PASCAL DECOMPOSITIONS OF GEOMETRIC ARRAYS IN MATRICES

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1. INTRODUCTION

In a previous paper [6], we discussed decompositions of certain matrices whose rows are generalized arithmetic progressions or whose columns are successive convolutions of an integer sequence. This work was an extension of the explorations made by Bicknell and Hoggatt in several papers [2, 5] in the 1970s. In this paper, we extend our prior results to include matrices whose are generalized geometric progressions. We then show how this applies to a class of matrices studied by Ollerton and Shannon [7] as well as some novel matrices. We finally employ our results to easily calculate generating functions for these matrices.

2. PRIOR RESULTS AND MOTIVATION

Our previous work ended with a discussion of the convolution matrices for sequences with first term not equal to one, such as the following matrix based on convolutions of the familiar Lucas numbers:

$$L = \begin{pmatrix} 2 & 4 & 8 & 16 & 32 & 64 & \cdots \\ 1 & 4 & 12 & 32 & 80 & 192 & \cdots \\ 3 & 13 & 42 & 120 & 320 & 816 & \cdots \\ 4 & 22 & 85 & 280 & 840 & 2368 & \cdots \\ 7 & 45 & 195 & 705 & 2290 & 6924 & \cdots \\ 11 & 82 & 399 & 1588 & 5601 & 18204 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1}$$

To review, the convolution of two sequences $\{a_n\}$ and $\{b_n\}, (n = 0, 1, \dots)$, is the sequence $\{c_n\}$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. The convolution matrix of a sequence is the matrix whose i^{th} column $(i = 1, 2, \dots)$ is the $(i-1)^{th}$ convolution of the sequence with itself.

The final theorem in this previous paper guarantees that we can decompose such matrices into a product of two triangular matrices, one of which is raised to a positive integer power. Here we propose to generalize this theorem to any integer exponent using methods developed by Call and Velleman [3].

Let $P_U[x]$ be the matrix defined by

$$(P_U[x])_{i,j} = \begin{cases} (x)^{j-i}\binom{j-1}{i-1}, & \text{if } j \geq i, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where x is any nonzero integer. Then it is proved in [3] that

$$(P_U)^r = P_U[r], \tag{3}$$

where $P_U = P_U[1]$, the standard upper triangular Pascal matrix, and r is any nonzero integer. If $r = 0$, then we define $P_U^0 = I$, where I is an identity matrix.

Using these facts, we can prove the following generalization of our previous theorem:

Theorem 1(Strong Convolution Decomposition Theorem): *Let $\{v_n\}$ be a sequence whose first term is any integer v_0, and let V be the convolution matrix of that sequence. Then $V = S \cdot P_U^{v_0}$ for some lower triangular matrix S and the upper triangular Pascal matrix P_U. Moreover, successive columns of S are successive convolutions of the sequence $\{v_n\}$ with the sequence $\{0, v_1, v_2, v_3, \dots\}$.*

The proof is similar to the one given in [6].

In our Lucas example,

$$L = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 0 & 0 & 0 & 0 & \dots \\ 3 & 7 & 2 & 0 & 0 & 0 & \dots \\ 4 & 14 & 13 & 2 & 0 & 0 & \dots \\ 7 & 31 & 43 & 19 & 2 & 0 & \dots \\ 11 & 60 & 115 & 90 & 25 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 2 & 3 & 4 & 5 & \dots \\ 0 & 0 & 1 & 3 & 6 & 10 & \dots \\ 0 & 0 & 0 & 1 & 4 & 10 & \dots \\ 0 & 0 & 0 & 0 & 1 & 5 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^2 . \tag{4}$$

Three things are important to notice here. First, the seed matrix is related to the original sequence that is convoluted. Second, the exponent of the upper triangular Pascal matrix is equal to the first element in the original sequence. Third, the first row of the original convolution matrix L is clearly a geometric sequence.

It is this final observation that will lead to a generalization of this approach. Theorem 1 now deals with all convolution matrices of integer sequences, whereas previously we were limited to (AP) matrices whose rows were all generalized arithmetic progressions. The natural tack to take is to exploit the insight gained with convolution matrices in order to extend these previous results to a class of matrices based on geometric progressions.

3. GEOMETRIC PROGRESSION MATRICES

The first step towards such a goal is to extend the common definition of a geometric sequence in a manner analagous to the generalization of arithmetic progressions we capitalized on in [6]. To this end, we define the first order geometric difference with respect to a ratio r of a sequence $\{a_k\}$ to be the sequence $\Delta_r^1\{a_k\} = \{a_{k+1} - r(a_k)\}$. If we let $\Delta_r^i\{a_k\}$ represent the i^{th} order difference, i.e., $\Delta_r^i\{a_k\} = \{\Delta_r^{i-1}a_{k+1} - r(\Delta_r^{i-1}a_k)\}$, we can then define a geometric progression of n^{th} order with ratio r, or $(GP)_{n,r}$, to be a sequence whose n^{th} order geometric difference with ratio r is an ordinary nonzero geometric sequence with ratio r, while the $(n-1)^{th}$ order geometric difference is not.

After such a load of definitions, an example is in order. Consider the sequence $\{3, 13, 42, 120, 320, 816, \dots\}$: it is a second order progression with ratio 2, that is, a $(GP)_{2,2}$. We can see this by constructing its geometric difference table with ratio 2:

$$\begin{array}{lccccccccccccc} \Delta_2^0 a_k & 3 & & 13 & & 42 & & 120 & & 320 & & 816 & \dots \\ \Delta_2^1 a_k & & 7 & & 16 & & 36 & & 80 & & 176 & \dots \\ \Delta_2^2 a_k & & & 2 & & 4 & & 8 & & 16 & \dots \\ \Delta_2^3 a_k & & & & 0 & & 0 & & 0 & \dots \end{array} . \tag{5}$$

Extending the notation in [6], we call the first element in the j^{th} row of the geometric difference table of a $(GP)_{n,r}$, that is, $\Delta_r^j a_1$, the j^{th} order initial constant, and we call the $(n+1)^{th}$ order initial constant – in our example, 2 – the constant of the progression. The common ratio of rows, r, is called the ratio of the progression. Note that the sequence itself is the zeroth row of geometric differences, so a regular nonzero geometric sequence with ratio r is a $(GP)_{0,r}$. Also, it is plain that the $(n+2)^{th}$ and all subsequent rows in the geometric difference table are zero sequences.

Moving on, a generalized geometric progression matrix, or (GP) matrix, is an $n \times n$ matrix whose i^{th} row ($i = 1, 2, \ldots$) is n terms of a $(GP)_{(i-1),r}$. That is, each subsequent row is a (GP) of ascending order all of which have the same ratio. These (GP) matrices are generalizations of the (AP) matrices of [2] and [6], since whenever $r = 1$ the sequences become arithmetic progressions and we have an (AP) matrix. We will prove that these (GP) matrices decompose according to the following rule:

Theorem 2(Geometric Decomposition Theorem): *$G_{n\times n}$ is a (GP) matrix with ratio r if and only if $G_{n\times n} = S_{n\times n} \cdot (P_U)^r$, where $S_{n\times n}$ is some lower triangular seed matrix with nonzero diagonal elements, P_U is the $n \times n$ upper triangular Pascal matrix, and r is any nonzero integer ratio of G's progressions.*

Proof: Given that G is an $n \times n$ (GP) matrix with ratio r, we will prove it has a seed matrix S by constructing it. The key observation is that the leading diagonal of any sequence's geometric difference table is algebraically determined by the original sequence. Symbolically, if $\{g_k\}$ is some sequence and $\{b_m\}$ is the leading diagonal of its geometric difference table, i.e.,

$$\begin{array}{lcccccccc} \Delta_r^0\{g_k\} & b_1 = g_1 & & g_2 & & g_3 & g_4 & g_5 & g_6 & \cdots \\ \Delta_r^1\{g_k\} & & b_2 & & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta_r^2\{g_k\} & & & b_3 & & \cdot & \cdot & \cdot & \cdot & \\ \vdots & & & & \ddots & \cdot & \cdot & \cdot \end{array}, \tag{6}$$

then the relationships between the two sequences are

$$g_i = \sum_{j=1}^{i} (r)^{i-j} \binom{i-1}{j-1} b_j \text{ and } b_i = \sum_{j=1}^{i} (-r)^{i-j} \binom{i-1}{j-1} g_j. \tag{7}$$

This is a generalization of Sloane and Plouffe's formula for arithmetic difference tables [8, p. 13]. An inductive proof of these two equations follows readily from the definition of geometric differences and a little algebra.

Returning to our (GP) matrix G, we let

$$G = \begin{pmatrix} \overline{G}_1 \\ \overline{G}_2 \\ \vdots \\ \overline{G}_n \end{pmatrix}, \tag{8}$$

where $\overline{G}_i$ is the i^{th} row of G, that is, $\overline{G}_i = (g_{i,1}, g_{i,2}, \ldots, g_{i,n})$, where $i = 1, 2, \ldots, n$. Then, since G is a (GP) matrix, $\overline{G}_i$ is a $(GP)_{(i-1),r}$ as defined above. Writing out the geometric difference table of this i^{th} row and labeling the leading diagonal $\{b_{i,1}, b_{i,2}, \dots\}$ yields the following diagram:

$$\begin{array}{lcccccccccccc}
\Delta_r^0\overline{G}_i & b_{i,1}=g_{i,1} & & g_{i,2} & & g_{i,3} & & g_{i,4} & & g_{i,5} & & g_{i,6} & \cdots \\
\Delta_r^1\overline{G}_i & & b_{i,2} & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot \\
\Delta_r^2\overline{G}_i & & & b_{i,3} & & \cdot & & \cdot & & \cdot & & \cdot & \\
\vdots & & & & \ddots & & \cdot & & \cdot & & \cdot & & \cdot \\
\Delta_r^{i-1}\overline{G}_i & & & & & b_{i,i} & & \cdot & & \cdot & & & \\
\Delta_r^{i}\overline{G}_i & & & & & & 0 & & \cdot & & & &
\end{array} \tag{9}$$

Now since $\overline{G}_i$ is a $(GP)_{(i-1),r}$, its $(i-1)^{th}$ order geometric difference must be equal to a regular geometric sequence with ratio r. Moreover, $b_{i,i}$ equals the nonzero constant of the progression. By the definition of $(GP)_{(i-1),r}$, any elements below row i (on the leading diagonal, all $b_{i,j}, j > i$) must equal zero. Substituting from equation (8) for the elements $g_{i,k}$ that make up our matrix G and using equation (7), we have

$$G = \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n,1} & g_{n,2} & \cdots & g_{n,n} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{1}\binom{0}{j-1}(r)^{1-j}b_{1,j} & \sum_{j=1}^{2}\binom{1}{j-1}(r)^{2-j}b_{1,j} & \cdots & \sum_{j=1}^{n}\binom{n-1}{j-1}(r)^{n-j}b_{1,j} \\ \sum_{j=1}^{1}\binom{0}{j-1}(r)^{1-j}b_{2,j} & \sum_{j=1}^{2}\binom{1}{j-1}(r)^{2-j}b_{2,j} & \cdots & \sum_{j=1}^{n}\binom{n-1}{j-1}(r)^{n-j}b_{2,j} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{1}\binom{0}{j-1}(r)^{1-j}b_{n,j} & \sum_{j=1}^{2}\binom{1}{j-1}(r)^{2-j}b_{n,j} & \cdots & \sum_{j=1}^{n}\binom{n-1}{j-1}(r)^{n-j}b_{n,j} \end{pmatrix}. \tag{10}$$

Remembering that many of the b_{ij} are zero and then applying equation (3), we see that

$$G = \begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & b_{n,3} & \dots & b_{n,n} \end{pmatrix} \begin{pmatrix} \binom{0}{0}(r)^0 & \binom{1}{0}(r)^1 & \binom{2}{0}(r)^2 & \dots & \binom{n-1}{0}(r)^{n-1} \\ 0 & \binom{1}{1}(r)^0 & \binom{2}{1}(r)^1 & \dots & \binom{n-1}{1}(r)^{n-2} \\ 0 & 0 & \binom{2}{2}(r)^0 & \dots & \binom{n-1}{2}(r)^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \binom{n-1}{n-1}(r)^0 \end{pmatrix}$$

$$= \begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & b_{n,3} & \dots & b_{n,n} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & \binom{n-1}{0} \\ 0 & 1 & 2 & \dots & \binom{n-1}{1} \\ 0 & 0 & 1 & \dots & \binom{n-1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \binom{n-1}{n-1} \end{pmatrix}^r \tag{11}$$

$$= S_{n\times n} \cdot (P_U)^r.$$

G can therefore be split into a product of a lower triangular matrix S and the $n \times n$ upper triangular Pascal matrix raised to the r^{th} power, and the first half of the theorem is proved. The converse follows from a reversal of this argument. Start with any lower triangular seed matrix S with nonzero diagonal elements, multiply r times by the upper triangular Pascal matrix (or by its inverse, if r is negative), and a (GP) matrix with ratio r will result.

What's more, given a (GP) matrix G, we now know the exact structure of the seed matrix S and can calculate it from our original matrix G. We call this process the *Pascal decomposition* of G. It turns out that the elements on S's main diagonal are the progression constants of each row of G, and – since the determinant of P is always one – we can make a corollary.

Corollary (Generalization of Eves' Theorem): *For any* (GP) *matrix* $G_{n\times n}$,

$$|G_{n\times n}| = \prod_{j=1}^{n} c_j,$$

where c_j *is the progression constant of the* j^{th} *row of* $G_{n\times n}$.

When the ratio of G's progressions equals one, this agrees with both Bicknell and Hoggatt's work [2] and our own previous results [6] concerning arithmetic progression matrices and convolution matrices with first term one. It also explains what we observed when decomposing convolution matrices with first term greater than one.

4. SPECIAL CASES OF (GP) MATRICES

Ollerton and Shannon [7] explore matrices generated under a class of recursion relations. If $\{n,p\}$ represents the entry in the n^{th} row and p^{th} column of a matrix, their relation,

$$\{n,p\} = b\{n-1,p\} + \sum_{i=r}^{s} a_i\{n+i,p-1\}, \text{ with } \{n,-1\} = 0 \quad \forall n, \tag{12}$$

allows one to fill in the rest of a matrix so long as its top row is known. We have found that the transposes of such matrices are always (GP) matrices with ratio b, and hence are decomposable. Furthermore, by translating notations it can be verified algebraically that Property 12 in [7] is a special case of our Corollary. To do so, first note that if $\{0,0\} = a$ then

$$c_j = a\left(b\sum_{i=r}^{s} a_i b^i\right)^{j-1}, \quad \text{for} \quad j = 1,2,\ldots,n, \tag{13}$$

is the progression constant of the j^{th} column of this recurrence relation array. It follows that the transpose of such an array must be a (GP) matrix. Applying Theorem 2 and its Corollary then yields Property 12.

It is not difficult to show that a convolution matrix of a sequence $\{v_n\} = \{v_0, v_1, v_2, \ldots\}$ in [5,6] is also a (GP) matrix with ratio v_0. It can be verified algebraically that

$$c_j = v_0 v_1^{j-1}, \quad \text{for} \quad j = 1,2,\ldots,n, \tag{14}$$

is the progression constant of the j^{th} row of the convolution array. Thus Theorem 4 and its Corollary in [6] are direct results of the Theorem 2 and its Corollary, respectively.

For example, we can invent the following matrix Q based on the Lucas-like numbers $\{1,4,5,9,14,\dots\}$, which form its top row, using the recursion $\{n,p\} = 2\{n-1,p\}+\{n,p-1\}$:

$$Q = \begin{pmatrix} 1 & 4 & 5 & 9 & 14 & 23 & \cdots \\ 2 & 10 & 20 & 38 & 66 & 112 & \cdots \\ 4 & 24 & 64 & 140 & 272 & 496 & \cdots \\ 8 & 56 & 184 & 464 & 1008 & 2000 & \cdots \\ 16 & 128 & 496 & 1424 & 3440 & 7440 & \cdots \\ 32 & 288 & 1280 & 4128 & 11008 & 25888 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{15}$$

We can decompose its transpose like so:

$$Q^T = \begin{pmatrix} 1 & 2 & 4 & 8 & 16 & 32 & \cdots \\ 4 & 10 & 24 & 56 & 128 & 288 & \cdots \\ 5 & 20 & 64 & 184 & 496 & 1280 & \cdots \\ 9 & 38 & 140 & 464 & 1424 & 4128 & \cdots \\ 14 & 66 & 272 & 1008 & 3440 & 11008 & \cdots \\ 23 & 112 & 496 & 2000 & 7440 & 25888 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 10 & 4 & 0 & 0 & 0 & \cdots \\ 9 & 20 & 24 & 8 & 0 & 0 & \cdots \\ 14 & 38 & 64 & 56 & 16 & 0 & \cdots \\ 23 & 66 & 140 & 184 & 128 & 32 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ 0 & 0 & 1 & 3 & 6 & 10 & \cdots \\ 0 & 0 & 0 & 1 & 4 & 10 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 5 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^2, \tag{16}$$

which gives us $|Q^T_{n\times n}| = |Q_{n\times n}| = 2^{n(n-1)/2}$, in agreement with Ollerton and Shannon's prediction.

Now, Ollerton and Shannon's results only hold for arrays with recursion relations. We can generate other (GP) matrices without such relations. For instance,

$$N = \begin{pmatrix} 1 & 3 & 9 & 27 & 81 & 243 & \cdots \\ 3 & 11 & 39 & 135 & 459 & 1539 & \cdots \\ 11 & 35 & 114 & 378 & 1269 & 4293 & \cdots \\ 50 & 152 & 471 & 1489 & 4800 & 15750 & \cdots \\ 274 & 824 & 2511 & 7753 & 24245 & 76737 & \cdots \\ 1764 & 5294 & 16038 & 49036 & 151263 & 470553 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{17}$$

is a (GP) matrix with ratio 3 based on the Stirling numbers of the first kind. It cannot be generated by an Ollerton-Shannon type recursion, as the constants of the progressions do not form a regular geometric sequence. However, since

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 11 & 2 & 3 & 0 & 0 & 0 & \cdots \\ 50 & 2 & 9 & 4 & 0 & 0 & \cdots \\ 274 & 2 & 33 & 4 & 5 & 0 & \cdots \\ 1764 & 2 & 150 & 4 & 15 & 6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ 0 & 0 & 1 & 3 & 6 & 10 & \cdots \\ 0 & 0 & 0 & 1 & 4 & 10 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 5 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^3, \tag{18}$$

it can be easily seen that $|N_{n\times n}| = n!$.

5. PRESERVATIVE DECOMPOSITIONS

An immediate consequence of Theorem 2 is that the rectangular Pascal matrix can be decomposed into the product of the lower triangular Pascal matrix and the upper triangular Pascal matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & 4 & \cdots \\ 0 & 0 & 1 & 3 & 6 & \cdots \\ 0 & 0 & 0 & 1 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{19}$$

This matrix is special in that its seed matrix is just a shifted copy of the original matrix. It is natural to ask if other matrices share this property.

Further study of (16) shows that the seed matrix of Q^T is the exact lower triangular form of Q^T. Similarly, for Vieta's array (c.f. Table 2 in [7]):

$$\begin{pmatrix} 1 & 2 & 2 & 2 & 2 & \cdots \\ 1 & 3 & 5 & 7 & 9 & \cdots \\ 1 & 4 & 9 & 16 & 25 & \cdots \\ 1 & 5 & 14 & 30 & 55 & \cdots \\ 1 & 6 & 20 & 50 & 105 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & \cdots \\ 0 & 1 & 3 & 5 & 7 & \cdots \\ 0 & 0 & 1 & 4 & 9 & \cdots \\ 0 & 0 & 0 & 1 & 5 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{20}$$

Finally, we decompose the Fibonacci array given by Table 3 in [7]:

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 5 & \cdots \\ 1 & 2 & 4 & 7 & 12 & \cdots \\ 1 & 3 & 7 & 14 & 26 & \cdots \\ 1 & 4 & 11 & 25 & 51 & \cdots \\ 1 & 5 & 16 & 41 & 92 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 & 3 & 5 & \cdots \\ 0 & 1 & 2 & 4 & 7 & \cdots \\ 0 & 0 & 1 & 3 & 7 & \cdots \\ 0 & 0 & 0 & 1 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{21}$$

Clearly, there are many such matrices. We would like to characterize them, and so we consider an array $T = (t_{i,j})$, where $t_{i,j}$ is the entry in the i^{th} row and j^{th} column of T.

Theorem 3 (Preservation Theorem): *Let b be a nonzero integer, let T_U stand for the upper triangular matrix form of T, and let P_L^b denote the b^{th} power of the lower triangular Pascal matrix. Then, $T = P_L^b \cdot T_U$ if and only if $t_{i,j} = bt_{i-1,j} + t_{i,j-1}$, for $i \geq 2$ and $j \geq 1$.*

We call such a decomposition a preservative decomposition, and say that T^T is a preservative (GP) matrix.

Proof: Suppose that $t_{i,j} = bt_{i-1,j} + t_{i,j-1}$, for $i \geq 2$ and $j \geq 1$. By Theorem 2 and the discussion in Section 4, we know that the transpose of T is a (GP) matrix, and hence that T itself can be decomposed into $P_L^b \cdot S$. Moreover,

$$\begin{aligned} S_{i,j} &= \Delta_b^{i-1} t_{1,j} = \Delta_b^{i-2}\{t_{2,j} - bt_{1,j}\} = \Delta_b^{i-2} t_{2,j-1} \\ &= \Delta_b^{i-3}\{t_{3,j-1} - bt_{2,j-1}\} = \Delta_b^{i-3} t_{3,j-2} \\ &\vdots \\ &= \begin{cases} \Delta_b^{i-i} t_{i,j-i+1}, & \text{if } i \leq j \\ 0, & \text{if } i > j \end{cases} = \begin{cases} t_{i,j-i+1}, & \text{if } i \leq j \\ 0, & \text{if } i > j \end{cases}. \end{aligned} \tag{22}$$

Thus, we have $T = P_L^b \cdot S = P_L^b \cdot T_U$.

Conversely, suppose that $T = P_L^b \cdot T_U$. We want to prove that $t_{i,j} = bt_{i-1,j} + t_{i,j-1}$, for $i \geq 2$ and $j \geq 1$. Since $T = P_L^b \cdot T_U$, then we have

$$\begin{aligned} T &= \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,n} \\ t_{2,1} & t_{2,2} & t_{2,3} & \cdots & t_{2,n} \\ t_{3,1} & t_{3,2} & t_{3,3} & \cdots & t_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n,1} & t_{n,2} & t_{n,3} & \cdots & t_{n,n} \end{pmatrix} \\ &= \begin{pmatrix} \binom{0}{0}b^0 & 0 & 0 & \cdots & 0 \\ \binom{1}{0}b^1 & \binom{1}{1}b^0 & 0 & \cdots & 0 \\ \binom{2}{0}b^2 & \binom{2}{1}b^1 & \binom{2}{2}b^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n-1}{0}b^{n-1} & \binom{n-1}{1}b^{n-2} & \binom{n-1}{2}b^{n-3} & \cdots & \binom{n-1}{n-1}b^0 \end{pmatrix} \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,n} \\ 0 & t_{2,1} & t_{2,2} & \cdots & t_{2,n-1} \\ 0 & 0 & t_{3,1} & \cdots & t_{3,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t_{n,1} \end{pmatrix}. \end{aligned} \tag{23}$$

Equating the corresponding entries of matrices in the two sides of equation (23) and solving the system of equations for $t_{i,j}$ yields

$$T = \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,n} \\ t_{2,1} & t_{2,2} & t_{2,3} & \cdots & t_{2,n} \\ t_{3,1} & t_{3,2} & t_{3,3} & \cdots & t_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n,1} & t_{n,2} & t_{n,3} & \cdots & t_{n,n} \end{pmatrix} = \mathrm{diag}(1, b^1, b^2, \ldots, b^{n-1})$$
$$\cdot \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{n-1}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots & \binom{n}{1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n-2}{n-2} & \binom{n-1}{n-2} & \binom{n}{n-2} & \cdots & \binom{2n-3}{n-2} \end{pmatrix} \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,n} \\ 0 & t_{1,1} & t_{1,2} & \cdots & t_{1,n-1} \\ 0 & 0 & t_{1,1} & \cdots & t_{1,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t_{1,1} \end{pmatrix}. \tag{24}$$

Therefore, we obtain

$$t_{i,j} = \begin{cases} \sum_{k=0}^{j-1} \binom{i-2+k}{i-2} b^{i-1} t_{1,j-k}, & \text{if } i \geq 2, j \geq 1 \\ t_{i,j}, & \text{if } i = 1, j \geq 1 \end{cases}. \tag{25}$$

Next, we calculate $bt_{i-1,j} + t_{i,j-1}$, for $i \geq 3$ and $j = 1, 2, 3, \ldots$. Using equation (25) and noting $t_{i,j} = 0$ for $j \leq 0$ leads to

$$\begin{aligned} bt_{i-1,j} + t_{i,j-1} &= \sum_{l=0}^{j-1} \binom{i-3+l}{i-3} b^{i-1} t_{1,j-l} + \sum_{m=0}^{j-2} \binom{i-2+m}{i-2} b^{i-1} t_{1,j-m-1} \\ &= \binom{i-3}{i-3} b^{i-1} t_{1,j} + \sum_{s=1}^{j-1} \left\{ \binom{i-3+s}{i-3} + \binom{i-3+s}{i-2} \right\} b^{i-1} t_{1,j-s}. \end{aligned} \tag{26}$$

By the combinatorial identity $\binom{n-1}{m-1} + \binom{n-1}{m} = \binom{n}{m}$, we obtain

$$bt_{i-1,j} + t_{i,j-1} = \binom{i-3}{i-3} b^{i-1} t_{1,j} + \sum_{k=1}^{j-1} \binom{i-2+k}{i-2} b^{i-1} t_{1,j-k} = t_{i,j}. \tag{27}$$

For $i = 2$, we have

$$bt_{i,j} + t_{2,j-1} = bt_{1,j} + \sum_{k=0}^{j-2} \binom{k}{0} bt_{1,j-k-1} = \sum_{k=0}^{j-1} \binom{k}{0} bt_{1,j-k} = t_{2,j}. \tag{28}$$

This completes the proof, and characterizes a subset of the matrices defined in [7] which are preservative. In fact, this characterization is complete: all preservative (GP) matrices are of this sort.

6. PRESERVATIVE MATRICES AND THEIR GENERATING FUNCTIONS

Another important application of Pascal decompositions is to determine the generating functions of the columns of the preservative arrays discussed in the last section. Suppose T is such an array, and let $t_{i,j}$ be the entry in the i^{th} row and j^{th} column of T.

Theorem 4: *If T is a preservative (GP) matrix with ratio b, then the generating function for the j^{th} column of T is the convolution*

$$\sum_{k=1}^{j} t_{k,j-k+1} \frac{x^{k-1}}{(1-bx)^k}.$$

Proof: By Theorem 3, we have

$$\overline{X} \cdot T = (1, x, x^2, x^3, \dots) \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & \cdots \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & \cdots \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & \cdots \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= (1, x, x^2, x^3, \dots) P_L^b \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & \cdots \\ 0 & t_{2,1} & t_{2,2} & t_{2,3} & \cdots \\ 0 & 0 & t_{3,1} & t_{3,2} & \cdots \\ 0 & 0 & 0 & t_{4,1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{29}$$

$$= \left(\frac{1}{1-bx}, \frac{x}{(1-bx)^2}, \frac{x^2}{(1-bx)^3}, \frac{x^3}{(1-bx)^4}, \dots \right) \begin{pmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & \cdots \\ 0 & t_{2,1} & t_{2,2} & t_{2,3} & \cdots \\ 0 & 0 & t_{3,1} & t_{3,2} & \cdots \\ 0 & 0 & 0 & t_{4,1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This last step follows from the generating functions given for P_L in [1]. From here, we can see that the generating function for the k^{th} column of T is

$$C_k(x) = t_{1,k} \frac{1}{1-bx} + t_{2,k-1} \frac{x}{(1-bx)^2} + t_{3,k-2} \frac{x^2}{(1-bx)^3} + \cdots + t_{k,1} \frac{x^{k-1}}{(1-bx)^k}. \tag{30}$$

This completes the proof of the theorem.

As an example, suppose we want to determine the generating functions of the columns of Vieta's array from equation (2). Because it satisfies the conditions of Theorem 3, it is a preservative matrix. Hence we can immediately write the generating functions of its columns using equation (29):

$$\left(\frac{1}{1-x}, \frac{1}{1-x} \left(\frac{x}{1-x} \right)^1, \frac{1}{1-x} \left(\frac{x}{1-x} \right)^2, \dots \right) \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 & \cdots \\ 0 & 1 & 3 & 5 & 7 & 9 & \cdots \\ 0 & 0 & 1 & 4 & 9 & 16 & \cdots \\ 0 & 0 & 0 & 1 & 5 & 14 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 6 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{31}$$

From [4], we know that the k^{th} column generating function of the upper triangular array in equation (31) is $f_k(x) = (x+2)(x+1)^{k-2}$, for $k \geq 2$. Thus, the column generating functions of Vieta's array are

$$\frac{1}{1-x}\left(1, f_2\left(\frac{x}{1-x}\right), f_3\left(\frac{x}{1-x}\right), f_4\left(\frac{x}{1-x}\right), \dots\right)$$

$$= \left(\frac{1}{1-x}, \frac{(2-x)}{(1-x)^2}, \frac{(2-x)}{(1-x)^3}, \frac{(2-x)}{(1-x)^4}, \dots\right). \tag{32}$$

The new technique provides a highly desirable alternative: consider how much work was needed in [5] to derive the row generating functions for a convolution matrix. Possibilities for further investigation include reanalyzing the row generating functions for a convolution matrix and finding the column generating functions for a recursion relation matrix. We hope that the Pascal decompositions developed here may also shed some light on these problems.

7. CONCLUSION

Pascal decompositions, generalized to the geometric case, are an interesting new tool for understanding a rather broad category of matrices. Within the broad category, several subcategories with handy properties have been singled out, such as convolution matrices, recursion relation matrices and preservative matrices. In the future, we hope to apply these techniques to understand the inverses of these matrices.

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