SOME PROPERTIES OF THE SEQUENCES
$C_{n,3} = C_{n-1,3} + C_{n-3,3} + r$

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1. INTRODUCTION

In this note we introduce the sequence $\{C_{n,3}(a,b,r)\}$ defined by the recurrence relation

$$C_{n,3}(a,b,r) = C_{n-1,3}(a,b,r) + C_{n-3,3}(a,b,r) + r \tag{1.1}$$

with $C_{0,3}(a,b,r) = b - a - r,\ C_{1,3}(a,b,r) = a,\ C_{2,3}(a,b,r) = b$ where r is a constant.

Recall that the squences $\{C_n(a,b,r)\}$ were studied in [1], [2], [3], and [4].

Observe that the sequences $\{C_{n,3}(a,b,r)\}$, defined by (1.1), are a generalization of the sequences $\{C_n(a,b,r)\}$. Further, we use the shorter notation $\{C_{n,3}\}$ instead of $\{C_{n,3}(a,b,r)\}$.

The purpose of this note is to establish some properties of $C_{n,3}$ by using methods similar to Zh. Zhang [4], and Hsu and Maosen [3].

First, we introduce the following operators:

I will be the identity operator;

E represents the shift operator;

E_i is the "i^{th} coordinate" operator $(i = 1, 2)$;

$\nabla = I - 2E_1 + E_2$;

$\nabla_1 = I - E_1 + 2E_2$;

$\nabla_2 = I + 4E_1 + E_2$.

$\nabla_3 = I + 4E_1 + 2E_2$.

Also, we use the notation

$$\binom{n}{i,j} = \frac{n!}{i!j!(n-i-j)!}.$$

Now, by using the identity

$$(a+b+c)^n = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}a^i b^j c^{n-i-j} = \sum_{i+j+l=n}\binom{n}{i,j}a^i b^j c^{n-i-j},$$

we get respectively:

$$\nabla^n = \sum_{i+j+l=n}\binom{n}{i,j}(-1)^i 2^i E_1^i E_2^j; \tag{1.2}$$

$$\nabla_1^n = \sum_{i+j+l=n}\binom{n}{i,j}(-1)^i 2^j E_1^i E_2^j; \tag{1.3}$$

$$\nabla_2^n = \sum_{i+j+l=n} \binom{n}{i,j} 4^i E_1^i E_2^j. \tag{1.4}$$

$$\nabla_3^n = \sum_{i+j+l=n} \binom{n}{i,j} 2^{2i+j} E_1^i E_2^j. \tag{1.5}$$

Namely, when we apply the operators in (1.2), (1.3), (1.4) and (1.5) to any function $f(i,j)$ (Proposition 1, [4] or [1]), we get respectively:

$$g(n,k) = \nabla^n f(0,k) = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^i f(i, j+k);$$

$$g(n,k) = \nabla_1^n f(0,k) = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^j f(i, j+k);$$

$$g(n,k) = \nabla_2^n f(0,k) = \sum_{i+j+l=n} \binom{n}{i,j} 4^i f(i, j+k);$$

$$g(n,k) = \nabla_3^n f(0,k) = \sum_{i+j+l=n} \binom{n}{i,j} 2^{2i+j} f(i, j+k).$$

2. PROPERTIES OF THE SEQUENCE $\{C_{n,3}\}$

Lemma 1:

$$C_{k,3} + 2C_{k+1,3} + C_{k+7,3} = 4C_{k+4,3}, \tag{2.1}$$
$$C_{k,3} = C_{k-1,3} + C_{k-2,3} - C_{k-5,3}, \tag{2.2}$$
$$C_{k,3} = 2C_{k-1,3} - 2C_{k-4,3} + C_{k-7,3}, \tag{2.3}$$

where k is a nonnegative integer.

Proof: Using relation (1.1), we get

$$\begin{aligned}
&C_{k,3} + 2C_{k+1,3} + C_{k+7,3} = C_{k+3,3} \\
&- C_{k+2,3} - r + 2(C_{k+4,3} - C_{k+3,3} - r) + C_{k+6,3} + C_{k+r,3} + r = \\
&3C_{k+4,3} - C_{k+3,3} - C_{k+2,3} - 2r + C_{k+6,3} = \\
&3C_{k+4,3} - C_{k+3,3} - 2r - C_{k+2,3} + C_{k+5,3} + C_{k+3,3} + r = \\
&3C_{k+4,3} - r - C_{k+2,3} + C_{k+4,3} + C_{k+2,3} + r = 4C_{k+4,3}.
\end{aligned}$$

Hence, it follows that (2.1) is true.

Again, using recurrence relation (1.1), it is easy to prove equalities (2.2) and (2.3). □

Theorem 1:

$$C_{4n+k,3} = \sum_{i+j+l-n} \binom{n}{i,j} 2^{i-2n} C_{i+7(j+k),3}, \tag{2.4}$$

$$(-1)^n C_{n+7k,3} = \sum_{i+j+l=n} \binom{n}{i,j} 2^{i-n} C_{4i+7(j+k),3}, \tag{2.5}$$

where n and k are nonnegative integers.

Proof: Let $f(i,j) = (-1)^i C_{7i+j,3}$. Then

$$\begin{aligned}\nabla_i f(i,j) =& (-1)^i (C_{7i+j,3} + C_{7i+7+j,3} + 2C_{7i+j+1,3}) = \\ & (-1)^i 4C_{7i+4+j,3} = 4E_2^4 f(i,j).\end{aligned}$$

Thus, we get

$$g(n,k) = \nabla_1^n f(0,k) = 4^n E_2^{4n} f(0,k).$$

Moreover, by (1.3), we have

$$4^n C_{4n+k,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^j C_{7i+k+j,3}.$$

Or, if $f(i,j) = (-1)^i C_{i+7j,3}$, then

$$\begin{aligned}\nabla f(i,j) =& (-1)^i (C_{i+7j,3} + 2C_{i+1+7j,3} + C_{i+7j+7,3}) = \\ & (-1)^i 4C_{i+4+7j,3} = 4E_1^4 f(i,j).\end{aligned}$$

Hence, from (1.2), we get

$$4^n E_1^{4n} f(0,k) = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^j f(i,j+k) = \sum_{i+j+l=n} \binom{n}{i,j} 2^i C_{i+7(j+k),3}.$$

Namely, we get the following identity

$$4^n C_{4n+7k,3} = \sum_{i+j+l=n} \binom{n}{i,j} 2^i C_{i+7(j+k),3}.$$

Now, let $g(i,j) = (-1)^i C_{4i+7j,3}$. Applying the operator ∇_2 to $g(i,j)$, we get

$$\nabla_2 g(i,j) = (-1)^i (C_{4i+7j,3} - 4C_{4i+4+7j,3} + C_{4i+7j+7,3}) =$$

$$(-1)^i ((-2) C_{4i+7j+1,3}) = -2E_1^{\frac{1}{4}} g(i,j).$$

So,

$$(-2)^n E_1^{\frac{n}{4}} g(0,k) = \nabla_2^n g(0,k) = (-2)^n C_{n+7k,3} = \sum_{i+j+l=n} \binom{n}{i,j} 4^i C_{4i+7(j+k),3}.$$

Hence, it follows that

$$(-1)^n C_{n+7k,3} = \sum_{i+j+l=n} \binom{n}{i,j} 2^{2i-n} C_{4i+7(j+k),3}. \quad \square$$

Corollary 1: *For $k=0$ in (2.4) and (2.5), we get respectively:*

$$C_{4n,3} = \sum_{i+j+l=n} \binom{n}{i,j} 2^{i-2n} C_{i+7j,3},$$

and

$$(-1)^n C_{n,3} = \sum_{i+j+l=n} \binom{n}{i,j} 2^{i-n} C'_{4i+7j,3}$$

where n is a nonnegative integer.

Theorem 2:

$$4^n C_{4n+k,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^j C_{7i+j+k,3}, \tag{2.6}$$

$$(-1)^n C_{7n+k,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^{2i+j} C_{4i+j+k,3}, \tag{2.7}$$

where n is a nonnegative integer.

Proof: Let $f(i,j) = (-1)^i C_{7i+j,3}$. Then

$$\begin{aligned}\nabla_1 f(i,j) =& (-1)^i (C_{7i+j,3} + C_{7i+7+j,3} + 2C_{7i+j+1,3}) = \\ & (-1)^i (4C_{7i+4+j,3} - C_{7i+j+7,3} + C_{7i+7+j,3}) \\ & (-1)^i 4C_{7i+4+j,3} = 4E_2^4 f(i,j).\end{aligned}$$

It follows that

$$\nabla_1^n f(0,k) = 4^n E_2^{4n} f(0,k) = 4^n C_{4n+k,3}.$$

By (1.3), we have

$$4^n C_{4n+k,3} = \sum_{i+j+l=n} \binom{n}{i,j} 2^i C_{7i+j+k,3}.$$

Let $g(i,j) = (-1)^i C_{4i+j,3}$. Then

$$\begin{aligned}\nabla_3 g(i,j) =& (-1)^i (C_{4i+j,3} - 4C_{4i+4+j,3} + 2C_{4i+j+1,3}) = \\ & (-1)^i (4C_{4i+j+4,3} - C_{4i+j+7,3} - 4C_{4i+4+j,3}) = \\ & (-1)^i (-1) C_{4i+j+7,3} = -1 E_2^7 g(i,j).\end{aligned}$$

¿From the other side, by (1.4), we get

$$\nabla_3^n g(0,k) = (-1)^n E_2^{7n} g(0,k) = (-1)^n C_{k+7n,3} = \sum_{i+j+l=n} \binom{n}{i,j} 2^{2i+j} (-1)^i C_{4i+j+k,3}. \quad \square$$

Corollary 2: *If $k = 0$ in (2.6) and (2.7), then we obtain respectively*

$$4^n C_{4n,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^j C_{7i+j,3},$$

$$C_{7n,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^{i+n} 2^{2i+j} C_{4i+j,3},$$

where n is a nonnegative integer.

Proposition 1: *If a sequence $\{X_n\}$ satisfies the relations*

$$\begin{aligned} X_n &= X_{n-1} + X_{n-2} - X_{n-5}, \qquad n \geq 5, \\ X_n &= 2X_{n-1} - 2X_{n-4} + X_{n-7}, \qquad n \geq 7, \end{aligned}$$

then the operators

$$\begin{aligned} I &= E^{-1} + E^{-2} - E^{-5}, \\ I &= 2E^{-1} - 2E^{-4} + E^{-7}, \end{aligned}$$

are the identity operators. Hence, we get the following identity operators

$$I^n (= I) = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^{n-i-j} E^{-5n+4i+3j}, \tag{2.8}$$

$$I^n (= I) = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^{i+j} E^{-7n+3j+6i}, \tag{2.9}$$

where n is a nonnegative integer. Applying the operators in (2.8) and (2.9) to the sequences $\{X_{5n+k}\}$ and $\{X_{7n+k}\}$, where n and k are nonnegative integers, we get

$$X_{5n+k} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^{n-i-j} X_{4i+3j+k},$$

$$X_{7n+k} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^{i+j} X_{6i+3j+k}.$$

Theorem 3:

$$C_{5n+k,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^{n-i-j} C_{4i+3j+k,3}; \tag{2.10}$$

$$C_{7n+k,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^{i+j} C_{6i+3j+k,3}; \tag{2.11}$$

where n is a nonnegative integer.

The proof of the last theorem can be realized by using Lemma 1 and Proposition 1. □

Corollary 3: *If $k = 0$, then (2.10) and (2.11) become the following equalities respectively:*

$$C_{5n,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^{n-i-j} C_{4i+3j,3}$$

and

$$C_{7n,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^{n-i-j} C_{4i+3j,3},$$

where n is a nonnegative integer.

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