, the circle through C, D, and H is orthogonal to the circle with center G and passing through H, and \overline{HG} is tangent to the circle through C, D, and H.

Let H be any point on the circle of Apollonius. Then $CH/HD = \phi$, $CG/HG = \phi$, and $\triangle CHG$ is similar to $\triangle HDG$. The area of $\triangle CHG$ is

$$h(1 + \phi)/2 = h\phi^2/2,$$

and when $\triangle HDG$ is removed, the area of the remaining $\triangle CHD$ is $h\phi/2$, so that the areas have the ratio ϕ . Then, $\triangle CHG$ is a Golden Triangle, and there are an infinite number of Golden Triangles because H can take an infinite number of positions on circle G.

If we choose H so that $CH = \phi + 1$, then we have the isosceles 36-72-72 Golden Triangle of decagon fame. If we erect a perpendicular at D and let H be the intersection with the circle of Apollonius, then we have a right golden triangle by applying the Pythagorean theorem and its converse. In our right golden triangle $\triangle CHG$, $CH = \phi\sqrt{\phi}$, $HG = \phi$, and $CG = \phi^2$. The two smaller right triangles formed by the altitude to \overline{CG} are each similar to $\triangle CHG$, so that all three triangles are golden. The areas of $\triangle HDG$, $\triangle CDH$, and $\triangle CHG$ form the geometric progression, $\sqrt{\phi}/2$, $(\sqrt{\phi}/2)\phi$, $(\sqrt{\phi}/2)\phi^2$.



Before going on, notice that the right golden triangle $\triangle CHG$ provides an unusual and surprising configuration. While two pairs of sides and all three pairs of angles of $\triangle CHG$ and $\triangle CDH$ are congruent, yet $\triangle CHG$ is not congruent to $\triangle CDH$! Similarly for $\triangle CDH$ and $\triangle HDG$. (See Holt [4].)

5. THE GOLDEN CUBOID

H. E. Huntley [5] has described a Golden Cuboid (rectangular parallelepiped) with lengths of edges a , b , and c , such that

$$a : b : c = \phi : 1 : \phi^{-1} .$$

The ratios of the areas of the faces are

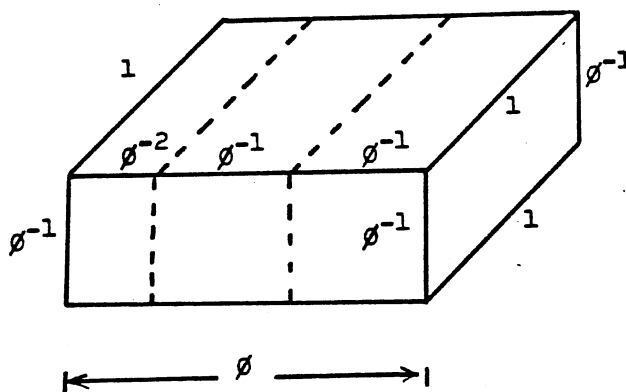
$$\phi : 1 : \phi^{-1} ,$$

and four of the six faces of the cuboid are Golden Rectangles.

If two cuboids of dimension

$$\phi^{-1} \times 1 \times \phi^{-1}$$

are removed from the Golden Cuboid, the remaining cuboid is similar to the original and is also a golden cuboid,



If a cuboid similar to the original is removed and has sides b , c , and d , then $b : c : d = \phi$, so that

$$c = d\phi , \quad b = d\phi^2 , \quad a = d\phi^3 .$$

The volume of the original is $abc = \phi^6 d^3$, and the volume removed is $bcd = \phi^3 d^3$. The remaining volume is $(\phi^6 - \phi^3)d^3$. The ratio of the volume of the original to the volume of the remaining cuboid is

$$\frac{\phi^6 d^3}{(\phi^6 - \phi^3)d^3} = \frac{\phi^3}{\phi^3 - 1} = \frac{2 + \sqrt{5}}{1 + \sqrt{5}} = \frac{3 + \sqrt{5}}{4} = \phi^2/2 .$$

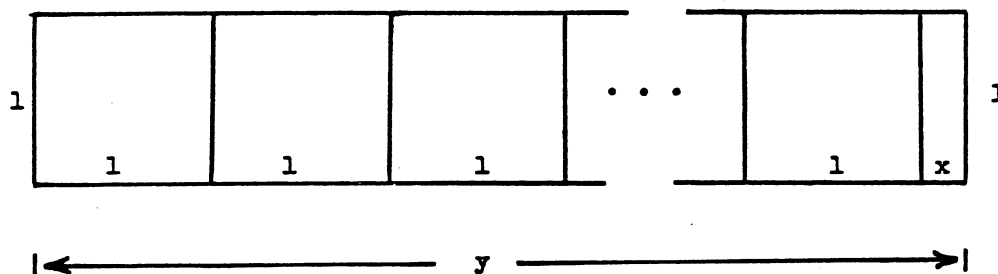
6. LUCAS GOLDEN-TYPE RECTANGLES

Now, in a Golden Rectangle, if one square with side equal to the width is removed, the resulting rectangle is similar to the original. Suppose that we have a rectangle in which when k squares with side equal to the width are removed, a rectangle similar to the original is formed, as discussed by J. A. Raab [6]. In the figure below, the ratio of length to width in the original rectangle and in the similar one formed after removing k squares is $y : 1 = 1 : x$ which gives $x = 1/y$. Since each square has side 1,

$$y - x = y - 1/y = k ,$$

or,

$$(6.1) \quad y^2 - ky - 1 = 0 .$$



Let us consider only Lucas golden-type rectangles. That is, let $k = L_{2m+1}$, where L_{2m+1} is the $(2m + 1)$ st Lucas number defined by

$$L_0 = 2 , \quad L_1 = 1 , \quad L_n = L_{n-1} + L_{n-2} , \quad n \geq 2 .$$

A known identity is

$$L_k = \left(\frac{1 + \sqrt{5}}{2}\right)^k + \left(\frac{1 - \sqrt{5}}{2}\right)^k = \alpha^k + \beta^k,$$

where α and β are the roots of $x^2 - x - 1 = 0$. In our problem, when $k = L_{2m+1}$, then (6.1) becomes

$$y^2 - L_{2m+1}y - 1 = 0$$

so that

$$y = \alpha^{2m+1} \quad \text{or} \quad y = \beta^{2m+1}$$

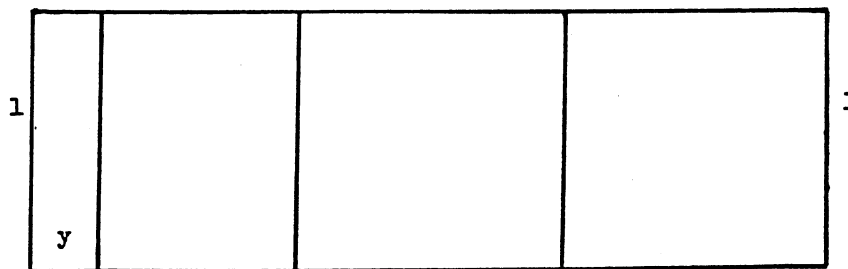
but $y = \alpha^{2m+1}$ is the only positive root. Then

$$x = 1/\alpha^{2m+1} = -\beta^{2m+1}.$$

On the other hand, suppose we insist that to a given rectangle we add one similar to it such that the result is k squares long. Illustrated for $k = 3$, the equal ratios of length to width in the similar rectangles give

$$\frac{1}{y} = \frac{k - y}{1} \quad \text{or} \quad ky - y^2 = 1 \quad \text{or} \quad y^2 - ky + 1 = 0.$$

$$\left\langle \longleftarrow \text{---} k - y \text{---} \longrightarrow \right\rangle$$



$$\left\langle \longleftarrow \text{---} k \text{---} \longrightarrow \right\rangle$$

Now, let $k = L_{2m}$; then $y = \alpha^{2m}$ or $y = \beta^{2m}$. Here, of course, $y = \alpha^{2m}$, so that

$$k - y = L_{2m} - \beta^{2m} = \alpha^{2m}.$$

Both of these cases are, of course, in the plane; the reader is invited to extend these ideas into the third dimension.

(Section 7, entitled "Generalized Golden-Type Cuboids," is omitted here.)

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2. Verner E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton Mifflin Company, Boston, 1969.
3. Tobias Dantzig, Bequest of the Greeks, Charles Scribner's Sons, New York, 1955, pp. 61-62.
4. Marvin H. Holt, "Mystery Puzzler and Phi," The Fibonacci Quarterly, Vol. 3, No. 2, April, 1965, pp. 135-138.
5. H. E. Huntley, "The Golden Cuboid," The Fibonacci Quarterly, Vol. 2, No. 3, October, 1964, p. 184⁺.
6. Joseph A. Raab, "A Generalization of the Connection Between the Fibonacci Sequence and Pascal's Triangle," The Fibonacci Quarterly, Vol. 1, No. 3, October, 1963, pp. 21-31.

A SPECIAL GEOMETRY PROBLEM

H-19 (Proposed by Charles R. Wall) In the triangle below [drawn for the case (1, 1, 3)], the trisectors of angle B divide side AC into segments of length F_n , F_{n+1} , and F_{n+3} . Find:

$$(i) \lim_{n \rightarrow \infty} \theta \quad \text{and} \quad (ii) \lim_{n \rightarrow \infty} \phi$$

