ON THE E-ALGORITHM FOR POLYNOMIAL ROOTS AND LINEAR RECURRENCE RELATIONS

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1. INTRODUCTION

The approximation of solutions of the polynomial equations $P(x) = x^r - a_0x^{r-1} - \cdots - a_{r-2}x - a_{r-1} = 0$, where $a_0, a_1, \ldots, a_{r-1}$ ($r \geq 2$ and $a_{r-1} \neq 0$) are real or complex numbers, is still playing a central role in many branches of mathematics. To solve this problem several theoretical and numerical methods have been developed. The iterative Bernoulli process is one of the most usual methods for the approximation of the solutions of $P(x) = 0$ (see [7, 8, 9, 10, 11, 12, 13], for example). The main idea consists of associating with $P(x) = x^r - a_0x^{r-1} - \cdots - a_{r-2}x - a_{r-1}$ a sequence $\{V_n\}_{n\geq 0}$ defined by the following linear recurrence relation of order r

$$V_{n+1} = a_0V_n + a_1V_{n-1} + \cdots + a_{r-1}V_{n-r+1} \text{ for } n \geq r-1, \tag{1}$$

where $V_0, V_1, \ldots, V_{r-1}$ are specified by the initial conditions. Such sequences, called r-generalized Fibonacci sequences, are largely studied in the literature (see [7, 8, 13, 14, 15, 16, 18], for example). We shall refer to them in the sequel as *sequences (1).* If the sequence of ratios $\left\{\frac{V_{n+1}}{V_n}\right\}_{n\geq 0}$ converges, then $q = \lim_{n\to+\infty}\frac{V_{n+1}}{V_n}$ is a root of $P(x)$. Therefore, sequences (1) may be used in the approximation of roots of algebraic equations (see [12, 17], for example), like Newton's method or the secant method (see [10]).

For a convergent sequence $\{S_n\}_{n\geq 0}$ one of the most important problems consists of accelerating its convergence to $S = \lim_{n\to+\infty}S_n$. The extrapolation methods are a powerful tool for producing a new sequence $\{T_n\}_{n\geq 0}$ converging to the same limit S, faster than $\{S_n\}_{n\geq 0}$, namely $\lim_{n\to+\infty}(T_n - S)/(S_n - S) = 0$ (see [1, 3, 4, 5, 6], for example). The most well known of these methods are the Aitken's Δ^2 process, ε-algorithm and the E-algorithm (see [3, 4, 5, 6], for example). This later extrapolation method generalizes the ε-algorithm and represents the more powerful extrapolation method for accelerating the convergence (see [3, 5, 6]).

When $r = 2$ McCabe and Phillips have studied a theoretical application of Aitken acceleration for the convergence of the sequence of ratios $\{W_n = \frac{V_{n+1}}{V_n}\}_{n\geq 0}$, where $\{V_n\}_{n\geq 0}$ is a sequence (1) (see [17]). This gives an application of the Aitken acceleration to the Bernoulli method for solving $x^2 - a_0x - a_1 = 0$ (see [17]). The extension of McCabe-Phillips's idea to the general case of sequence (1) is studied in [2]. More precisely, the ε-algorithm method, which generalizes the Aitken acceleration, has been applied to accelerate the convergence of the sequence of ratios $\{W_n\}_{n\geq 0}$ associated with a sequence (1) (see [2]). The hypothesis that the dominant root λ_0 of $P(X)$ (namely $|\lambda| < |\lambda_0|$ for every other root λ) is simple plays an important role in [2].

In this paper we apply the general E-algorithm extrapolation method to accelerate the convergence of the sequence of ratios $\{W_n = \frac{V_{n+1}}{V_n}\}_{n\geq 0}$, associated with a sequence (1). Therefore, we extend and perform results of [2, 17]. In particular, it is not necessary to suppose that the dominant root of $P(X)$ is simple. Moreover, since the ε-algorithm is a subcase of the E-algorithm, we discuss the difference between these two extrapolation methods. Some examples allow us to show that the E-algorithm method is more powerful for accelerating the convergence of $\{W_n\}_{n\geq 0}$.

Note that there is a large literature on the extrapolation methods. In this paper, our basic reference is Brezinski's papers and monographs.

This paper is organized as follows. In Section 2 we give a connection between sequences (1) and the E-algorithm. In Section 3 we apply the E-algorithm to the sequence of the ratios $\{W_n = \frac{V_{n+1}}{V_n}\}_{n\geq 0}$. The last section is devoted to some results on the vectorial case.

2. E-ALGORITHM AND SEQUENCES (1).

2.1 The E-Algorithm:

The extrapolation E-algorithm method is an extension of the ε-algorithm (see [3, 4, 5, 6]). Indeed, it is a more general extrapolation algorithm. Its main idea consists of associating with each convergent sequence $\{Sn\}_{n\geq 0}$ a sequence $\{T_n\}_{n\geq 0}$, which converges to $S = \lim_{n\to+\infty} S_n$ faster than $\{S_n\}_{n\geq 0}$. Therefore, we have $\lim_{n\to+\infty}|\frac{T_n - S}{S_n - S} = 0$ (see [3, 4, 5, 6], for example). The kernel of the transformation $T : \{S_n\}_{n\geq 0} \longrightarrow T(\{S_n\}_{n\geq 0}) = \{T_n\}_{n\geq 0}$, namely $\mathcal{K}_T = \{\{S_n\}_{n\geq 0}; \exists N > 0, T_n = S \text{ for every } n \geq N\}$, plays a central role in the extrapolation methods (see [3, 4, 5, 6]).

Let $\{S_n\}_{n\geq 0}$ be a convergent sequence of real numbers, with $S = \lim_{n\to+\infty} S_n$, such that

$$S_n = S + \alpha_1 g_1(n) + \cdots + \alpha_k g_k(n),$$

where $\{g_s(n)\}_{n\geq 0}(1 \leq s \leq k)$ are some real sequences and $\alpha_1, \ldots, \alpha_k$ are real numbers. In summary, the E-algorithm associated with $\{S_n\}_{n\geq 0}$, consists in considering the following sequence $\{E_j(S_n)\}_{j\geq 0, n\geq 0}$, defined as follows,

$$E_k(S_n) = \begin{vmatrix} S_n & \ldots & S_{n+k} \\ g_1(n) & \ldots & g_1(n+k) \\ \vdots & \ldots & \vdots \\ g_k(n) & \ldots & g_k(n+k) \end{vmatrix} \quad \begin{vmatrix} 1 & \ldots & 1 \\ g_1(n) & \ldots & g_1(n+k) \\ \vdots & \ldots & \vdots \\ g_k(n) & \ldots & g_k(n+k) \end{vmatrix}^{-1}. \tag{2}$$

Let $\{E_j^{(n)}\}_{j\geq 0, n\geq 0}$ be the sequence defined by,

$$E_k^{(n)} = \frac{E_{k-1}^{(n)} g_{k-1,k}^{(n+1)} - E_{k-1}^{(n+1)} g_{k-1,k}^{(n)}}{g_{k-1,k}^{(n+1)} - g_{k-1,k}^{(n)}}, n \geq 0, k \geq 1, \tag{3}$$

where

$$E_0^{(n)} = S_n, g_{k,i}^{(n)} = \frac{g_{k-1,i}^{(n)} g_{k-1,k}^{(n+1)} - g_{k-1,i}^{(n+1)} g_{k-1,k}^{(n)}}{g_{k-1,k}^{(n+1)} - g_{k-1,k}^{(n)}}, \text{ for every } n \geq 0, i \geq k+1, \tag{4}$$

such that $g_{0,i}^{(n)} = g_i(n)$ for every $i \geq 1$ and $n \geq 0$. It was proved in Theorem 1 of [4] (see also [3]) that $E_k(S_n) = E_k^{(n)}$.

Recall that the ε-algorithm is a particular case of the E-algorithm, corresponding to $g_i(n) = \Delta^i S_n$ in (2) - (3) (see [3, 4, 6]). Therefore, the sequence $\{\varepsilon_k^{(n)}\}_{k\geq -1, n\geq 0}$ associated with the convergent sequence $\{S_n\}_{n\geq 0}$ satisfies $\varepsilon_{-1}^{(n)} = 0, \varepsilon_0^{(n)} = S_n$ and $\varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n)} + \frac{1}{\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)}}$, for $n \geq 0$ and for $k \geq 0$. The application of the ε-algorithm method considered with some constraints, for example the condition $\varepsilon_k^{(n)} \neq \varepsilon_k^{(n+1)}$ is assumed. On the other hand, the $\varepsilon_{2k}^{(n)}$ are the only interesting quantities and the $\varepsilon_{2k+1}^{(n)}$ are only used for intermediate computations (see [3, 4, 5, 6]).

2.2 E-Algorithm for sequences (1):

Let $\{V_n\}_{n\geq 0}$ be a sequence (1) and $\lambda_0, \ldots, \lambda_l$ the roots of $P(X) = X^r - a_0X^{r-1} - \cdots - a_{r-1}$, whose multiplicities are $m_0, \ldots, m_l$ (respectively). Suppose that $0 < |\lambda_l| < |\lambda_{l-1}| < \cdots < |\lambda_1| < |\lambda_0|$. If the limit $V = \lim_{n\to+\infty} V_n$ exists, for every choice of the initial conditions $V_0, \ldots, V_{r-1}$ and $V \neq 0$ for some $V_0, \ldots, V_{r-1}$, then $\lambda_0 = 1$ is a simple root of $P(X)$ and $|\lambda| < 1$ for every other root λ of $P(X)$. Therefore, we have

$$V_n = V + \sum_{k=1}^{l} P_k(n)\lambda_k^n,$$

where $P_k(n) = \sum_{j=0}^{m_k-1} \beta_{k,j} n^j$. Thus, $V_n = V + \sum_{k=1}^{l} g_k(n)$, where $g_k(n) = P_k(n)\lambda_k^n (1 \leq k \leq l)$. We verify easily that, $\lim_{n\to+\infty} \frac{g_k(n+1)}{g_k(n)} = \lambda_k \neq 1$. Since $\lambda_i \neq \lambda_j (1 \leq i \neq j \leq l)$, Theorem 2.8 of [3] shows that $\lim_{n\to+\infty} E_k^{(n)} = lim_{n\to+\infty} E_k(V_n) = V$, for every k.

Since $0 < |\lambda_l| < |\lambda_{l-1}| < \cdots < |\lambda_1| < \lambda_0 = 1$, we derive that $\lim_{n\to+\infty} \frac{g_{k+1}(n)}{g_k(n)} = 0$. Hence, Theorem 2.10 of [3] allows us to see that, for every $k \geq 1$, the sequence $\{E_k^{(n)}\}_{n\geq 0}$ converges to V faster than $\{V_n\}_{n\geq 0}$.

Suppose now that $\lambda_0 = 1, \lambda_1, \ldots, \lambda_{r-1}$ are the distinct roots of $P(X)$. Then, we have $V_n = V + \sum_{k=1}^{r-1} \beta_k g_k(n)$, where $g_k(n) = \lambda_k^n (1 \leq k \leq r-1)$. For every $n \geq 0$ and $k \geq j+1$, we set $g_{0,j}^{(n)} = g_j(n) (1 \leq j \leq r-1)$ and

$$g_{k,j}^{(n)} = \frac{g_{k-1,j}^{(n)} g_{k-1,k}^{(n+1)} - g_{k-1,j}^{(n+1)} g_{k-1,k}^{(n)}}{g_{k-1,k}^{(n+1)} - g_{k-1,k}^{(n)}}, \text{ for } j \geq k+1.$$

Therefore, a straightforward computation shows that $g_{k,j}^{(n)} = \alpha_{k,j}\lambda_j^n$ for every $j > k$, where $\alpha_{k,j} = \frac{\lambda_k - \lambda_j}{\lambda_k - 1}\alpha_{k-1,j}$, with $\alpha_{1,j} = \frac{\lambda_1 - \lambda_j}{\lambda_1 - 1}$. Application of Theorem 2.2 of [3] (see also Theorem 3 of [4]) allows us to derive that

$$E_k^{(n)} = V + \sum_{j=k+1}^{r-1} C_{k,j}\lambda_j^n,$$

where $C_{k,j} = \alpha_{k,j}\beta_j$. Hence, for every $k \geq 1$, the sequence $\{E_k^{(n)}\}_{n\geq 0}$ is also a sequence (1), which converges to V faster than $\{V_n\}_{n\geq 0}$. Moreover, in this case the characteristic polynomial of $\{E_k^{(n)}\}_{n\geq 0}$ is given by $Q_k(X) = (X-1)(X-\lambda_{k+1})\dots(X-\lambda_{r-1})$.

Summarizing, the preceding discussions give rise to the following proposition.

Proposition 2.1: *Let $\{V_n\}_{n\geq 0}$ be a sequence (1) such that $V = \lim_{n\to+\infty} V_n$ exists, for every choice of the initial conditions. Consider the sequence $\{E_k^{(n)}\}_{n\geq 0} (k \geq 0)$, where $E_k^{(n)} = E_k(V_n)$. Then, we have*

(i) $\lim_{n\to\infty} E_k^{(n)} = V$.

(ii) $\{E_k^{(n)}\}_{n\geq 0}$ *converges to V faster than $\{V_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ is in the kernel of the transformation $T = E$, namely $E_{r-1}^{(n)} = V$, for every $n \geq 0$.*

(iii) Suppose that $P(X)$ has distinct roots $\lambda_0 = 1, \lambda_1, \dots, \lambda_{r-1}$. Then, the sequence $\{E_k^{(n)}\}_{n\geq 0}$ is a sequence (1), which converges to V faster than $\{V_n\}_{n\geq 0}$.

Remark 2.1: *Why the E-algorithm for sequences* (1)*?* Let $\{V_n\}_{n\geq 0}$ be a sequence (1). Suppose that we have $\lambda \in \mathbb{C}$ and $\nu \in \mathbb{N}^*$ such that $\lim_{n\to+\infty} \frac{V_n}{n^{\nu-1}\lambda^n} = L_A$ exists and $L_A \neq 0$ for some choice of $A = (V_0, \dots, V_{r-1})$. It was shown in Theorem 3 of [8] that the preceding assertion is equivalent to the following: λ is a characteristic root of (1) of maximum modulus and multiplicity ν, and any other characteristic root μ of modulus $|\mu| = |\lambda|$ has multiplicity m_μ strictly less than ν. Therefore, the E-algorithm is very useful for accelerating the convergence of the sequence $\{\frac{V_n}{n^{\nu-1}\lambda^n}\}_{n\geq 1}$.

On the other hand, Theorem 7 of [8] implies that under one of the conditions of Theorem 3 of [8] we have $\lim_{n\to+\infty} W_n = \lambda$, where $W_n = \frac{V_{n+1}}{V_n}$. Therefore, the E-algorithm allows us again to accelerate the convergence of the sequence of ratios $\{W_n\}_{n\geq 0}$.

3. APPLICATION OF THE E-ALGORITHM TO $\lim_{n\to+\infty} \frac{V_{n+1}}{V_n}$.

Let $\{V_n\}_{n\geq 0}$ be a sequence (1) and $\lambda_1, \dots, \lambda_l$ the roots of $P(X) = X^r - a_0X^{r-1} - \dots - a_{r-1}$. The Binet formula shows that $V_n = \sum_{j=1}^{l} P_j(n)\lambda_j^n$, where m_j is the multiplicity of $\lambda_j (1 \leq j \leq l)$ and $P_j(n) = \sum_{i=0}^{m_j-1} \beta_{ij} n^i$. The β_{js} are derived from the initial conditions by solving the linear system of equations $\sum_{j=1}^{l} P_j(n)\lambda_j^n = V_n, n = 0, 1, \dots, r-1$ (see [14, 8, 13], for example). Suppose that $P_1(n)$ is not identically vanishing. Set $W_n = \frac{V_{n+1}}{V_n}$ and $S_n = \frac{P_1(n)}{P_1(n+1)} W_n$. Then, we have $\lim_{n\to+\infty} W_n = \lim_{n\to+\infty} S_n$. The Binet formula implies that

$$S_n = \lambda_1 + g_2(n) + \dots + g_l(n), \tag{5}$$

where

$$g_{j+1}(n) = \left(\frac{P_{j+1}(n+1)}{P_1(n+1)}\lambda_{j+1} - \frac{P_{j+1}(n)}{P_1(n)} S_n\right)\left(\frac{\lambda_{j+1}}{\lambda_1}\right)^n, \text{ for } j \geq 1. \tag{6}$$

The application of the E-algorithm to $\{S_n\}_{n\geq 0}$ allow us to accelerate the convergence of $\{W_n\}_{n\geq 0}$ to λ_1. More precisely, we have the main result of this Section.

Theorem 3.1: *Let $\{V_n\}_{n\geq 0}$ be a sequence (1) and $\lambda_1, \ldots, \lambda_1$ its characteristic roots, such that $|\lambda_l| < \cdots < |\lambda_1|$. Let $\{S_n\}_{n\geq 0}$ be defined by (5) - (6). Then, we have the following assertions.*
(i) $\lim_{n\to+\infty} E_k(S_n) = \lambda_1$.
(ii) $\{E_k(S_n)\}_{n\geq 0}$ *converges to* λ_1 *faster than* $\{W_n\}_{n\geq 0}$.
(iii) $E_{l-1}(S_n) = \lambda_1$ *for every* $n \geq 0$.

Proof: (i) From (5) - (6) we derive that $\lim_{n\to+\infty} \frac{g_{j+1}(n+1)}{g_j(n)} = \frac{\lambda_{j+1}}{\lambda_j} \neq 1$, for every $j(2 \leq j \leq l)$. Since $\lambda_i \neq \lambda_j (i \neq j)$, we show that $\lim_{n\to+\infty} \frac{g_{i+1}(n+1)}{g_i(n)} \neq \lim_{n\to+\infty} \frac{g_{j+1}(n+1)}{g_j(n)}$. Therefore, Theorem 2.8 of [3] implies that $\lim_{n\to+\infty} E_k(S_n) = \lambda_1$, for every $k \geq 0$.

(ii) Since $|\lambda_l| < \cdots < |\lambda_1|$ a straightforward computation implies that $\lim_{n\to+\infty} \frac{g_{i+1}(n)}{g_i(n)} = 0$. Therefore, Theorem 2.10 of [3] shows that $\{E_k(S_n)\}_{n\geq 0}$ converges to λ_1 faster than $\{E_{k-1}(S_n)\}_{n\geq 0}$, for every $k \geq 1$. Particularly, $\{E_k(S_n)\}_{n\geq 0}$ converges to λ_1 faster than $\{W_n\}_{n\geq 0}$.

(iii) Expression (4) and Theorem 2.1 of [3] imply that $E_{l-1}(S_n) = \lambda_1$, for every $n \geq 0$. More generally, we have $E_k(S_n) = \lambda_1$ for every $k \geq l-1$ and $n \geq 0$. □

Suppose that λ_1 is a simple root of $P(X)$. It was established that the application of the Aitken acceleration to the sequence of ratios $\{W_n\}_{n\geq 0}$ associated with the sequence (1), implies that $\{\epsilon_2^{(n)}\}_{n\geq 0}$ converges to λ_1 faster than $\{W_n\}_{n\geq 0}$ (see Proposition 3.1 of [2]). The application of the ϵ-algorithm to $\{W_n\}_{n\geq 0}$ shows also that $\{\epsilon_{2k}^{(n)}\}_{n\geq 0}$ converges to λ_1 faster than $\{W_{n+r-1}\}_{n\geq 0}$, for every $k \geq 1$ (see Proposition 3.3 of [2]). For the E-algorithm, we have the following corollary of Theorem 3.1.

Corollary 3.2: *Let $\{V_n\}_{n\geq 0}$ be a sequence (1) and $\lambda_l, \ldots, \lambda_1$ its characteristic root such that $|\lambda_l| < \cdots < |\lambda_1|$. Suppose that λ_1 is a simple root. Then, for every $k \geq 1$, the sequence $\{E_k^{(n)}\}_{n\geq 0}$ converges to λ_1 faster than $\{W_n\}_{n\geq 0}$ and we have $E_{l-1}^{(n)} = E_{l-1}(W_n) = \lambda_1$, for every $n \geq 0$.*

Remark 3.1: *The advantage of the E-algorithm for* $\lim_{n\to+\infty} \frac{V_{n+1}}{V_n}$. Let $\{V_n\}_{n\geq 0}$ be a sequence (1), whose characteristic roots satisfy $|\lambda_l| < \cdots < |\lambda_1|$ and λ_1 is simple. Proposition 2.1 of [2] shows that there exists $N > 0$ such that $\epsilon_{2k}^{(n)} = \lambda_1$, for every $n \geq N$ and some $k \geq 1$, if and only if $\{W_{n+N} - \lambda_1\}_{n\geq 0}$ is a sequence (1). However, Corollary 3.2 shows that by the E-algorithm we have $E_k(W_n) = \lambda_1$, for every $n \geq 0$ and $k \geq l-1$. More generally, for the E-algorithm the root λ_1 is not necessarily simple.

Remark 3.2: Let $\{V_n\}_{n\geq 0}$, be a sequence (1) defined by $V_{n+1} = aV_n - bV_{n-1}$, where $V_0 = 1$ and $V_1 = a$. In [17] the Aitken acceleration has been applied to the sequence of ratios $\{W_n\}_{n\geq 0}$, with the aim to approximate the larger root of the polynomial equation $x^2 - ax + b = 0$. Consider now the ϵ-algorithm for $\{W_n\}_{n\geq 0}$. A straightforward computation allows us to obtain that $\epsilon_{-1}^{(n)} = 0$, $\epsilon_0^{(n)} = W_n$, $\epsilon_1^{(n)} = \frac{1}{W_{n+1} - W_n}$, $\epsilon_2^{(n)} = W_{2(n+1)}$, $\epsilon_3^{(n)} = \frac{1}{W_{n+2} - W_{n+1}} + \frac{1}{W_{2(n+2)} - W_{2(n+1)}}$, $\epsilon_4^{(n)} = \frac{W_{2(n+2)}^2 - W_{n+2}W_{4(n+2)}}{W_{4(n+2)} - W_{n+2}}$. The application of the E-algorithm to $\{W_n\}_{n\geq 0}$, shows that $E_0^{(n)} = W_n$ and $E_1^{(n)} = \lim_{n\to+\infty} W_n$ according to the initial conditions $V_0 = 1, V_1 = a$. This example shows that he E-algorithm is a more powerful tool in the acceleration of convergence of $\{W_n\}_{n\geq 0}$ to the solution of the equation $x^2 - ax + b = 0$.

Remark 3.3: The preceding results of Section 3 can be applied to the sequence of ratios $\{W_n = V_A(n)/V_B(n)\}_{n\geq 0}$, where $\{V_A(n)\}_{n\geq 0}$ and $\{V_B(n)\}_{n\geq 0}$ are two sequences (1), whose initial conditions are $A = (\alpha_0, \ldots, \alpha_{r-1}), B = (\beta_0, \ldots, \beta_{r-1})$ respectively. More precisely, suppose that we have $\lambda \in \mathbb{C}$ and $\nu \in IN^*$ such that $\lim_{n\to+\infty} \frac{V_A(n)}{n^{\nu-1}\lambda^n} = L_A$, $\lim_{n\to+\infty} \frac{V_B(n)}{n^{\nu-1}\lambda^n} = L_B$ exist and $L_B \neq 0$. Theorem 6 of [8] shows that $S = \lim_{n\to+\infty} V_A(n)/V_B(n) = L_A/L_B$. For reason of simplicity, suppose that $\nu = 1$ (or equivalently λ is simple) and $0 < |\lambda_l| < \cdots < |\lambda_1| < |\lambda|$. The Binet formulas $V_A(n) = L_A + \sum_{k=1}^{l} P_k(n)\lambda_k^n, V_B(n) = L_B + \sum_{k=1}^{l} Q_k(n)\lambda_k^n$ show that

$$W_n = V_A(n)/V_B(n) = S + \sum_{j=1}^{l} g_k(n),$$

where $g_k(n) = \frac{P_k(n) - S_n Q_k(n)}{L_B}(\lambda_k/\lambda)^n (1 \leq k \leq l)$. A straightforward computation shows that $\lim_{n\to+\infty} g_k(n+1)/g_k(n) = \lambda_k$. Suppose that there exists $A = (\alpha_0, \ldots, \alpha_{r-1}), B = (\beta_0, \ldots, \beta_{r-1})$ such that $L_A/L_B \neq 0$ and $P_k(n) \neq SQ_k(n)(1 \leq k \leq l)$. Then, we have $\lim_{n\to+\infty} g_{k+1}(n)/g_k(n) = 0$. Since $\lambda_k \neq \lambda$ and $\lambda_k \neq \lambda_j (j \neq k)$, we have $\lim_{n\to+\infty} E_k(V_A(n)/V_B(n)) = L_A/L_B$ and $E_k(V_A(n)/V_B(n)) = E_k^{(n)} = L_A/L_B$. Therefore, the sequence $\{E_k^{(n)}\}_{n\geq 0}$ $(1 \leq k \leq l)$ converges to L_A/L_B faster than $\{V_A(n)/V_B(n)\}_{n\geq 0}$.

4. THE VECTORIAL CASE.

In [3, 4] the E-algorithm is applied to the case of convergent sequences in $\mathbb{R}^p$ (or $\mathbb{C}^p$), with $p \geq 2$. We extend here some results of the preceding Sections 2 and 3 to the vectorial sequences (1). More precisely, let $\{V_n\}_{n\geq 0}$ be a sequence of $\mathbb{R}^p$ (or $\mathbb{C}^p$) such that $V_{n+1} = a_0 V_n + \cdots + a_{r-1} V_{n-r+1}$ for $n \geq r-1$, where $V_0, \ldots, V_{r-1}$ are the initial values. Therefore, we have a family $\{V_{j,n}\}_{n\geq 0}(0 \leq j \leq p)$ of scalar convergent sequences (1), and the procedure of Section 2 allows us to have $V_n = V + \sum_{j=1}^{l} g_k(n)$, where $g_k(n) \in \mathbb{R}^p$ (or $\mathbb{C}^p$). Hence, Theorems 9 and 10 of [4] show that we have $\lim_{n\to+\infty} E_k(V_n) = V$, for every $k \geq 0$ and $E_k(V_n) = V + g_{k,k+1}^{(n)} + \cdots + g_{k,l}^{(n)}$, for every k, n, where the $g_{k,j}^{(n)}$ are given in the same way as in [4] (see p. 184 of [4]).

Moreover, suppose that $0 < |\lambda_l| < \cdots < |\lambda_1| < \lambda_0 = 1$ are the distinct roots of $P(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$. The Binet formula implies that $g_k(n) = (g_k^{(1)}(n), \ldots, g_k^{(p)}(n))^T = \lambda_k^n (P_k^{(1)}(n), \ldots, P_k^{(p)}(n))^T$. Therefore, we have

$$\lim_{n\to+\infty} \frac{< Y, g_k(n+1) >}{< Y, g_k(n) >} = \lambda_k,$$

for every nonvanishing $Y = (y_1, \ldots, y_p)^T$. Let j be the first integer such that g_j is not identically 0, then we have $\lim_{n\to+\infty} \frac{<Y, V_{n+1} - V>}{<Y, V_n - V>} = \lambda_j$. Since $\lambda_k \neq \lambda_i$ for $k \neq i$, Theorem 4.6 of [3] allows us to have the following proposition,

Proposition 1: *Let $\{V_n\}_{n\geq 0}$ be a sequence (1) such that $V = \lim_{n\to+\infty} V_n$ exists and $\lambda_l, \ldots, \lambda_1, \lambda_0$ its characteristic root such that $|\lambda_l| < \cdots < |\lambda_1| < \lambda_0 = 1$. Then, we have*

$$\lim_{n\to+\infty} \frac{< Y, E_k^{(n)} - V >}{< Y, V_n - V >} = 0,$$

for every $k \geq 0$.

On the other hand, for every $i, j(1 \leq i \leq l, 1 \leq j \leq p)$, we have $\lim_{n\to+\infty} g_i^{(j)}(n) = 0$. Consider j_i such that $degP_i^j \leq degP_i^{j_i}$ for every $j(1 \leq j \leq p)$. Then, there exists $C > 0$ and $N \in IN$ such that $\left|\frac{g_i^{(j)}(n)}{g_i^{(j_i)}(n)}\right| < C$, for every $n \geq N$. Finally, we have $\lim_{n\to+\infty} \frac{g_i^{(j_i)}(n+1)}{g_i^{(j_i)}(n)} = \lambda_i$ and $\lim_{n\to+\infty} \frac{g_{i+1}^{(j_{i+1})}(n)}{g_i^{(j_i)}(n)} = 0$, since $|\lambda_{i+1}| < |\lambda_i|$. The preceding discussion allows us to derive from Theorem 4.7 of [3] the following proposition.

Proposition 2: *Let $\{V_n\}_{n\geq 0}$ be a vectorial sequence (1). Suppose that there exists $V_0, \ldots, V_{r-1}$ such that, for every i, we have $\lim_{n\to+\infty} \frac{<Y, g_i(n)>}{g_i^{j_i}(n)} = \alpha_i \neq 0$, for some fixed $Y = (y_1, \ldots, y_p)^T$. Then*

$$\lim_{n\to+\infty} \frac{\| E_i^{(n)} - V \|}{\| E_{i-1}^{(n)} - V \|} = 0,$$

for every $i(1 \leq i \leq l)$.

The choice of the initial values $V_0, \ldots, V_{r-1}$ and $Y = (y_1, \ldots, y_p)^T$ satisfying the hypotheses of Proposition 2 is always possible. For example, if $V_j = u_j(1, \ldots, 1)^T$ with $u_j \in \mathbb{R}$ (or $\mathbb{C}$) and $Y = (1, \ldots, 1)^T$, we have $\lim_{n\to+\infty} \frac{<Y, g_i(n)>}{g_i^{j_i}(n)} = p \neq 0$.

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