YE OLDE FIBONACCI CURIOSITY SHOPPE REVISITED

R.S. Melham
Department of Mathematical Sciences, University of Technology, Sydney
PO Box 123, Broadway, NSW 2007 Australia
(Submitted September 2001)

1. INTRODUCTION

There are many Fibonacci identities to be found in short informal articles in the early editions of *The Fibonacci Quarterly*. See, for example, [1] and [2]. The aim of the authors was to gather Fibonacci identities from diverse sources and display them for all to see. Many of the identities that appeared were quite old and possessed beautiful symmetry. As such, some were already classics, and today appear regularly in research papers. Perhaps the best example is Simson's identity, which has undergone many generalizations. For an up to date account see [7]. However, other identities have received little or no attention, and have not featured in the literature since those early days. The purpose of this paper is to present some new insight into two such identities. We assume throughout that the sequences in this paper are defined for all integers, and henceforth we do not restate this.

2. CANDIDO'S IDENTITY

The lovely identity

$$\left(F_n^2+F_{n+1}^2+F_{n+2}^2\right)^2=2\left(F_n^4+F_{n+1}^4+F_{n+2}^4\right) \tag{2.1}$$

is quoted in [2] and first appeared in 1951 [3]. We decided to increase the number of terms on the left, and were pleasantly surprised to find the following.

$$2\left(F_n^2+F_{n+1}^2+F_{n+2}^2+F_{n+3}^2\right)^2=3\left(F_n^4+4F_{n+1}^4+4F_{n+2}^4+F_{n+3}^4\right); \tag{2.2}$$

$$\left(F_n^2+F_{n+1}^2+F_{n+2}^2+F_{n+3}^2+F_{n+4}^2\right)^2=F_n^4+7F_{n+1}^4+25F_{n+2}^4+7F_{n+3}^4+F_{n+4}^4; \tag{2.3}$$

$$3\left(F_n^2+F_{n+1}^2+F_{n+2}^2+F_{n+3}^2+F_{n+4}^2+F_{n+5}^2\right)^2=32\left(F_{n+1}^4+4F_{n+2}^4+4F_{n+3}^4+F_{n+4}^4\right) \tag{2.4}$$

Before proceeding we shed a little more light on the nature of (2.1)-(2.4). In fact, the sequence $\left\{F_n^4\right\}$ is generated by a fifth order linear recurrence, namely

$$r_n=5r_{n-1}+15r_{n-2}-15r_{n-3}-5r_{n-4}+r_{n-5}. \tag{2.5}$$

Indeed, for fixed integers k and m, sequences such as $\left\{F_{n+k}^2F_{n+m}^2\right\}$ and $\left\{F_{n+k}^3F_{n+m}\right\}$ are also generated by this recurrence. Thus, by linearity, the sequences whose n^{th} terms are the left or right sides of (2.1)-(2.4) are generated by (2.5) as well. So to discover (2.2), for example, we simply put

$$\left(F_n^2+F_{n+1}^2+F_{n+2}^2+F_{n+3}^2\right)^2=aF_n^4+bF_{n+1}^4+cF_{n+2}^4+dF_{n+3}^4+eF_{n+4}^4,$$

and solve the set of five linear equations that arise after the substitution of $n=0,1,2,3$, and 4. In so doing we are simply choosing the constants a,b,c,d,e that yield the same initial values for the sequences whose n^{th} terms are defined by the left and right sides. If such an identity exists, the equations that arise are solvable. Of course the resulting identity would be memorable if the coefficients on the right displayed a pleasing symmetry.

By using a similar approach we have found the following:

$$10\left(F_n^2+F_{n+1}^2+F_{n+2}^2+F_{n+3}^2\right)^3=27\left(F_n^6+8F_{n+1}^6+8F_{n+2}^6+F_{n+3}^6\right); \tag{2.6}$$

$$5\left(F_n^2+F_{n+1}^2+F_{n+2}^2+F_{n+3}^2+F_{n+4}^2+F_{n+5}^2\right)^3=256\left(F_{n+1}^6+8F_{n+2}^6+8F_{n+3}^6+F_{n+4}^6\right); \tag{2.7}$$

$$10\left(F_n^2+F_{n+1}^2+\cdots+F_{n+6}^2+F_{n+7}^2\right)^3=9261\left(F_{n+2}^6+8F_{n+3}^6+8F_{n+4}^6+F_{n+5}^6\right). \tag{2.8}$$

In the sequel we require the following lemma which is contained in [8].

Lemma 1: Let $\{w_n\}$ be a sequence of complex numbers defined by

$$w_n=\sum_{i=1}^{k}c_i w_{n-i},$$

where $c_1,\ldots,c_k$ and $w_0,\ldots,w_{k-1}$ are given complex numbers with $c_k\neq 0$. Let $h\geq 1$ be an integer. Then $\left\{w_n^h\right\}$ is generated by a linear recurrence of order $\binom{h+k-1}{h}$.

Lemma 1 shows that $\left\{F_n^h\right\}$ is generated by a linear recurrence of order $h+1$, and so $\left\{F_n^6\right\}$ is generated by a linear recurrence of order 7. This gives an insight as to why identities (2.6)-(2.8) can be considered to be special. First, there are only four terms on the right instead of a possible seven terms, and second, the coefficients on the right have a pleasing symmetry. Notice also that there is an even number of squares on the left of (2.6)-(2.8). Similar observations can also be made about identities (2.2) and (2.4).

We are now ready to state our main results in two theorems. With hardly any extra effort we can formulate and prove our results for the more general sequence

$$H_n=H_{n-1}+H_{n-2},\quad H_0=a,\quad H_1=b, \tag{2.9}$$

which includes the Fibonacci and Lucas sequences as special cases (see [6]).

Theorem 1: Let k be a positive integer. Then

$$6\left(\sum_{i=0}^{2k-1}H_{n+i}^2\right)^2=F_{2k}^2\left(H_{n+k-2}^4+4H_{n+k-1}^4+4H_{n+k}^4+H_{n+k+1}^4\right); \tag{2.10}$$

$$10\left(\sum_{i=0}^{2k-1}H_{n+i}^2\right)^3=F_{2k}^3\left(H_{n+k-2}^6+8H_{n+k-1}^6+8H_{n+k}^6+H_{n+k+1}^6\right). \tag{2.11}$$

Theorem 1, which yields (2.2), (2.4), and (2.6)-(2.8), addresses the case in which there are an *even* number of squares inside the brackets on the left. Our investigations have uncovered results for higher powers (that is, where the sum of squares on the left is raised to higher powers), however they are not as succinct as the results in Theorem 1. Specifically, for powers

$2m, m \geq 2$, the number of $4m^{th}$ powers on the right is $4m+1$, which is the maximum number predicted by Lemma 1, and there is no symmetry in the coefficients. For powers $2m+1, m \geq 2$. the situation is slightly better. For this case we have found identities in which the number of $(4m+2)^{th}$ powers on the right is $4m+2$, one less than the maximum given by Lemma 1. Furthermore, the coefficients on the right possess symmetry, and can be expressed as polynomials in F_{k+1} and F_k. However, even for small m these identities become rather unwieldy, and for this reason we do not present them here.

A hint as to the nature of the identities discussed in the paragraph above can be found in our next theorem, which is analogous to Theorem 1. First, we need to define certain polynomials in F_{k+1} and F_k. For $0 \leq i \leq 4$, we define polynomials $P_i = P_i(F_{k+1}, F_k)$ by

$$\begin{cases} P_0 = P_4 = F_{k+1}^3 F_k - F_{k+1} F_k^3 \\ P_1 = P_3 = -F_{k+1}^3 F_k + 12 F_{k+1}^2 F_k^2 + F_{k+1} F_k^3 \\ P_2 = 6F_{k+1}^4 + 8F_{k+1}^3 F_k - 8F_{k+1} F_k^3 + 6F_k^4 \end{cases}$$

Similarly, for $0 \leq i \leq 6$, we define polynomials $Q_i = Q_i(F_{k+1}, F_k)$ by

$$\begin{cases} Q_0 = Q_6 = -F_{k+1}^5 F_k + 2F_{k+1}^4 F_k^2 - 4F_{k+1}^3 F_k^3 - 2F_{k+1}^2 F_k^4 - F_{k+1} F_k^5 \\ Q_1 = Q_5 = 24F_{k+1}^5 F_k + 12F_{k+1}^4 F_k^2 + 36F_{k+1}^3 F_k^3 - 12F_{k+1}^2 F_k^4 + 24F_{k+1} F_k^5 \\ Q_2 = Q_4 = 40F_{k+1}^5 F_k - 140F_{k+1}^4 F_k^2 + 1420F_{k+1}^3 F_k^3 + 140F_{k+1}^2 F_k^4 + 40F_{k+1} F_k^5 \\ Q_3 = 300F_{k+1}^6 + 930F_{k+1}^5 F_k + 600F_{k+1}^4 F_k^2 - 2040F_{k+1}^3 F_k^3 - 600F_{k+1}^2 F_k^4 \\ \qquad\qquad +930F_{k+1} F_k^5 - 300F_k^6 \end{cases}$$

Theorem 2: Let k be a positive integer. Then, with the polynomials P_i and Q_i as defined above, we have

$$6\left(\sum_{i=0}^{2k} H_{n+i}^2\right)^2 = \sum_{i=0}^{4} P_i H_{n+k+i-2}^4; \tag{2.12}$$

$$300\left(\sum_{i=0}^{2k} H_{n+i}^2\right)^3 = \sum_{i=0}^{6} Q_i H_{n+k+i-3}^6. \tag{2.13}$$

Here, in contrast to Theorem 1, there are an *odd* number of squares inside the brackets on the left. Furthermore, while the coefficients on the right are not as succinct as in Theorem 1, they are symmetric in the sense that $P_i = P_{4-i}$ and $Q_i = Q_{6-i}$. In addition, each of the P_i and Q_i are homogeneous polynomials in F_{k+1} and F_k of degree 4 and 6, respectively, and, ignoring sign, there is symmetry in their coefficients. We note that Candido's identity arises from (2.12) with $k = 1$.

Interestingly, unlike the situation in Theorem 1, the pattern of the identities in Theorem 2 seems to continue for higher powers. For the integers $4 \le m \le 10$, we have checked that there are identities of the form

$$C\left(\sum_{i=0}^{2k} H_{n+i}^2\right)^m = \sum_{i=0}^{2m} R_i^{(m)}(F_{k+1}, F_k) H_{n+k+i-m}^{2m}. \tag{2.14}$$

In (2.14), for $0 \le i \le 2m$, each polynomial $R_i^{(m)} = R_i^{(m)}(F_{k+1}, F_k)$ is homogeneous of degree $2m$ in F_{k+1} and F_k, and the types of symmetry discussed in the previous paragraph carry over. Also, as we can see if we compare the P_i and Q_i, the coefficients of the $R_i^{(m)}$ increase in size quite rapidly with m. We illustrate this by considering the case $m = 4$. For this case the value of C in (2.14) is 163800, and the coefficients of $R_i^{(4)}, 0 \le i \le 4$, are given below. Reading from the left, the first number is the coefficient of F_{k+1}^8, the second is the coefficient of $F_{k+1}^7 F_k$, and so on. Recall also that $R_i^{(4)} = R_{8-i}^{(4)}$.

$$\begin{cases} R_0^{(4)} : 0, -10, -3, -34, 198, 34, -3, 10, 0 \\ R_1^{(4)} : 0, 270, -36, 1542, -6984, -1542, -36, -270, 0 \\ R_2^{(4)} : 0, 10530, 14391, 22698, -144846, -22698, 14391, -10530, 0 \\ R_3^{(4)} : 0, -14040, 14508, -194376, 1944072, 194376, 14508, 14040, 0 \\ R_4^{(4)} : 163800, 755300, 971880, -1157260, -542880, 1157260, 971880, -755300, 163800 \end{cases}$$

In order to prove Theorems 1 and 2 we have found it convenient to make use of a powerful technique developed by Dresel [4]. Although Dresel's paper focuses mainly on the Fibonacci and Lucas sequences, his Verification Theorem carries over to more general sequences, and, in particular, to the sequence $\{H_n\}$. See Sections 3 and 8 in [4]. We prove only (2.13) since the proofs of our remaining results are similar. We require the following sum, which can be proved by mathematical induction.

$$\sum_{i=1}^{n} H_i^2 = H_n H_{n+1} - H_0 H_1. \tag{2.15}$$

With the use of (2.15), the left side of (2.13) becomes

$$300(H_{n+2k}H_{n+2k+1} - H_{n-1}H_n)^3 \tag{2.16}$$

and, upon expansion, this yields

$$300\left(H_{n+2k}^3 H_{n+2k+1}^3 - 3H_{n+2k}^2 H_{n+2k+1}^2 H_{n-1}H_n + 3H_{n+2k}H_{n+2k+1}H_{n-1}^2 H_n^2 - H_{n-1}^3 H_n^3\right). \tag{2.17}$$

Following Dresel, we see that (2.17) is homogeneous of degree 6 in the variable n. The right side of (2.13) is also homogeneous of degree 6 in the variable n.

We next consider the variable k. The right side of (2.13) is homogeneous of degree 12 in k. However, as it stands, (2.17) is not homogeneous of degree 12 in k. This can be remedied by inserting appropriate powers of $\alpha\beta = -1$, where α and β are the roots of $x^2 - x - 1 = 0$. To this end, working from left to right, we insert the coefficients $(\alpha\beta)^0, (\alpha\beta)^{2k}, (\alpha\beta)^{4k}$, and $(\alpha\beta)^{6k}$.

In summary, (2.13) is an equation that is homogeneous of degree 6 in n, and homogeneous of degree 12 in k. To complete the proof, it suffices to verify that (2.13) is true for seven distinct values of n, say $1 \le n \le 7$. To verify that (2.13) is true for $n = 1$, say, one simply substitutes $n = 1$ and verifies that the resulting equation is true for thirteen distinct values of k, for example $1 \le k \le 13$. This procedure is repeated for the other six values of n. We have managed to accomplish these verifications quite quickly with the help of the computer algebra system *Mathematica* 3.0, where the command "Fibonacci[n]" calls up the n^{th} Fibonacci number. We also used the fact that $H_n = aF_{n-1} + bF_n$ for all integers n(see (8) in [6]). In essence, then, we have proved the validity of (2.13) by verifying its validity for ninety-one distinct ordered pairs (n, k). Finally, we cannot overstate our reliance on *Mathematica* 3.0 during the discovery process, during which, as we hinted earlier, it was used to solve formidable systems of simultaneous equations.

3. AURIFEUILLE'S IDENTITY

An old and beautiful identity of Aurifeuille states

$$L_{5n} = L_n \left(L_{2n} + 5F_n + 3\right)\left(L_{2n} - 5F_n + 3\right), \quad n \text{ odd.} \tag{3.1}$$

Aurifeuille's identity gives certain factors of L_{5n} when n is odd, and, according to Maxey Brooke [1] it dates back to 1879. We have found a generalization of Aurifeuille's identity that holds for the sequences

$$\begin{cases} U_n = pU_{n-1} + U_{n-2}, & U_0 = 0, \quad U_1 = 1, \\ V_n = pV_{n-1} + V_{n-2}, & V_0 = 2, \quad V_1 = p, \end{cases} \tag{3.2}$$

provided the integer p is suitably restricted. The sequence $\{U_n\}$ and $\{V_n\}$ generalize the Fibonacci and Lucas sequences, respectively. Our result is contained in the following theorem.

Theorem 3: In the sequences (3.2) let $p = L_{2k+1}$, where k is any integer. Then

$$V_{5n} = V_n(V_{2n} + 5F_{2k+1}U_n + 3)(V_{2n} - 5F_{2k+1}U_n + 3), \quad n \text{ odd.} \tag{3.3}$$

Proof: We make use of the closed forms

$$U_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad V_n = \gamma^n + \delta^n \tag{3.4}$$

in which $\gamma = \frac{p+\sqrt{p^2+4}}{2}$ and $\delta = \frac{p-\sqrt{p^2+4}}{2}$. Then $(\gamma - \delta)^2 = p^2 + 4 = L_{2k+1}^2 + 4 = 5F_{2k+1}^2$, where the last equality follows from I_{12} on page 56 in [5]. Keeping in mind that n is assumed to be odd, and that $\gamma\delta = -1$, we have

$$\begin{aligned}
V_{5n} &= \gamma^{5n} + \delta^{5n} \\
&= (\gamma^n + \delta^n)\left(\gamma^{4n} - \gamma^{3n}\delta^n + \gamma^{2n}\delta^{2n} - \gamma^n\delta^{3n} + \delta^{4n}\right) \\
&= V_n\left(\gamma^{4n} + \delta^{4n} + 1 - (\gamma\delta)^n\left(\gamma^{2n} + \delta^{2n}\right)\right) \\
&= V_n\left(\gamma^{4n} + \delta^{4n} + 1 + \gamma^{2n} + \delta^{2n}\right) \\
&= V_n\left(\gamma^{4n} + \delta^{4n} + 2 + 6\left(\gamma^{2n} + \delta^{2n}\right) + 9 - 5\left(\gamma^{2n} + \delta^{2n} + 2\right)\right) \\
&= V_n\left(\left(\gamma^{2n} + \delta^{2n}\right)^2 + 6\left(\gamma^{2n} + \delta^{2n}\right) + 9 - 5\left(\gamma^n - \delta^n\right)^2\right) \\
&= V_n\left(V_{2n}^2 + 6V_{2n} + 9 - 5(\gamma - \delta)^2\left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right)^2\right) \\
&= V_n\left((V_{2n} + 3)^2 - (5F_{2k+1}U_n)^2\right), \text{ and Theorem 3 follows.} \quad \square
\end{aligned}$$

When $k = 0$ we see that $p = 1, U_n = F_n, V_n = L_n$, and (3.3) reduces to (3.1).

REFERENCES

[1] Maxey Brooke. "Fibonacci Formulas." *The Fibonacci Quarterly* **1.2** (1963): 60.

[2] Brother Alfred Brousseau. "Ye Olde Fibonacci Curiosity Shoppe." *The Fibonacci Quarterly* **10.4** (1972): 441-43.

[3] G. Candido. "A Relationship Between the Fourth Powers of the Terms of the Fibonacci Series." *Scripta Mathematica* **xvii No. 3-4** (1951): 230.

[4] L.A.G. Dresel. "Transformations of Fibonacci-Lucas Identities." *Applications of Fibonacci Numbers.* Volume 5, 169-84. Ed. G.E. Bergum, et. al. Dordrecht: Kluwer, 1993.

[5] V.E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Boston: Houghton Mifflin, 1969; rpt. The Fibonacci Association, 1979.

[6] A.F. Horadam. "A Generalized Fibonacci Sequence." *The American Mathematical Monthly* **68.5** (1961): 455-59.

[7] Stanley Rabinowitz. "Algorithmic Manipulation of Second-Order Linear Recurrences." *The Fibonacci Quarterly* **37.2** (1999): 162-77.

[8] A.G. Shannon. "Explicit Expressions for Powers of Linear Recursive Sequences." *The Fibonacci Quarterly* **12.3** (1974): 281-87.

AMS Classification Numbers: 11B39, 11B37

✠ ✠ ✠