CONGRUENCES FOR EULER NUMBERS

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1. INTRODUCTION AND NOTATIONS

The Bernoulli polynomials $B_n(X)$ are defined by

$$\frac{te^{Xt}}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n(X)t^n}{n!},\quad |t|<2\pi.$$

And the Bernoulli numbers B_n can be defined by $B_n=B_n(0)$. The Euler polynomials $E_n(X)$ are defined by

$$\frac{2e^{Xt}}{e^t+1}=\sum_{n=0}^{\infty}\frac{E_n(X)t^n}{n!},\quad |t|<\pi.$$

And the Euler numbers E_n are defined by $E_n=2^nE_n(1/2)$. These numbers and polynomials arise in some combinatorial contexts, and have been investigated by many authors. For example, see Powell [6], Young [7], and Zhang [8]. The well-known congruences among these numbers or polynomials are the classical Kummer's congruences:

Theorem 1.1: (ref. page 239 in [4]). *Suppose p is prime, and m,n and e are positive integers such that m and n are even, and $p-1\nmid m,n$. Then one has*

$$\frac{1}{m}(1-p^{m-1})B_m\equiv\frac{1}{n}(1-p^{n-1})B_n \pmod{p^e},$$

if $m\equiv n\ (\mathrm{mod}\,\varphi(p^e))$.

Kummer's congruences play important roles in the p-adic interpolation of the Riemann zeta function [5]. In 1997, Eie and Ong [3] generalized Kummer's congruences to Bernoulli polynomials via p-adic interpolation on p-adic spaces.

Theorem 1.2: (ref. [3]). *Suppose that p is an odd prime, and $m,n,$ and e are positive integers such that $p-1\nmid m,n$. Then for any positive integer k relatively prime to p and positive integers $0\le\alpha,\beta\le k-1$ such that $\alpha+jk=p\beta$ for some j with $0\le j\le p-1$, one has*

$$\frac{1}{m}\left\{B_m\left(\frac{\alpha}{k}\right)-p^{m-1}B_m\left(\frac{\beta}{k}\right)\right\}\equiv\frac{1}{n}\left\{B_n\left(\frac{\alpha}{k}\right)-p^{n-1}B_n\left(\frac{\beta}{k}\right)\right\} \pmod{p^e},$$

if $m\equiv n \pmod{\varphi(p^e)}$.

In this paper, we prove the following theorem which is a generalization of Kummer's congruences on Euler polynomials.

Theorem 1.3: *Suppose that p is an odd prime, and m, n, and e are positive integers such that $p-1 \nmid m, n$. Then for any positive integer k relatively prime to p and positive integers α, β such that $\alpha + 2jk = p\beta$ with $0 \leq j \leq (p-1)/2$, one has*

$$E_{m-1}\left(\frac{\alpha}{k}\right) - p^{m-1}E_{m-1}\left(\frac{\beta}{k}\right) \equiv E_{n-1}\left(\frac{\alpha}{k}\right) - p^{n-1}E_{n-1}\left(\frac{\beta}{k}\right) \pmod{p^e},$$

if $m \equiv n \pmod{\varphi(p^e)}$.

The classical congruences on Euler numbers (ref. page 124 of [2])

$$E_{4n} \equiv 5 \pmod{60} \quad \text{and} \quad E_{4n-2} \equiv -1 \pmod{60}, \tag{1}$$

are normally attributed to Stern. In 1998, Zhang [8] deduced some other congruences on Euler numbers:

Proposition 1.4: (ref. Corollary 1 in [8]). *For any odd prime p, we have the congruence*

$$E_{p-1} \equiv \begin{cases} 0 \pmod{p}, & \text{if } 4|p-1, \\ 2 \pmod{p}, & \text{if } 4|p-3. \end{cases}$$

Proposition 1.5: (ref. Corollary 2 in [8]). *For any integer $n > 0$, we have the congruences*
(1) $E_{2n+2} - E_{2n} \equiv 0 \pmod{6}$,
(2) $E_{2n+4} - 10E_{2n+2} + 9E_{2n} \equiv 0 \pmod{24}$,
(3) $E_{2n+6} - E_{2n} \equiv 0 \pmod{42}$.

Here we derive some new congruences on Euler numbers:

Theorem 1.6: *Assume $\delta = p_1^{n_1}p_2^{n_2}\ldots p_r^{n_r}$, where p_i are odd primes, and n_i are positive integers for $i = 1, \ldots, r$. Let $N = \max_{1\leq i\leq r} n_i$ and $M = l.c.m.\{2^k, \varphi(p_1^{n_1}), \ldots, \varphi(p_r^{n_r})\}$, where k is a non-negative integer. Then for any positive integers m, n with $\min\{2m, 2n\} \geq N$, we have*

$$E_{2m} \equiv E_{2n} \pmod{2^k\delta}$$

if $2m \equiv 2n \pmod{M}$.

In the last section, we give an algorithm to treat the congruences of Euler numbers for any modulus. Using this algorithm we can easily derive all the above congruences.

2. CONGRUENCES WITH EULER POLYNOMIALS

Lemma 2.1: *Suppose that p is an odd prime, and m, k are positive integers. Then for any positive integers α, β, one has*

$$E_{m-1}\left(\frac{\alpha}{k}\right) - p^{m-1}E_{m-1}\left(\frac{\beta}{k}\right)$$

$$= \frac{2^m}{m}\left[B_m\left(\frac{\alpha+k}{2k}\right) - p^{m-1}B_m\left(\frac{\beta+k}{2k}\right)\right] - \frac{2^m}{m}\left[B_m\left(\frac{\alpha}{2k}\right) - p^{m-1}B_m\left(\frac{\beta}{2k}\right)\right].$$

Proof: By formula 23.1.27 of [1]

$$E_{m-1}(x) = \frac{2^m}{m}\left[B_m\left(\frac{x+1}{2}\right) - B_m\left(\frac{x}{2}\right)\right].$$

Utilizing this to find the following difference in values of $E_{m-1}(x)$ for $x = \alpha/k$ and $x = \beta/k$ yields

$$E_{m-1}\left(\frac{\alpha}{k}\right) - p^{m-1}E_{m-1}\left(\frac{\beta}{k}\right)$$

$$= \frac{2^m}{m}\left[B_m\left(\frac{\alpha+k}{2k}\right) - B_m\left(\frac{\alpha}{2k}\right)\right] - p^{m-1}\frac{2^m}{m}\left[B_m\left(\frac{\beta+k}{2k}\right) - B_m\left(\frac{\beta}{2k}\right)\right].$$

This completes our proof. □

Now we prove the generalization of Kummer's congruences on Euler polynomials in the following theorem which is stated as the same with Theorem 1.3 in Section 1.

Theorem 2.2: *Suppose that p is an odd prime, and m, n, and e are positive integers such that $p-1 \nmid m, n$. Then for any positive integer k relatively prime to p and positive integers α, β such that $\alpha + 2jk = p\beta$ with $0 \le j \le (p-1)/2$, one has*

$$E_{m-1}\left(\frac{\alpha}{k}\right) - p^{m-1}E_{m-1}\left(\frac{\beta}{k}\right) \equiv E_{n-1}\left(\frac{\alpha}{k}\right) - p^{n-1}E_{n-1}\left(\frac{\beta}{k}\right) \pmod{p^e},$$

if $m \equiv n \pmod{\varphi(p^e)}$.

Proof: Suppose p is an odd prime, and k is relatively prime to p. This implies that $(2k, p) = 1$. Applying Theorem 1.2 (ref. [3]), and then for any positive integers α, β such that $\alpha + j \cdot (2k) = p\beta$ with $0 \le j \le \frac{p-1}{2}$, one has

$$\frac{1}{m}\left\{B_m\left(\frac{\alpha}{2k}\right) - p^{m-1}B_m\left(\frac{\beta}{2k}\right)\right\} \equiv \frac{1}{n}\left\{B_n\left(\frac{\alpha}{2k}\right) - p^{n-1}B_n\left(\frac{\beta}{2k}\right)\right\} \pmod{p^e},$$

if $m \equiv n \pmod{\varphi(p^e)}$.

Since $\frac{p-1}{2} \le j + \frac{p-1}{2} \le p-1$ and $(\alpha+k) + \left(j + \frac{p-1}{2}\right)\cdot(2k) = (k+\beta)p$. Applying Theorem 1.2 again, it follows that

$$\frac{1}{m}\left\{B_m\left(\frac{\alpha+k}{2k}\right) - p^{m-1}B_m\left(\frac{\beta+k}{2k}\right)\right\} \equiv \frac{1}{n}\left\{B_n\left(\frac{\alpha+k}{2k}\right) - p^{n-1}B_n\left(\frac{\beta+k}{2k}\right)\right\} \pmod{p^e}.$$

From Fermat's Little Theorem we have $2^m \equiv 2^n \pmod{p^e}$. Combining these three congruences together, we find that

$$\frac{2^m}{m}\left[B_m\left(\frac{\alpha+k}{2k}\right) - p^{m-1}B_m\left(\frac{\beta+k}{2k}\right)\right] - \frac{2^m}{m}\left[B_m\left(\frac{\alpha}{2k}\right) - p^{m-1}B_m\left(\frac{\beta}{2k}\right)\right]$$

$$\equiv \frac{2^n}{n}\left[B_n\left(\frac{\alpha+k}{2k}\right) - p^{n-1}B_n\left(\frac{\beta+k}{2k}\right)\right] - \frac{2^n}{n}\left[B_n\left(\frac{\alpha}{2k}\right) - p^{n-1}B_n\left(\frac{\beta}{2k}\right)\right] \pmod{p^e}.$$

Applying Lemma 2.1, we conclude our assertion. □

In particular, we let $\alpha = \beta = k = 1$ in Theorem 2.2. And we apply the fact that (see e.g. formula 23.1.20 of [1])

$$E_n(0) = -2(n+1)^{-1}(2^{n+1}-1)B_{n+1}.$$

We obtain

$$(1-p^{m-1})E_{m-1}(1) \equiv (1-p^{n-1})E_{n-1}(1) \pmod{p^e}$$

$$(1-p^{m-1})E_{m-1}(0) \equiv (1-p^{n-1})E_{n-1}(0) \pmod{p^e}$$

$$(1-p^{m-1})\frac{(-2)(2^m-1)B_m}{m} \equiv (1-p^{n-1})\frac{(-2)(2^n-1)B_n}{n} \pmod{p^e}.$$

By Euler's generalization of Fermat's Little Theorem we can divide

$$-2(2^m-1) \equiv -2(2^n-1) \pmod{p^e}$$

from the above congruence and this gives the classical Kummer's congruences.

3. CONGRUENCES WITH EULER NUMBERS

Since $E_n(1/2) = 2^{-n}E_n$, if we let $k = 2$ in Theorem 2.2, then we could reformulate congruences in terms of Euler numbers.

Theorem 3.1: *Suppose that p is an odd prime and m, n be non-negative integers. Then if $2m \equiv 2n$ (mod $\varphi(p^e)$), we have*

$$\begin{cases} (1-p^{2m})E_{2m} \equiv (1-p^{2n})E_{2n} \pmod{p^e}, \ \textit{if } 4|p-1, \\ (1+p^{2m})E_{2m} \equiv (1+p^{2n})E_{2n} \pmod{p^e}, \ \textit{if } 4|p-3. \end{cases}$$

Proof: First, let p be an odd prime with $4|p-1$, that is, $p = 4j+1$ for some positive integer j. Clearly, it is the case that $\alpha = \beta = 1$ in Theorem 2.2, therefore

$$E_{m-1}\left(\frac{1}{2}\right) - p^{m-1}E_{m-1}\left(\frac{1}{2}\right) \equiv E_{n-1}\left(\frac{1}{2}\right) - p^{n-1}E_{n-1}\left(\frac{1}{2}\right) \pmod{p^e}$$

$$2^{1-m}(1-p^{m-1})E_{m-1} \equiv 2^{1-n}(1-p^{n-1})E_{n-1} \pmod{p^e},$$

if $m \equiv n \pmod{\varphi(p^e)}$ and $p-1$ is not a divisor of m. Since Fermat's Little Theorem gives $2^{1-m} \equiv 2^{1-n}$ (mod p^e), we can divide it from the above congruence. And let $m = 2m'+1$ and $n = 2n'+1$ where m', n' are non-negative integers. It is clear that $p-1$ is not a divisor of $2m'+1$ for any odd prime p. Then we get our assertion for the case $4|p-1$.

Second, let p be an odd prime number with $4|p-3$, that is, $p = 4j+3$ for some non-negative integer j. Clearly, it is the case that $\alpha = 3, \beta = 1$ in Theorem 2.2, therefore

$$E_{m-1}\left(\frac{3}{2}\right) - p^{m-1}E_{m-1}\left(\frac{1}{2}\right) \equiv E_{n-1}\left(\frac{3}{2}\right) - p^{n-1}E_{n-1}\left(\frac{1}{2}\right) \pmod{p^e}$$

if $m \equiv n \pmod{\varphi(p^e)}$ and $p-1$ is not a divisor of m. By formula 23.1.6 of [1]

$$E_n\left(\frac{3}{2}\right) = 2^{-n+1} - E_n\left(\frac{1}{2}\right), \qquad \text{for } n \geq 0.$$

Substituting this in the above congruence, one has

$$2^{2-m}E_{m-1}\left(\frac{1}{2}\right) - p^{m-1}E_{m-1}\left(\frac{1}{2}\right) \equiv 2^{2-n} - E_{n-1}\left(\frac{1}{2}\right) - p^{n-1}E_{n-1}\left(\frac{1}{2}\right) \pmod{p^e}$$

$$2^{2-m} - 2^{1-m}(1+p^{m-1})E_{m-1} \equiv 2^{2-n} - 2^{1-n}(1+p^{n-1})E_{n-1} \pmod{p^e}.$$

Again using Fermat's Little Theorem, we can cancel

$$2^{2-m} \equiv 2^{2-n}, \quad 2^{1-m} \equiv 2^{1-n} \pmod{p^e},$$

from the above congruence. Similarly we let $m = 2m'+1$ and $n = 2n'+1$ where m', n' are non-negative integers. The condition $p-1$ is not a divisor of $2m'+1$ always holds for any odd prime p. Then the proof is complete. □

Now we treat the situation when $p = 2$.

Lemma 3.2: *For any non-negative integer n, we have*

$$E_n = 1 + \frac{1}{n+1}\sum_{k=2}^{n+1}\binom{n+1}{k}2^k(1-2^k)B_k. \tag{2}$$

Proof: By formula 23.1.7 of [1]

$$E_n(x+h) = \sum_{k=0}^{n}\binom{n}{k}E_k(x)h^{n-k}.$$

We substitute $x = 0$ and $h = 1/2$ in the above equation. Then

$$2^{-n}E_n = E_n\left(\frac{1}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}E_k(0)2^{k-n}.$$

Now (see e.g. formula 23.1.20 of [1])

$$E_n(0) = -2(n+1)^{-1}(2^{n+1}-1)B_{n+1},$$

and the above equality becomes

$$E_n = \sum_{k=0}^{n}\binom{n}{k}\frac{2^{k+1}(1-2^{k+1})}{k+1}B_{k+1}$$

$$= 1 + \frac{1}{n+1}\sum_{k=2}^{n+1}\binom{n+1}{k}2^k(1-2^k)B_k.$$

This gives our assertion. □

We set $n = 2m - 1$ in Eq. (2) and since $E_{2m-1} = 0$, we get an identity in Bernoulli numbers

$$\sum_{k=1}^{m} \binom{2m}{2k} 2^{2k}(2^{2k} - 1)B_{2k} = 2m,$$

for any positive integer m.

In the following we will discuss the congruences of Euler numbers modulo 2^r for some integer r. Here we introduce two notations. The notation $2^h \parallel t$ means 2^h divides t, but 2^{h+1} does not divide t. To simplify the writing we denote

$$C_k(n) = \frac{1}{n+1}\binom{n+1}{k} 2^k(1 - 2^k)B_k.$$

Proposition 3.3: *For any non-negative integers m and r with $2m \geq r$, we have*

$$E_{2m} \equiv 1 + \frac{1}{2m+1}\sum_{k=1}^{[r/2]} \binom{2m+1}{2k} 2^{2k}(1 - 2^{2k})B_{2k} \pmod{2^r},$$

where $[r/2]$ denotes the greatest integer not exceeding $r/2$.

Proof: We consider Eq. (2) in Lemma 3.2, i.e.

$$E_n = 1 + \frac{1}{n+1}\sum_{k=2}^{n+1} \binom{n+1}{k} 2^k(1 - 2^k)B_k$$

$$= 1 + \sum_{k=2}^{n+1} C_k(n).$$

Let $n = 2m$ and from the Staudt-Clausen Theorem we know $2B_k$ is 2-integral, thus the number h with $2^h \parallel C_k(n)$ satisfies $h \geq k - 1$. Thus when $2m \geq k - 1 \geq r$,

$$C_k(2m) \equiv 0 \pmod{2^r}.$$

Therefore

$$E_{2m} \equiv 1 + \sum_{k=2}^{r} C_k(2m) \pmod{2^r}$$

$$\equiv 1 + \sum_{k=2}^{r} \frac{1}{2m+1}\binom{2m+1}{k} 2^k(1 - 2^k)B_k \pmod{2^r}$$

$$\equiv 1 + \frac{1}{2m+1}\sum_{k=1}^{[r/2]} \binom{2m+1}{2k} 2^{2k}(1 - 2^{2k})B_{2k} \pmod{2^r},$$

for $2m \geq r$.

Lemma 3.4: *If $2^h \parallel k!$, then $h \leq k-1$.*

Proof: Clearly, the power of 2 that divides $k!$ is given by

$$h = \left[\frac{k}{2}\right] + \left[\frac{k}{2^2}\right] + \cdots .$$

Since k is finite, $[k/2^m] = 0$ for all m sufficiently large. Thus

$$h < \frac{k}{2} + \frac{k}{2^2} + \cdots = k.$$

Therefore $h \leq k-1$. □

Proposition 3.5: *For any non-negative integers m, n, and r, we have*

$$E_{2m} \equiv E_{2n} \pmod{2^r},$$

if $2m \equiv 2n \pmod{2^r}$.

Proof: From the Staudt-Clausen Theorem we know $2B_{2k}$ is 2-integral and the result of Lemma 3.4 gives $2^{2k-1}/(2k)!$ is also 2-integral. Therefore $2^{2k-1}(1-2^{2k})2B_{2k}/(2k)!$ is 2-integral. This gives for $2k \leq \min\{2m, 2n\}$,

$$\begin{aligned} C_{2k}(2m) &= 2m \cdot (2m-1) \cdots (2m+2-2k) \cdot \frac{2^{2k-1}}{(2k)!}(1-2^{2k}) \cdot (2B_{2k}) \\ &\equiv 2n \cdot (2n-1) \cdots (2n+2-2k) \cdot \frac{2^{2k-1}}{(2k)!}(1-2^{2k}) \cdot (2B_{2k}) \pmod{2^r} \\ &= C_{2k}(2n). \end{aligned}$$

We assume $2m < r$, so $2n = 2m + (a \cdot 2^r) \geq 2^r > r$ for some positive integer a. If $k \geq m+1$, then $(2n-2m)$ divides $2n \cdot (2n-1) \cdots (2n-2k+2)$. This gives $C_{2k}(2n) \equiv 0$ modulo 2^r. Hence by Proposition 3.3

$$\begin{aligned} E_{2n} &\equiv 1 + \sum_{k=1}^{[r/2]} C_{2k}(2n) \pmod{2^r} \\ &\equiv 1 + \sum_{k=1}^{m} C_{2k}(2n) \pmod{2^r} \\ &\equiv 1 + \sum_{k=1}^{m} C_{2k}(2m) \equiv E_{2m} \pmod{2^r}. \end{aligned}$$

And the remaining case $\min\{2m, 2n\} \geq r$ is found by applying $C_{2k}(2m) \equiv C_{2k}(2n)$ for $k = 1, \ldots, [r/2]$. The proof is complete. □

Now we combine the above proposition and Theorem 3.1, we give a congruence relation between Euler numbers for any modulus which is stated as the same with Theorem 1.6 in Section 1.

Theorem 3.6: *Assume $\delta = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$, where p_i are odd primes, and n_i are positive integers for $i = 1, \cdots, r$. Let $N = \max_{1 \leq i \leq r} n_i$ and $M = l.c.m.\{2^k, \varphi(p_1^{n_1}), \cdots, \varphi(p_r^{n_r})\}$, where k is a non-negative integer. Then for any positive integers m, n with $\min\{2m, 2n\} \geq N$, we have*

$$E_{2m} \equiv E_{2n} \pmod{2^k \delta}$$

if $2m \equiv 2n \pmod{M}$.

Proof: Since $2m \equiv 2n \pmod{\varphi(p_i^{n_i})}$ for $i = 1, \cdots, r$, we may apply Theorem 3.1 and obtain

$$\begin{cases} (1 - p_i^{2m})E_{2m} \equiv (1 - p_i^{2n})E_{2n} \pmod{p_i^{n_i}}, & \text{if } 4|p_i - 1; \\ (1 + p_i^{2m})E_{2m} \equiv (1 + p_i^{2n})E_{2n} \pmod{p_i^{n_i}}, & \text{if } 4|p_i - 3. \end{cases}$$

However, both $2m$ and $2n$ are not less than $N = \max\{n_1, \cdots, n_r\}$, thus

$$E_{2m} \equiv E_{2n} \pmod{p_i^{n_i}}, \quad \text{for } i = 1, \cdots, r.$$

Combining the congruences in Proposition 3.5, we complete our proof. □

Remark 3.7: *In particular we let $n = 0$ in the congruences in Theorem 3.1 and Proposition 3.5. We obtain*

$$\begin{cases} E_{2^k m} \equiv 1 & \pmod{2^k}; \\ (1 - p_i^{\varphi(p_i^{n_i})m})E_{\varphi(p_i^{n_i})m} \equiv 0 & \pmod{p_i^{n_i}}, \text{ if } 4|p_i - 1; \\ (1 + p_i^{\varphi(p_i^{n_i})m})E_{\varphi(p_i^{n_i})m} \equiv 2 & \pmod{p_i^{n_i}}, \text{ if } 4|p_i - 3, \end{cases}$$

for any non-negative integer m. Since $\varphi(p_i^{n_i}) > n_i$ for any odd prime p_i and non-negative integer n_i, we have

$$\begin{cases} E_{2^k m} \equiv 1 & \pmod{2^k}; \\ E_{\varphi(p_i^{n_i})m} \equiv 0 & \pmod{p_i^{n_i}}, \text{ if } 4|p_i - 1; \\ E_{\varphi(p_i^{n_i})m} \equiv 2 & \pmod{p_i^{n_i}}, \text{ if } 4|p_i - 3, \end{cases}$$

for any positive integer m.

Letting $n_i = 1$ in the above congruences, we have the following corollary which is a generalization of Corollary 1 in [8].

Corollary 3.8: *For any odd prime p and any positive integer m, we have*

$$E_{(p-1)m} \equiv \begin{cases} 0 \pmod{p}, & \text{if } 4|p - 1, \\ 2 \pmod{p}, & \text{if } 4|p - 3. \end{cases}$$

It is clearly we have following corollary which can be stated in a similar manner as Theorems 1.1, 1.2, and 1.3.

Corollary 3.9: *Let* $\prod_{i=1}^{t} p_i^{n_i}$ *be the prime-power factorization of an odd integer* D. *Let* $N = \max_{1\le i\le t} n_i$, *and* $M = \operatorname{lcm}_{1\le i\le t} \varphi(p_i^{n_i})$. *Then*

$$E_m \equiv E_n \pmod{D},$$

if $m \equiv n \pmod{M}$ *and* $\min\{m,n\} \ge N$.

¿From Eq. (1) we know

$$E_{4n} \equiv 1 \pmod 4 \quad \text{and} \quad E_{4n-2} \equiv 3 \pmod 4.$$

Therefore, the condition that D is an odd integer in the above corollary cannot be changed to that D is an integer.

4. ALGORITHM AND APPLICATIONS

Combining Theorem 3.6 and Remark 3.7, we can given an algorithm to list all the congruences of Euler numbers for any modulus.

Algorithm 4.1: *Given an arbitrary positive integer* m.

Step 1: *Write down the prime factorization of* m *as*

$$m = 2^k p_1^{a_1} p_2^{a_2} \dots p_r^{a_r} q_1^{b_1} q_2^{b_2} \dots q_s^{b_s},$$

where 4 *divides both* $(p_i - 1)$ *and* $(q_j - 3)$ *for* $i = 1, \dots, r; j = 1, \dots, s$.

Step 2: *Compute*

$$M = l.c.m.\{2^k, \varphi(p_1^{a_1}), \dots, \varphi(p_r^{a_r}), \varphi(q_1^{b_1}), \dots, \varphi(q_s^{b_s})\}.$$

and let

$$N = \max\{a_i, b_j | i = 1, \dots, r, j = 1, \dots, s\}.$$

Step 3: *Use the Chinese Remainder Theorem to solve*

$$\begin{cases} E_{M(n+1)} \equiv 1 \pmod{2^k}; \\ E_{M(n+1)} \equiv 0 \pmod{p_i^{a_i}}, \ \textit{for } i = 1, \dots, r; \\ E_{M(n+1)} \equiv 2 \pmod{q_j^{b_j}}, \ \textit{for } j = 1, \dots, s, \end{cases}$$

for any non-negative integer n, *and denote the solution by* x_0 *modulo* m.

Step 4: *For any non-negative integer* n, *we can list all the congruences of Euler numbers modulo* m *as*

$$\begin{cases} E_{M(n+1)} \equiv x_0 \pmod{m}, \\ E_{Mn+2i} \equiv E_{2i} \pmod{m}, \ \textit{for } N \le 2i \le N + M - 2 \textit{ and } 2i \not\equiv 0 \pmod{M}. \end{cases}$$

Using Algorithm 4.1 we can easily derive the classical congruences in Euler numbers which is attributed to Stern (ref. page 124 of [2]).

Proposition 4.2: *For any positive integer n, we have*

$$\begin{cases} E_{4n} \equiv 5 & \pmod{60} \\ E_{4n-2} \equiv -1 & \pmod{60}. \end{cases}$$

Proof: Since $60 = 2^2 \cdot 3 \cdot 5$, $M = l.c.m.\{2^2\varphi(3), \varphi(5)\} = 4$, and $N = 1$. We apply the Chinese Remainder Theorem to solve

$$\begin{cases} E_{4(n+1)} \equiv 1 & \pmod{2^2}; \\ E_{4(n+1)} \equiv 0 & \pmod{5}; \\ E_{4(n+1)} \equiv 2 & \pmod{3}, \end{cases}$$

and we obtain

$$E_{4(n+1)} \equiv 5 \pmod{60}.$$

Since $1 = N \leq 2i \leq N + M - 2 = 3$ and $2i \not\equiv 0 \pmod{4}$, this forces $i = 1$. Therefore we have

$$\begin{cases} E_{4n+4} \equiv 5 & \pmod{60} \\ E_{4n+2} \equiv E_2 \equiv -1 & \pmod{60}, \end{cases}$$

for any non-negative integer n. □

In fact, the above congruences give

$$E_{2n} \equiv 5 \pmod{6},$$

for any positive integer n. This is exactly the congruences in Corollary 2(a) in [8].

Proposition 4.3: *For any positive integer n,*

$$\begin{cases} E_{6n} \equiv 65 & \pmod{126}, \\ E_{6n-2} \equiv 5 & \pmod{126}, \\ E_{6n-4} \equiv -1 & \pmod{126}. \end{cases}$$

Proof: Since $126 = 2 \cdot 3^2 \cdot 7$, $M = l.c.m.\{2, \varphi(3^2), \varphi(7)\} = 6$, and $N = 2$. We apply the Chinese Remainder Theorem to solve

$$\begin{cases} E_{6(n+1)} \equiv 1 & \pmod{2}; \\ E_{6(n+1)} \equiv 2 & \pmod{7}; \\ E_{6(n+1)} \equiv 2 & \pmod{3^2}, \end{cases}$$

and we obtain

$$E_{6(n+1)} \equiv 65 \pmod{126}.$$

Since $2 = N \leq 2i \leq N + M - 2 = 6$ and $2i \not\equiv 0 \pmod{6}$, this forces $i = 1$ and 2. Therefore

$$\begin{cases} E_{6n+6} \equiv 65 & \pmod{126} \\ E_{6n+2} \equiv E_2 \equiv -1 & \pmod{126} \\ E_{6n+4} \equiv E_4 \equiv 5 & \pmod{126}, \end{cases}$$

for any non-negative integer n. $\square$

The above congruences give us

$$E_{2n+6} \equiv E_{2n} \pmod{126},$$

for any positive integer n. This is clearly a generalization of Corollary 2(c) in [8].

Proposition 4.4: *For any positive integer n,*

$$\begin{cases} E_{8n} \equiv 65 & \pmod{120}, \\ E_{8n-2} \equiv -61 & \pmod{120}, \\ E_{8n-4} \equiv 5 & \pmod{120}, \\ E_{8n-6} \equiv -1 & \pmod{120}. \end{cases}$$

Proof: Since $120 = 2^3 \cdot 3 \cdot 5$, $M = l.c.m.\{2^3, \varphi(3), \varphi(5)\} = 8$, and $N = 1$. We apply the Chinese Remainder Theorem to solve

$$\begin{cases} E_{8(n+1)} \equiv 1 & \pmod{8}; \\ E_{8(n+1)} \equiv 0 & \pmod{5}; \\ E_{8(n+1)} \equiv 2 & \pmod{3}, \end{cases}$$

and we obtain

$$E_{8(n+1)} \equiv 65 \pmod{120}.$$

Since $1 = N \leq 2i \leq N + M - 2 = 7$ and $2i \not\equiv 0 \pmod{8}$, this forces $i = 1$, 2, and 3. Therefore

$$\begin{cases} E_{8n+8} \equiv 65 & \pmod{120} \\ E_{8n+2} \equiv E_2 \equiv -1 & \pmod{120} \\ E_{8n+4} \equiv E_4 \equiv 5 & \pmod{120} \\ E_{8n+6} \equiv E_6 \equiv -61 & \pmod{120}, \end{cases}$$

for any non-negative integer n. $\square$

Corollary 4.5: *For any integer $n > 0$,*

$$E_{2n} - 9E_{2n-2} \equiv 14 \pmod{120}.$$

Proof: We just need to substitute $n = 4k$, $4k + 1$, $4k + 2$, and $4k + 3$ in the above congruences. Applying the results in Proposition 4.4, the assertion is proved. $\square$

Also the above result is a generalization of Corollary 2(b) in [8].

Proposition 4.6: *For any non-negative integer n,*

$$\begin{cases} E_{144n+2i} \equiv E_{2i} & \pmod{323}, \quad \textit{for } 1 \leq i \leq 71, \\ E_{144n+144} \equiv 306 & \pmod{323}. \end{cases}$$

Proof: Since $323 = 17 \cdot 19$, $M = l.c.m.\{\varphi(17), \varphi(19)\} = 144$, and $N = 1$. We apply the Chinese Remainder Theorem to solve

$$\begin{cases} E_{144(n+1)} \equiv 0 \pmod{17}; \\ E_{144(n+1)} \equiv 2 \pmod{19}, \end{cases}$$

and we obtain

$$E_{144(n+1)} \equiv 306 \pmod{323}.$$

Since $1 = N \leq 2i \leq N + M - 2 = 143$ and $2i \not\equiv 0 \pmod{144}$, this gives $E_{144n+2i} \equiv E_{2i} \pmod{323}$, for $1 \leq i \leq 71$. Therefore we complete the proof. □

Proposition 4.7: *For any non-negative integer n,*

$$\begin{cases} E_{16n+2} \equiv -1 \pmod{68}, \\ E_{16n+4} \equiv 5 \pmod{68}, \\ E_{16n+6} \equiv 7 \pmod{68}, \\ E_{16n+8} \equiv 25 \pmod{68}, \end{cases} \qquad \begin{cases} E_{16n+10} \equiv 3 \pmod{68}, \\ E_{16n+12} \equiv -31 \pmod{68}, \\ E_{16n+14} \equiv -9 \pmod{68}, \\ E_{16n+16} \equiv 17 \pmod{68}. \end{cases}$$

Proof: Since $68 = 2^2 \cdot 17$, $M = l.c.m.\{2^2, \varphi(17)\} = 16$, and $N = 1$. We apply the Chinese Remainder Theorem to solve

$$\begin{cases} E_{16(n+1)} \equiv 1 \pmod{4}; \\ E_{16(n+1)} \equiv 0 \pmod{17}, \end{cases}$$

and we obtain

$$E_{16(n+1)} \equiv 17 \pmod{68}.$$

Since $1 = N \leq 2i \leq N + M - 2 = 15$ and $2i \not\equiv 0 \pmod{16}$, this gives $E_{16n+2i} \equiv E_{2i}$ (mod 68), for $i = 1, 2, \ldots, 7$. This completes our proof. □

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