# ADVANCED PROBLEMS AND SOLUTIONS

*Edited by*
**Florian Luca**

*Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.*

## PROBLEMS PROPOSED IN THIS ISSUE

**H-612 Proposed by Mario Catalani, University of Torino, Italy**

Let $a_r$ be the sequence $a_r = a_{r-1} + 2r$ for $r \geq 1$, with $a_0 = 0$. Let $\boldsymbol{A}_n$ be the matrix elements $a_{ij} = \min(i,j)$, $1 \leq i,j \leq n$, and let $\boldsymbol{I}$ be the identity matrix. Find

$$b_n = |\boldsymbol{A}_n + a_r\boldsymbol{I}|$$

as a function of $r$ and $n$, where $|\cdot|$ is the determinant operator.

**H-613 Proposed by Jayantibhai M. Patel, Ahmedabad, India**

For any positive integer $n$, prove that

$$\begin{vmatrix} F_n^2 & -F_nF_{n+3} & F_{n+3}^2 & F_nF_{n+3} \\ -F_nF_{n+3} & F_{n+3}^2 & F_nF_{n+3} & F_n^2 \\ F_{n+3}^2 & F_nF_{n+3} & F_n^2 & -F_nF_{n+3} \\ F_nF_{n+3} & F_n^2 & -F_nF_{n+3} & F_{n+3}^2 \end{vmatrix} = -(2F_{2n+3})^4.$$

**H-614 Proposed by R.S. Melham, Sydney, Australia**

Prove the identity

$$F_{a_2-a_3}F_{a_2-a_4}F_{a_3-a_4}F_{n+a_1}^4 + (-1)^{a_1+a_2+1}F_{a_1-a_3}F_{a_1-a_4}F_{a_3-a_4}F_{n+a_2}^4$$

$$+(-1)^{a_1+a_2}F_{a_1-a_2}F_{a_1-a_4}F_{a_2-a_4}F_{n+a_3}^4 + (-1)^{a_1+a_2+a_3+a_4+1}F_{a_1-a_2}F_{a_1-a_3}F_{a_2-a_3}F_{n+a_4}^4$$
$$= F_{a_1-a_2}F_{a_1-a_3}F_{a_1-a_4}F_{a_2-a_3}F_{a_2-a_4}F_{a_3-a_4}F_{4n+a_1+a_2+a_3+a_4}.$$

## SOLUTIONS

### Fibonacci meets Catalan

**H-599 Proposed by the Editor**
**(Vol. 41, no. 4, August 2003)**

For every $n \geq 0$ let $C_n := \frac{1}{n+1}\binom{2n}{n}$ be the $n$th Catalan number. Show that all the solutions of the diophantine equation $F_m = C_n$ have $m \leq 5$.

**Solution by the Editor**

The inequality

$$\binom{2n}{n} \geq \frac{2^{2n}}{n+1} \tag{1}$$

can be immediately shown to hold by induction on $n$. Indeed, (1) is an equality at $n = 0,\ 1$ while assuming that (1) holds for $n$ then

$$\binom{2(n+1)}{(n+1)} = \frac{(2n+1)(2n+2)}{(n+1)^2} \cdot \binom{2n}{n} \geq \frac{2(2n+1)}{n+1} \cdot \frac{2^{2n}}{n+1},$$

and it suffices to check that

$$\frac{2(2n+1)}{n+1} \cdot \frac{2^{2n}}{n+1} > \frac{2^{2n+2}}{n+2},$$

which is equivalent to

$$(2n+1)(n+2) > 2(n+1)^2,$$

which in turn is equivalent to

$$2n^2 + 5n + 2 > 2n^2 + 4n + 2,$$

which obviously holds. So, (1) holds for all $n \geq 0$. In particular, with $F_m = C_n$, we get

$$F_m = C_n \geq \frac{1}{n+1} \cdot \frac{2^{2n}}{n+1} = \frac{2^{2n}}{(n+1)^2}. \tag{2}$$

Let $\alpha = (1+\sqrt{5})/2$. The inequality $F_m < \alpha^m$ holds for all $m \geq 0$ and it can be checked by induction on $m$, while the inequality

$$\left(\frac{2}{\alpha}\right)^n > \alpha(n+1) \tag{3}$$

holds for all $n \geq 16$. And so, assuming that $n \geq 16$, inequality (3) implies that

$$\frac{2^{2n}}{(n+1)^2} > \alpha^{2n+2},$$

therefore

$$\alpha^m > F_m = C_n > \frac{2^{2n}}{(n+1)^2} > \alpha^{2n+2},$$

which implies that $m > 2n+2$. By the *Primitive Divisor Theorem* (see, for example, [1]), we know that for any $k > 12$, $F_k$ is divisible by a prime number $p$ with $p \equiv \pm 1 \pmod{k}$. In particular, $p \geq k-1$. Thus, since $m \geq 2n+2$ and $n \geq 16$, it follows that $F_m$ is divisible by a prime number $p \geq m-1 \geq 2n+1$. Of course, such a prime can not divide $C_n$ because $C_n$ is a divisor of $(2n)!$. This contradiction shows that $n \leq 15$. Listing all the Catalan numbers $C_n$ up to $n = 15$, we get that the largest value of $n$ for which $C_n = F_m$ for some $m$ is $C_3 = F_5 = 5$.

1. Minoru Yabuta, "A Simple Proof of Carmichael's Theorem on Primitive Divisors", The Fibonacci Quarterly **39.5** (2001): 439–443.

## The One-Third Squares in the Pseudo Fibonacci Sequence

**H-600 Proposed by Arulappah Eswarathasan, Hofstra University, Hempstead, NY**
**(Vol. 41, no. 4, August 2003)**

The Pseudo-Fibonacci numbers $u_n$ are defined by $u_1 = 1$, $u_2 = 4$ and $u_{n+2} = u_{n+1} + u_n$. A number of the form $3s^2$, where $s$ is an integer, is called a one-third square. Show that $u_0 = 3$ and $u_{-4} = 12$ are the only one-third squares in the sequence.

**Solution by the Proposer**

Assume that $u_n = 3x^2$. The proof is achieved in three stages.

**(a)** Assume that $n \equiv 1, 4, 6, -3, -2 \pmod{14}$, $n \equiv 2, 5, 10 \pmod{28}$ and $n \equiv -9, 19 \pmod{42}$. In this case, using congruence (11) of [1], we obtain $u_n \equiv u_1, u_4, u_6, u_{-3}, u_{-2} \pmod{L_7}$, $u_n \equiv \pm u_2, \pm u_5, \pm u_{10} \pmod{L_{14}}$, and $u_n \equiv u_{-9}, u_{19} \pmod{L_{21}}$, respectively, so that $u_n \equiv 30, 9, -6, 51, -24 \pmod{29}$, $u_n \equiv \pm 285, \mp 267, \pm 438 \pmod{281}$ and $u_n \equiv 291, 117 \pmod{211}$. In all these cases, the equation becomes $x^2 \equiv 10, 3, -2, 17, -8 \pmod{29}$, $x^2 \equiv \pm 95, \mp 89, \pm 146 \pmod{281}$, and $x^2 \equiv 97, 39 \pmod{211}$, all of which are impossible.

**(b)** Assume that $n \equiv -1, 3, 7, 8, 9 \pmod{14}$, $n \equiv 7, 11 \pmod{16}$, $n \equiv 14 \pmod{28}$, and $n \equiv -1, -13, 3 \pmod{48}$. In this case, using congruence (12) of [1], we find that $u_n \equiv \pm u_{-1}, \pm u_3, \pm u_7, \pm u_8, \pm u_9 \pmod{F_7}$, $u_n \equiv u_7, u_{11} \pmod{F_8}$, $u_n \equiv u_{14} \pmod{F_{14}}$ and $u_n \equiv u_{-1}, u_{-13}, u_3 \pmod{F_{24}}$, respectively, so that $u_n \equiv \pm 24, \pm 18, \pm 24, \pm 60, \pm 6 \pmod{13}$, $u_n \equiv 9, 240 \pmod{7}$, $u_n \equiv 1050 \pmod{13}$, and $u_n \equiv 21, -852, -18 \pmod{23}$. In all these cases, the equation becomes $x^2 \equiv \pm 8, \pm 6, \pm 8, \pm 20, \pm 2 \pmod{13}$, $x^2 \equiv 3, 80 \pmod{7}$, $x^2 \equiv 350 \pmod{13}$, and $x^2 \equiv 7, -284, -6 \pmod{23}$, all of which are impossible.

**(c)** We finally show that the given equation is impossible if $n = -4 + 2^t r$ or $n = 2^t r$, where $r$ is odd and $t \geq 3$ is a positive integer. By (11) of [1], in these cases we have $u_n \equiv -u_{-4}$

(mod $L_{2^{t-1}}$) and $u_n \equiv -u_0 \pmod{L_{2^{t-1}}}$. Hence, $u_n \equiv -12, -3 \pmod{L_{2^{t-1}}}$, which leads to $x^2 \equiv -4, -1 \pmod{L_{2^{t-1}}}$, which is impossible because $L_{2^{t-1}} \equiv 3 \pmod 4$. The only cases which are left are $n = -4, 0$ for which $u_n = 12, 3$, which are one-third squares.

1. A. Eswarathasan, "On Square Pseudo-Fibonacci Numbers", The Fibonacci Quarterly **16.4** (1978): 310–314.

**Solution by the Editor**

It is not hard to prove that $u_n = (7F_{n-1} + L_{n-1})/2$ holds for all integers $n$. Putting $v_n = (7L_{n-1} + 5F_{n-1})/2$, the formula

$$v_n^2 - 5u_n^2 = (-1)^{n-1} \cdot 44$$

is an immediate consequence of the known formula $L_n^2 - 5F_n^2 = (-1)^n \cdot 4$. When $3|u_n$, we get that $3|(7F_{n-1} + L_{n-1})$, and this shows that $n \equiv 0, 4 \pmod 8$. In particular, $n - 1$ is odd. Thus, with $u_n = 3x^2$ and $v_n = y$, we get the diophantine equation $y^2 = 45x^4 - 44$. This reduces to an elliptic curve and its integer solutions $(x, y) = (\pm 1, \pm 1)$, $(\pm 2, \pm 26)$ can be easily computed with one of the standard packages like magma, PARI, SIMATH, etc.

## A Decreasing Sequence

**H-601 Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria (Vol. 41, no. 4, August 2003)**

Prove or disprove that the sequence

$$\left\{\frac{\sqrt[n]{L_2 \cdot \ldots \cdot L_{n+1}}}{\alpha^{(n+3)/2}}\right\}_{n \geq 1}$$

strictly decreases to its limit 1. Here, $\alpha$ is the golden section.

**Solution by V. Mathe, Marseille, France**

Let

$$u_n = \frac{\sqrt[n]{L_2 \cdot \ldots \cdot L_{n+1}}}{\alpha^{(n+3)/2}}.$$

We have

$$\log u_n = \frac{\log L_2 + \cdots + \log L_{n+1}}{n} - \frac{n+3}{2} \log \alpha.$$

Here, for a positive real number $y$ we use $\log y$ for the natural logarithm of $y$. Since $L_k = \alpha^k + \beta^k = \alpha^k(1 + (-1/\alpha^2)^k)$, where $\beta = (1 - \sqrt{5})/2$ is the conjugate of $\alpha$, one gets

$$\log u_n = \frac{1}{n} \sum_{k=2}^{n+1} \log\Big(1 + \Big(-\frac{1}{\alpha^2}\Big)^k\Big). \tag{1}$$

Since

$$\left|\log\Big(1+\Big(-\frac{1}{\alpha^2}\Big)^k\Big)\right| < \frac{1}{\alpha^{2k}},$$

it follows that the series appearing in the right hand side of equation (1) converges absolutely as $n \to \infty$. Therefore $\log u_n$ tends to zero, and so $u_n$ tends to 1 as $n \to \infty$. We will now show that the sequence is strictly decreasing. For that purpose, we compute

$$\log u_n - \log u_{n+1} = \frac{\log L_2 + \cdots + \log L_{n+1}}{n(n+1)} - \frac{\log L_{n+2}}{n+1} + \frac{\log \alpha}{2}$$

whose sign is the same as the sign of

$$A_n = \log L_2 + \cdots + \log L_{n+1} - n \log L_{n+2} + \frac{n(n+1)}{2} \log \alpha$$

$$= \sum_{k=2}^{n+1} \log\Big(1+\Big(-\frac{1}{\alpha^2}\Big)^k\Big) - n\log\Big(1+\Big(-\frac{1}{\alpha^2}\Big)^{n+2}\Big).$$

We note that, for $n \geq 1$,

$$\sum_{k=2}^{n+1} \log\Big(1+\Big(-\frac{1}{\alpha^2}\Big)^k\Big) \geq \log\Big(1+\Big(-\frac{1}{\alpha^2}\Big)^2\Big) + \log\Big(1+\Big(-\frac{1}{\alpha^2}\Big)^3\Big),$$

where the last inequality above follows from the inequality

$$\log\Big(1+\Big(-\frac{1}{\alpha^2}\Big)^{2k}\Big) + \log\Big(1+\Big(-\frac{1}{\alpha^2}\Big)^{2k+1}\Big) > 0 \qquad \text{for } k = 1, 2, \ldots,$$

whose proof is straightforward, together with the inequality $\log(1+(-1/\alpha^2)^{2k}) > 0$, which is obvious. Therefore, we get

$$A_n \geq \log\Big(1+\Big(\frac{1}{\alpha^4}\Big) + \log\Big(1-\frac{1}{\alpha^6}\Big) - n\log\Big(1+\Big(-\frac{1}{\alpha^2}\Big)^{n+2}\Big).$$

If $n$ is odd, we have $\log(1+(-1/\alpha^2)^{n+2}) < 0$, and therefore

$$A_n > \log\Big(1+\frac{1}{\alpha^4}\Big) + \log\Big(1-\frac{1}{\alpha^6}\Big) > 0.$$

Assume now that $n$ is even, and let $B = \log(1 + 1/\alpha^4) + \log(1 - 1/\alpha^6)$. Using the inequality $\log(1 + x) < x$, which holds for all for $x > 0$, we get

$$A_n > B - \frac{n}{\alpha^{2(n+2)}}.$$

So, a sufficient condition for $A_n$ to be positive is

$$B - \frac{n}{\alpha^{2(n+2)}} \geq 0,$$

which is equivalent to

$$\frac{\alpha^{2n}}{n} \geq \frac{1}{B\alpha^4}$$

Since $1/B\alpha^4 < 1.86$, it suffices that $\alpha^{2n} \geq 1.86n$, and this last inequality holds for all $n \geq 2$.

Thus, the sequence $\{\log u_n\}_{n\geq 1}$ is strictly decreasing to its limit 0.

**Also solved by Paul Bruckman.**

**Please Send in Proposals!**