VARIETIES OF FIBONACCI TYPE

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1. INTRODUCTION

We consider the following family of cyclic presentations of groups depending on four positive integers:

$$F(r,n,s,p) =< x_1,\dots,x_n : \quad x_i x_{i+p} \cdots x_{i+p(r-1)} = x_{i+s} \quad (i=1,\dots,n) >$$

where the subscripts are reduced modulo n, and $r \geq 2$, $n \geq 2$. We also denote in short the above presentation by $F(r,n,s,p) = G_n(x_1 x_{1+p} \cdots x_{1+p(r-1)} x_{1+s}^{-1})$ according to notation in [13] and [16]. This family contains many classes of cyclic presentations of groups, previously considered by several authors. The groups $F(2,n,2,1)$ are the *Fibonacci groups* $F(2,n) = G_n(x_1x_2x_3^{-1})$ introduced by Conway in [9] (see also [10]). The groups $F(r,n,r,1) = G_n(x_1x_2\cdots x_r x_{1+r}^{-1})$ were introduced by Johnson, Wamsley and Wright in [17] (and denoted by $F(r,n)$, $r \geq 2$, $n \geq 3$) as a natural generalization of the Fibonacci groups $F(2,n)$ (and in fact they are called with the same name in the current literature). The groups $F(2,n,1,2) = G_n(x_1x_3x_2^{-1})$ are the *Sieradski groups* introduced in [22] (and denoted by $S(n)$) (see also [5] for some generalizations of them). The groups $F(r,n,r+k-1,1) = G_n(x_1x_2\cdots x_r x_{r+k}^{-1})$ were defined and algebraically studied by Campbell and Robertson in [1] (and denoted by $F(r,n,k)$) for any $r \geq 2$, $n \geq 3$, and $k \geq 1$. Obviously, they represent further generalizations of the Fibonacci groups $F(r,n)$. The groups $F(2,n,1,p) = G_n(x_1x_{1+p}x_2^{-1})$ are the Gilbert–Howie groups defined in [11], and denoted by $H(n,p)$. A natural generalization of them, that is, $F(2,n,s,p) = G_n(x_1x_{1+p}x_{1+s}^{-1})$, was considered in [6], and denoted by $G_n(p,s)$. Obviously, we have $G_n(2,1) = S(n)$ and $G_n(1,2) = F(2,n)$. In this paper we study algebraic systems, or briefly *algebras*, which are groups with an additional unary operation satisfying certain laws. These laws are directly suggested by the relators of the cyclically presented groups $F(r,n,s,p)$. More precisely, in addition to the group laws (we use notation $x \cdot y$, x^{-1}, and e for the product, inverse, and unit element, respectively) there is an unary operation ϕ (written as a right–hand operation) which satisfies the following properties:

$$\begin{aligned} (x\cdot y)\phi &= x\phi \cdot y\phi \\ x^{-1}\phi &= (x\phi)^{-1} \\ e\phi &= e \end{aligned} \tag{1.1}$$

$$x\phi^s = x\phi^{p(r-1)} \cdot x\phi^{p(r-2)} \cdot \ldots \cdot x\phi^p \cdot x \tag{1.2}$$

and

$$x\phi^n = x. \tag{1.3}$$

The laws in (1.1) say that ϕ is a morphism, and the first of them is well-known to imply the other two. Law (1.2) arises by taking the inverse relations of $F(r,n,s,p)$

$$x_{i+p(r-1)}^{-1} x_{i+p(r-2)}^{-1} \cdots x_{i+p}^{-1} x_i^{-1} = x_{i+s}^{-1}$$

and setting $a_i = x_i^{-1}$ to get

$$a_{i+p(r-1)} a_{i+p(r-2)} \cdots a_{i+p} a_i = a_{i+s}.$$

Law (1.3) derives from the fact that the group $F(r,n,s,p)$ (and its abelianization) has an obvious automorphism θ which permutes cyclically the generators, i.e., $a_i\theta = a_{i+1}$ (subscripts mod n). The operation ϕ imitates the action of θ (and, in particular, it is of order n), not only on the generators, but on all elements of the group. This gives some restrictions on the groups underlying the algebras considered in this paper. Let us denote the *variety* of algebras defined by laws (1.1)–(1.3) as $\mathcal{V}(r,n,s,p)$. These varieties of Fibonacci type generalize those considered in [14], [15], [18] and [19], so the present note can be considered as a sequel to the quoted papers. We shall study some properties of the underlying groups of monogenic free algebras in these varieties, and discuss connections between their orders and the Fibonacci and Lucas sequences.

2. THE VARIETIES $\mathcal{V}(2,n,s,p)$

Theorem 2.1: *If $\mathcal{A}$ is an arbitrary algebra in the variety $\mathcal{V}(2,n,s,p)$, then the underlying group $\mathcal{G}$ is abelian. The group underlying the monogenic (that is, one–generator) free algebra of $\mathcal{V}(2,n,s,p)$ is isomorphic to the abelianization of the cyclically presented group $G_n(p,s)$ (in the notation of [6]).*

Proof: Laws (1.1) and (1.2) imply

$$(x\cdot y)\phi^s = (x\cdot y)\phi^p \cdot x\cdot y = x\phi^p \cdot y\phi^p \cdot x \cdot y$$

and

$$(x\cdot y)\phi^s = x\phi^s \cdot y\phi^s = x\phi^p \cdot x \cdot y\phi^p \cdot y.$$

Equating these formulae and simplyfing $x\phi^p$ on the left and y on the right, we get $y\phi^p \cdot x = x\cdot y\phi^p$. Now we set $z = y\phi^p$, and note that z ranges with y over the whole carrier of the algebra $\mathcal{A}$ (since ϕ^p is an automorphism by (1.3)). Then we obtain $z\cdot x = x\cdot z$, that is, the underlying group $\mathcal{G}$ is abelian. The proof of the second part of the statement is the same given in [18] for the variety $\mathcal{V}(2,n,2,1)$ (denoted by $\underline{\underline{V}}(n)$ in that paper). We briefly report it to make the reading clear. Suppose that $\mathcal{A}$ is monogenic, i.e., it is generated, as algebra, by a single element, a say. Then the underlying group $\mathcal{G}$ is generated, as group, by the elements $a_0, \ldots,$ a_{n-1}, where $a_i = a\phi^i$ by (1.3) (here $a_0 = a$ as usual). Law (1.2) gives relations $a_{i+s} = a_{i+p}a_i$ (subscripts mod n). Thus, $\mathcal{G}$ satisfies the defining relations of the group $G_n(p,s)$ in terms of generators. This implies that $\mathcal{G}$ is an epimorphic image of the abelianization of $G_n(p,s)$ (note that $\mathcal{G}$ is abelian). The action of ϕ on the abelianized group, also denoted by the same symbol, is that induced by the automorphism θ of $G_n(p,s)$ permuting cyclically the generators. So, this action satisfies laws (1.1) and (1.3). It remains to show that law (1.2) is satisfied in the abelianized group not only by the generators, but by all elements of the group. This follows immediately from the commutative law. In fact, if (1.2) holds for two arbitrary elements, g and h say, then we have

$$(g \cdot h)\phi^s = g\phi^s \cdot h\phi^s = g\phi^p \cdot g \cdot h\phi^p \cdot h = g\phi^p \cdot h\phi^p \cdot g \cdot h = (g \cdot h)\phi^p \cdot (g \cdot h)$$

and

$$(g^{-1})\phi^s = (g\phi^s)^{-1} = (g\phi^p \cdot g)^{-1} = g^{-1} \cdot (g\phi^p)^{-1} = (g^{-1})\phi^p \cdot g^{-1}$$

hence (1.2) holds for products and inverses, and then it holds throughout the group.

We recall some algebraic properties of the groups $G_n(p,s)$ and their abelianizations. The structure of the abelianization of the Fibonacci group $F(2,n) = G_n(1,2)$ was described in [18], and we shall recall it in Section 5 to study relations with the Fibonacci and Lucas sequences. The following result was proved in [7].

Theorem 2.2: *The abelianization of the group $G_n(p,s)$ is infinite if and only if $n \equiv 0 \pmod 6$, $p + s \equiv 3 \pmod 6$, and $3p \equiv 0 \pmod 6$.*

For $s = 1$, $G_n(p,s)$ is the Gilbert–Howie group $H(n,p) = G_n(x_1 x_{1+p} x_2^{-1})$. So Theorem 2.2 immediately implies that $H(n,p)$ has infinite abelianization if and only if $n \equiv 0 \pmod 6$ and $p \equiv 2 \pmod 6$. This is a well-known result, proved by Odoni in [21]. Moreover, the abelianization of $H(n,p)$ is trivial if and only if either $\gcd(n,6) = 1$ and $p \equiv 1$ or $2 \pmod n$ or $\gcd(n,6) > 1$ and $p \equiv 1 \pmod n$ (see [21], Theorem 2). In particular, for $p = 2$, the group $G_n(2,1)$ is the Sieradski group $S(n) = G_n(x_1 x_3 x_2^{-1})$. The structure of the abelianized group of $S(n)$ was completely determined in [16] (where the group $S(n)$ was denoted by K_n).

3. THE VARIETIES $\mathcal{V}(3,n,s,p)$

Theorem 3.1: *If $\mathcal{A}$ is an arbitrary algebra in the variety $\mathcal{V}(3,n,s,p)$, then the underlying group $\mathcal{G}$ is abelian. The group underlying the monogenic free algebra of $\mathcal{V}(3,n,s,p)$ is isomorphic to the abelianization of $F(3,n,s,p)$.*

Proof: Law (1.2) becomes

$$x\phi^s = x\phi^{2p} \cdot x\phi^p \cdot x. \tag{3.1}$$

We apply (3.1) to a product $x \cdot z$, and obtain

$$(x \cdot z)\phi^s = (x \cdot z)\phi^{2p} \cdot (x \cdot z)\phi^p \cdot x \cdot z$$

hence

$$x\phi^s \cdot z\phi^s = x\phi^{2p} \cdot z\phi^{2p} \cdot x\phi^p \cdot z\phi^p \cdot x \cdot z \tag{3.2}$$

by (1.1). Applying (3.1) to x and z separately, and multiplying, we get

$$x\phi^s \cdot z\phi^s = x\phi^{2p} \cdot x\phi^p \cdot x \cdot z\phi^{2p} \cdot z\phi^p \cdot z. \tag{3.3}$$

Equating (3.2) and (3.3) and cancelling $x\phi^{2p}$ on the left and z on the right, we obtain

$$x\phi^p \cdot x \cdot z\phi^{2p} \cdot z\phi^p = z\phi^{2p} \cdot x\phi^p \cdot z\phi^p \cdot x. \tag{3.4}$$

Firstly, we put $z = x\phi^{-p}$, and obtain, after cancelling $x\phi^p$ on the left and x on the right,

$$x \cdot x\phi^p = x\phi^p \cdot x. \tag{3.5}$$

Secondly, we put $y = z\phi^p$ in (3.4) and note that y ranges with z over the whole carrier of our algebra (since ϕ^p is an automorphism by (1.3)). Then we have

$$y\phi^p \cdot x\phi^p \cdot y \cdot x = x\phi^p \cdot x \cdot y\phi^p \cdot y$$

hence

$$(y \cdot x)\phi^p \cdot y \cdot x = (x\phi^p \cdot x) \cdot (y\phi^p \cdot y). \tag{3.6}$$

Now we define a new operator ψ by setting $x\psi = x\phi^p \cdot x$. Then (3.6) says that ψ is an antimorphism, i.e., $(y \cdot x)\psi = x\psi \cdot y\psi$. Then ψ^2 is a morphism. Now we can express $x\psi^2$ in terms of ϕ as follows:

$$\begin{aligned} x\psi^2 &= (x\psi)\psi = (x\phi^p \cdot x)\psi \\ &= (x\phi^p \cdot x)\phi^p \cdot (x\phi^p \cdot x) \\ &= x\phi^{2p} \cdot x\phi^p \cdot x \cdot x\phi^p \\ &= x\phi^s \cdot x\phi^p \end{aligned}$$

where we have used (3.1) and (3.5). Applying ψ^2 to a product $x \cdot z$ yields

$$(x \cdot z)\psi^2 = (x \cdot z)\phi^s \cdot (x \cdot z)\phi^p = x\phi^s \cdot z\phi^s \cdot x\phi^p \cdot z\phi^p. \tag{3.7}$$

Since ψ^2 is a morphism, we also have

$$(x \cdot z)\psi^2 = x\psi^2 \cdot z\psi^2 = x\phi^s \cdot x\phi^p \cdot z\phi^s \cdot z\phi^p. \tag{3.8}$$

Equating (3.7) and (3.8) and cancelling $x\phi^s$ on the left and $z\phi^p$ on the right, we obtain $z\phi^s \cdot x\phi^p = x\phi^p \cdot z\phi^s$. We note that $z\phi^s$ and $x\phi^p$ range with z and x, respectively, over the whole carrier of our algebra. This gives the commutative law proving the first part of the statement. Now the second part of the statement can be proved exactly as done in the proof of Theorem 2.1, so we omit it.

4. THE VARIETIES $\mathcal{V}(r, n, s, p)$

In this section we shall assume that $r \geq 4$. Then there are non-abelian groups that underly algebras in the variety $\mathcal{V}(r, n, s, p)$ (see Theorem 3 of [19] for the case $s = r$ and $p = 1$). We show now that, under certain conditions on the parameters, the free one–generator algebras of $\mathcal{V}(r, n, s, p)$ turn out to be abelian, and have the abelianizations of $F(r, n, s, p)$ as their underlying groups. More precisely, we have the following result:

Theorem 4.1: *If n is coprime with $p + s - pr$, then the group underlying the monogenic free algebra of the variety $\mathcal{V}(r, n, s, p)$ is abelian, and isomorphic to the abelianized group $A(r, n, s, p)$ of $F(r, n, s, p)$.*

Proof: We apply ϕ^p to both sides of (1.2) to get

$$x\phi^{p+s} = x\phi^{pr} \cdot x\phi^{p(r-1)} \cdot \ldots \cdot x\phi^{2p} \cdot x\phi^p. \tag{4.1}$$

Then we have

$$x\phi^{p+s} \cdot x = x\phi^{pr} \cdot x\phi^{p(r-1)} \cdot \ldots \cdot x\phi^{p} \cdot x$$

hence

$$x\phi^{p+s} \cdot x = x\phi^{pr} \cdot x\phi^{s}. \tag{4.2}$$

Now we evaluate $(x\phi^p \cdot y)\phi^s$ by using (1.2) again, with $x\phi^p \cdot y$ and then $y \cdot x$ in place of x. Then we have

$$\begin{aligned}
(x\phi^p \cdot y)\phi^s &= (x\phi^p \cdot y)\phi^{p(r-1)} \cdot (x\phi^p \cdot y)\phi^{p(r-2)} \cdot \ldots \cdot (x\phi^p \cdot y)\\
&= x\phi^{pr} \cdot y\phi^{p(r-1)} \cdot x\phi^{p(r-1)} \cdot y\phi^{p(r-2)} \cdot \ldots \cdot x\phi^p \cdot y\\
&= x\phi^{pr} \cdot (y \cdot x)\phi^{p(r-1)} \cdot (y \cdot x)\phi^{p(r-2)} \cdot \ldots \cdot (y \cdot x)\phi^p \cdot (y \cdot x) \cdot x^{-1}\\
&= x\phi^{pr} \cdot (y \cdot x)\phi^s \cdot x^{-1}\\
&= x\phi^{pr} \cdot y\phi^s \cdot x\phi^s \cdot x^{-1}
\end{aligned}$$

hence

$$x\phi^{p+s} \cdot y\phi^s = x\phi^{pr} \cdot y\phi^s \cdot x\phi^s \cdot x^{-1}. \tag{4.3}$$

Setting $y = x\phi^p$, we get

$$x\phi^{p+s} \cdot x\phi^{p+s} \cdot x = x\phi^{pr} \cdot x\phi^{p+s} \cdot x\phi^s.$$

Using (4.2) and cancelling $x\phi^s$ on the right gives

$$x\phi^{p+s} \cdot x\phi^{pr} = x\phi^{pr} \cdot x\phi^{p+s}$$

or, equivalently,

$$x\phi^{p+s-pr} \cdot x = x \cdot x\phi^{p+s-pr}. \tag{4.4_1}$$

We now proceed by induction, and consider the laws

$$x\phi^{j(p+s-pr)} \cdot x = x \cdot x\phi^{j(p+s-pr)} \tag{4.4_j}$$

and

$$x\phi^{j(p+s-pr)} \cdot x\phi^{p+s-pr} = x\phi^{p+s-pr} \cdot x\phi^{j(p+s-pr)} \tag{4.5_j}$$

for any $j \geq 0$. Formula (4.4_1) has just been proved, and (4.5_1) is trivially satisfied. Applying ϕ^{p+s-pr} to (4.4_j) gives (4.5_{j+1}). To prove (4.4_{j+1}) we apply (4.4_j) to $x\phi^{p+s-pr} \cdot x$ in place of x. We get

$$(x\phi^{p+s-pr} \cdot x)\phi^{j(p+s-pr)} \cdot x\phi^{p+s-pr} \cdot x = x\phi^{p+s-pr} \cdot x \cdot (x\phi^{p+s-pr} \cdot x)\phi^{j(p+s-pr)}.$$

Thus we have

$$x\phi^{(j+1)(p+s-pr)} \cdot x\phi^{j(p+s-pr)} \cdot x\phi^{p+s-pr} \cdot x = x\phi^{p+s-pr} \cdot x \cdot x\phi^{(j+1)(p+s-pr)} \cdot x\phi^{j(p+s-pr)}.$$

Applying (4.5_j) and (4.5_{j+1}) we shift $x\phi^{p+s-pr}$ on the left hand side to the left and cancel it to get

$$x\phi^{(j+1)(p+s-pr)} \cdot x\phi^{j(p+s-pr)} \cdot x = x \cdot x\phi^{(j+1)(p+s-pr)} \cdot x\phi^{j(p+s-pr)}.$$

Now we use (4.4_j) to shift $x\phi^{j(p+s-pr)}$ to the right on the left hand side. Then we can cancel it, and obtain

$$x\phi^{(j+1)(p+s-pr)} \cdot x = x \cdot x\phi^{(j+1)(p+s-pr)}$$

which is (4.4_{j+1}). So we have proved by induction the validity of (4.4_j) and (4.5_j) for all positive j. Given arbitrary integers k and ℓ with $k > \ell$, we apply $\phi^{\ell(p+s-pr)}$ to $(4.4_{k-\ell})$ to get

$$x\phi^{k(p+s-pr)} \cdot x\phi^{\ell(p+s-pr)} = x\phi^{\ell(p+s-pr)} \cdot x\phi^{k(p+s-pr)}. \tag{4.6}$$

As (4.6) is symmetric in k and ℓ and the case $k = \ell$ is trivial, we have proved the following result:

Lemma 4.1: *In any variety $\mathcal{V}(r, n, s, p)$, law* (4.6) *is valid for arbitrary integers k and ℓ.*

Suppose now $\mathcal{A}$ is a monogenic free algebra in $\mathcal{V}(r, n, s, p)$. Then its underlying group $\mathcal{G}$ is generated, as group, by the set $\{a\phi^j : j \in \mathbb{Z}_n\}$ by (1.3). So (4.6) implies that $\mathcal{G}$ is abelian if $\gcd(p + s - pr, n) = 1$.

In [7] there were determined necessary and sufficient conditions for the parameters under which the abelianization of $F(r, n, s, p)$ is infinite.

Theorem 4.2: *The abelianized group $A(r, n, s, p)$ of $F(r, n, s, p)$ is infinite if and only if there exists $m \in \mathbb{Z}$, $m > 1$ such that m divides n, but it does not divide p, and either $p(1 - r) \equiv 0 \pmod{m}$ and $s \equiv 0 \pmod{m}$ or $p(r + 1) \equiv 0 \pmod{m}$ and $p + s \equiv m/2 \pmod{m}$, with m even in the second case.*

For $p = 1$, Theorem 4.2 implies that the abelianized group of the Campbell–Robertson group $F(r, n, s) = G_n(x_1x_2 \cdots x_r x_{1+s}^{-1})$, where $r \geq 2$, $n \geq 3$ and $s \geq 1$, is infinite if and only if there exists $m \in \mathbb{Z}$, $m > 1$, $m\backslash n$ such that either $s \equiv 0 \pmod{m}$ and $r \equiv 1 \pmod{m}$ or $s + 1 \equiv m/2 \pmod{m}$ and $r \equiv -1 \pmod{m}$, with m even in the second case.

We complete the section with further computations. We deduce from (4.2) the law

$$(x^{-1} \cdot x\phi^{p+s-pr})\phi^{pr} = x^{-1}\phi^{pr} \cdot x\phi^{p+s} = x\phi^s \cdot x^{-1}. \tag{4.7}$$

This implies that $g\phi^{p+s-pr} = g$ is equivalent with $g\phi^s = g$ for any element g of the carrier of an algebra in $\mathcal{V}(r, n, s, p)$. So we have

Lemma 4.2: *An element g of the carrier of an algebra in $\mathcal{V}(r, n, s, p)$ is fixed by ϕ^s if and only if g is fixed by ϕ^{p+s-pr}. In particular, the varieties $\mathcal{V}(r, s, s, p)$ and $\mathcal{V}(r, p + s - pr, s, p)$ coincide.*

Now we rewrite (4.3) in the form

$$(x\phi^{pr})^{-1} \cdot x\phi^{p+s} \cdot y\phi^s = y\phi^s \cdot x\phi^s \cdot x^{-1}.$$

Using (4.7) we obtain

$$(x^{-1} \cdot x\phi^{p+s-pr})\phi^{pr} \cdot y\phi^s = y\phi^s \cdot (x^{-1} \cdot x\phi^{p+s-pr})\phi^{pr}.$$

Applying ϕ^{-pr} yields

$$(x^{-1} \cdot x\phi^{p+s-pr}) \cdot y\phi^{s-pr} = y\phi^{s-pr} \cdot (x^{-1} \cdot x\phi^{p+s-pr}).$$

We put $z = y\phi^{s-pr}$ and note that z ranges with y over the whole carrier of our algebra. This gives

$$(x^{-1} \cdot x\phi^{p+s-pr}) \cdot z = z \cdot (x^{-1} \cdot x\phi^{p+s-pr}). \tag{4.8}$$

This law says that $x^{-1} \cdot x\phi^{p+s-pr}$ commutes with all elements, i.e., it is central in the underlying group $\mathcal{G}$ of an algebra $\mathcal{A}$ in $\mathcal{V}(r, n, s, p)$. The set $\{g^{-1} \cdot g\phi^{p+s-pr} : g \in \mathcal{G}\}$ generates a subgroup $\mathcal{H}$ of $\mathcal{G}$. The group $\mathcal{H}$ underlies a subalgebra $\mathcal{B}$ of $\mathcal{A}$. In fact, $\mathcal{B}$ is the kernel of the natural epimorphism from $\mathcal{A}$ onto the algebra $\mathcal{A}'$ with underlying group $\mathcal{G}' = \mathcal{G}/\mathcal{H}$, where ϕ acts on $\mathcal{G}'$ with the additional law $x\phi^{pr-p-s} = x$. So we have proved (use (4.8))

Lemma 4.3: *The subgroup $\mathcal{H}$ of $\mathcal{G}$ lies in the centre of $\mathcal{G}$. If n does not divide $p + s - pr$, then the underlying group of an arbitrary algebra in $\mathcal{V}(r, n, s, p)$ has a non–trivial centre.*

To complete the section we treat a special case given by conjugation. Let $\mathcal{G}$ be a group underlying an algebra $\mathcal{A}$ in $\mathcal{V}(r, n, s, p)$, and suppose that the operator ϕ on $\mathcal{G}$ is given by the conjugation with a fixed element $g \in \mathcal{G}$, i.e., $\phi(x) = g \cdot x \cdot g^{-1}$, for any $x \in \mathcal{G}$. Of course, ϕ satisfies the laws in (1.1), and it satisfies law (1.3) if and only if g^n belongs to the centre of $\mathcal{G}$. Substituting the relations $\phi^i(x) = g^i \cdot x \cdot g^{-i}$ in law (1.2) yields $g^{s-p(r-1)} \cdot x \cdot g^{-s-p} = (x \cdot g^{-p})^r$ for any $x \in \mathcal{G}$. If we put $x = g$, then the above relation implies that $g^{r-1} = e$, i.e., the order of g divides $r - 1$. In particular, if the centre of $\mathcal{G}$ is trivial and n is coprime with $r - 1$, then the action of ϕ is trivial. Let now $\mathcal{G}$ be a subgroup of $\mathrm{GL}(2; \mathbb{C})$ which contains at least one matrix $X = \begin{pmatrix} u & v \\ w & z \end{pmatrix}$ with $wv \neq 0$. Let ϕ be the conjugation on $\mathcal{G}$ with the matrix $g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where λ is a primitive n–th root of unity (hence n divides $r - 1$). Then there is an algebra in $\mathcal{V}(r, n, s, p)$ whose underlying group is $\mathcal{G}$. Furthermore, one can prove the following result.

Proposition 4.4: *If $\mathcal{G} \subset \mathrm{GL}(2; \mathbb{C})$ and ϕ are as above, then n divides $4s$. If n is coprime with $4s$, then the action of ϕ is trivial, hence $X^{r-1} = I_2$ for any $X \in \mathcal{G}$.*

5. CONNECTIONS WITH THE FIBONACCI AND LUCAS SEQUENCES

The structure of the abelianization $A(2, n)$ of the Fibonacci group $F(2, n)$ is well–known, and it is strictly related to the Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$ (see for example [18]). The following familiar relations describe some connections between the Fibonacci numbers F_n and the Lucas numbers L_n (see for example [2] and [20]):

$$L_{2g} = L_g^2 - 2(-1)^g \qquad \text{and} \qquad F_{2g} = F_g L_g \tag{5.1}$$

$$L_{-n} = (-1)^n L_n \qquad \text{and} \qquad F_{-n} = (-1)^{n+1} F_n \tag{5.2}$$

$$2L_{m+n} = L_m L_n + 5F_m F_n \tag{5.3}$$

$$L_n = F_{n-1} + F_{n+1} \tag{5.4}$$

$$L_{2gt+m} \equiv \pm L_{2g+m} \pmod{L_{2g}} \tag{5.5}$$

for any g, m, n, and t integers, t odd (here $F_0 = 0$ and $L_0 = 2$). If $m = n$, then (5.3) becomes $2L_{2n} = L_n^2 + 5F_n^2$. Substituting the first relation in (5.1) yields

$$2L_n^2 - 4(-1)^n = L_n^2 + 5F_n^2,$$

hence

$$L_n^2 = 5F_n^2 + 4(-1)^n \tag{5.6}$$

(see for example [20]). By [18] (see also [4]) the order of $A(2,n)$ is given by

$$g_n = F_{n-1} + F_{n+1} - 1 - (-1)^n = L_n - 1 - (-1)^n$$

for any $n \geq 1$. Thus $A(2,n)$ has order L_n for n odd, and $L_n - 2$ for n even. The group $A(2,n)$ is isomorphic to the direct product of two cyclic groups of orders h and k, where h divides k, i.e., $A(2,n) \cong \mathbb{Z}_h \oplus \mathbb{Z}_k$ with $hk = g_n$. If $\gcd(n,6) = 1$, then $h = 1$ and $k = g_n$. If $n \equiv 3 \pmod 6$, then $h = 2$ and $k = \frac{1}{2}g_n = \frac{1}{2}(F_{n-1} + F_{n+1}) = \frac{1}{2}L_n$. If $n \equiv 2 \pmod 4$, then $h = k = \sqrt{g_n} = \sqrt{L_n - 2} = L_{\frac{n}{2}} = F_{\frac{n}{2}-1} + F_{\frac{n}{2}+1}$ by (5.1) and (5.4). If $n \equiv 0 \pmod 4$, then $k = 5h$ and $h = \sqrt{\frac{1}{5}g_n} = \sqrt{\frac{1}{5}(L_n - 2)} = F_{\frac{n}{2}}$ by (5.1) and (5.6). The following table, related to that shown in [18], summarizes the above results in terms of the Lucas numbers.

$n \pmod{12}$	h	k
0	$\sqrt{\frac{1}{5}(L_n-2)}$	$\sqrt{5(L_n-2)}$
1	1	L_n
2	$L_{\frac{n}{2}}$	$L_{\frac{n}{2}}$
3	2	$\frac{1}{2}L_n$
4	$\sqrt{\frac{1}{5}(L_n-2)}$	$\sqrt{5(L_n-2)}$
5	1	L_n
6	$L_{\frac{n}{2}}$	$L_{\frac{n}{2}}$
7	1	L_n
8	$\sqrt{\frac{1}{5}(L_n-2)}$	$\sqrt{5(L_n-2)}$
9	2	$\frac{1}{2}L_n$
10	$L_{\frac{n}{2}}$	$L_{\frac{n}{2}}$
11	1	L_n

Let now $A(r,n,s,p)$ be the abelianization of the cyclically presented group $F(r,n,s,p)$. The polynomial $f_w(t)$ associated with the cyclic presentation is $f_w(t) = \sum_{i=0}^{r-1} t^{pi} - t^s$. By [16] and [21] it follows that $A(r,n,s,p)$ is infinite if and only if $f_w(t)$ vanishes on an n–th root of unity. Otherwise, the order of $A(r,n,s,p)$ is given by $|A(r,n,s,p)| = (r-1)|\prod_{i=2}^{n}(\sum_{j=0}^{r-1}\omega_i^{pj} - \omega_i^s)|$, where $\omega_1 = 1, \omega_2, \ldots, \omega_n$ are the distinct n–th roots of unity (note that $f_w(1) = r - 1$). The following questions arise in a natural way from the arguments discussed above.

Open questions. Determine the complete structure of the groups $A(r,n,s,p)$ in the finite case, and find formulae which express their orders in terms of the parameters. For which values of the parameters, do these formulae involve sequences of numbers related to Fibonacci and Lucas sequences? For which values of the parameters, is $A(r,n,s,p)$ cyclic?

Partial results in this direction are given in the next proposition which easily extends Corollaries 3 and 4 of [12] (see also [15]) to our case.

Proposition 5.1: *Suppose that p is coprime with n. Then we have*

(1) *If* $s \equiv p \pmod{n}$ *and* $pr \equiv p \pmod{n}$, *then* $|A(r,n,s,p)| = (r-1)n$.
(2) *If* $s \equiv -p \pmod{n}$ *and* $pr \equiv -p \pmod{n}$, *then* $|A(r,n,s,p)| = (r-1)2^{n-1}$.
(3) *If* $pr \equiv 2p \pmod{n}$ *and either* $s \equiv p \pmod{n}$ *or* $s \equiv 0 \pmod{n}$, *then*

$$|A(r,n,s,p)| = r - 1.$$

(4) *If* $s \equiv -2p \pmod{n}$ *and* $pr \equiv -2p \pmod{n}$, *then* $|A(r,n,s,p)| = \frac{1}{3}(r-1)(2^n - (-1)^n)$.

REFERENCES

[1] C. M. Campbell and E. F. Robertson. "A Class of Finitely Presented Groups of Fibonacci Type." *J. London Math. Soc.* **11** (1975): 249–255.

[2] C. M. Campbell, E. F. Robertson and R. M. Thomas. "Finite Groups of Deficiency Zero Involving the Lucas Numbers." *Proc. Edinb. Math. Soc.* **33** (1990): 1–10.

[3] C. M. Campbell and R. M. Thomas. "On Infinite Groups of Fibonacci Type." *Proc. Edinb. Math. Soc.* **29.2** (1986): 225–232.

[4] C. M. Campbell and R. M. Thomas. "On $(2,n)$–Groups Related to Fibonacci Groups." *Israel J. Math.* **58.3** (1987): 370–380.

[5] A. Cavicchioli, F. Hegenbarth and A. C. Kim. "A Geometric Study of Sieradski Groups." *Algebra Colloq.* **5.2** (1998): 203–217.

[6] A. Cavicchioli, F. Hegenbarth and D. Repovš. "On Manifold Spines and Cyclic Presentations of Groups." In: *Knot Theory* (Warszawa), Banach Center Publ. **42** (1998): 49–55.

[7] A. Cavicchioli, D. Repovš and F. Spaggiari. "Topological properties of cyclically presented groups." *J. Knot Theory and Ramifications* **12.2** (2003): 243–268.

[8] C. P. Chalk and D. L. Johnson. "The Fibonacci Groups. II." *Proc. Roy. Soc. Edinburgh Sect. A* **77** (1977): 79–86.

[9] J. H. Conway. "Advanced Problem 5327." *Amer. Math. Monthly* **72** (1965): 915.

[10] J. H. Conway. "Solution to Advanced Problem 5327." *Amer. Math. Monthly* **74** (1967): 91–93.

[11] N. Gilbert and J. Howie. "LOG Groups and Cyclically Presented Groups." *J. of Algebra* **174** (1995): 118–131.

[12] D. L. Johnson. "A Note on the Fibonacci Groups." *Israel J. Math.* **17** (1974): 277–282.

[13] D. L. Johnson. *Topics in the Theory of Group Presentations.* London Math. Soc. Lect. Note Ser. **42** Cambridge: Cambridge University Press, 1980.

[14] D. L. Johnson and A. C. Kim. "Periodic Fibonacci Algebras." *Proc. Edinb. Math. Soc.* **35** (1992): 169–172.

[15] D. L. Johnson and A. C. Kim. "Cyclic Fibonacci Algebras." *The Fibonacci Quarterly* **32.5** (1994): 441–444.

[16] D. L. Johnson and R. W. K. Odoni. "Some Results on Symmetrically Presented Groups." *Proc. Edinb. Math. Soc.* **37** (1994): 227–237.

[17] D. L. Johnson, J. W. Wamsley and D. Wright "The Fibonacci Groups." *Proc. London Math. Soc.* **29** (1974): 577–592.

[18] A. C. Kim. "Fibonacci Varieties." *Bull. Austral. Math. Soc.* **19** (1978): 191–196.

[19] A. C. Kim, B. H. Neumann and A. H. Rhemtulla. "More Fibonacci Varieties." *Bull. Austral. Math. Soc.* **22** (1980): 385–395.

[20] W. L. McDaniel. "On the Factorization of Lucas Numbers." *The Fibonacci Quarterly* **39.3** (2001): 206–210.
[21] R. W. K. Odoni. "Some Diophantine Problems Arising from the Theory of Cyclically Presented Groups." *Glasgow Math. J.* **41** (1999): 157–165.
[22] A. J. Sieradski. "Combinatorial Squashings, 3–Manifolds, and the Third Homotopy of Groups." *Invent. Math.* **84** (1986): 121–139.

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