p-ADIC INTERPOLATION OF THE FIBONACCI SEQUENCE VIA HYPERGEOMETRIC FUNCTIONS

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1. INTRODUCTION

Many authors have considered the problem of extending the Fibonacci sequence to arbitrary real or complex subscripts (cf. [1], [6], and references therein). Since the positive integers form a discrete subset of $\mathbb{R}$ the existence of multitudes of continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(n)=F_n$ for positive integers n is immediate and the question then becomes one of determining the various properties of such functions. In this paper we consider the extent to which the Fibonacci and Lucas sequences can be extended to arbitrary p-adic subscripts in a continuous way. In the process we determine several apparently new expressions, both p-adic and real, for the Fibonacci sequence in terms of hypergeometric functions and combinatorial sums.

For example, Dilcher ([3], eq. (3.3)) has proved that for positive integers n,

$$F_n = \frac{n}{2^{n-1}} F\left(\frac{1-n}{2}, \frac{2-n}{2}; \frac{3}{2}; 5\right), \tag{1.1}$$

where $F(a,b;c;z)$ is the Gauss hypergeometric function (see section 2). We have observed (Theorem 2.3 below) that for $p=5$ the hypergeometric function on the right in (1.1) in fact represents a continuous function of n from $\mathbb{Z}_p$ to $\mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the ring of p-adic integers. This means that the function $f:\mathbb{Z}_5\to\mathbb{Z}_5$ defined by

$$f(x) = 2xF\left(\frac{1-x}{2}, \frac{2-x}{2}; \frac{3}{2}; 5\right) \tag{1.2}$$

is 5-adically continuous and satisfies $f(n)=2^nF_n$ for all integers n, i.e., it 5-adically interpolates the sequence $\{2^nF_n\}$. In section 3 below we give generalizations of identity (1.1) which yield similar p-adic expressions for any prime p.

We say that a sequence $\{a_n\}_{n=0}^{\infty}$ of rational numbers is *p-adically interpolatable* if there exists a continuous function $f:\mathbb{Z}_p\to\mathbb{Q}_p$ such that $f(n)=a_n$ for all nonnegative integers n. Since the set of nonnegative integers is dense in $\mathbb{Z}_p$, for a given sequence $\{a_n\}$ there can be at most one such function, which will only exist under certain strong conditions on $\{a_n\}$. Specifically, an integer sequence is p-adically interpolatable if and only if it is purely periodic modulo p^M for all positive integers M, with each period a power of p (Proposition

2.1 below). While $\{F_n\}$ is purely periodic modulo p for every prime p, its period modulo p is never a power of p, which means that the Fibonacci sequence itself can never be p-adically interpolated. However, we show in section 3 that for odd primes p, the sequence $\{s^n F_{r_p n}\}$ can be p-adically interpolated, where s is a suitable integer and r_p is the rank of apparition of p in $\{F_n\}$. Our results illustrate many of the known periodicity properties of Fibonacci numbers; as one such illustration, we show that the sequence $\{F^4_{r_p n}\}$ can always be p-adically interpolated for any odd prime p.

While our primary interest in this topic has been the p-adic properties of these hypergeometric identities, we also present several real identities which we believe to be new, generalizing those in [3]. Our identities in section 3 show how the finite-sum rational-argument identities of [3] may be classified into six infinite families, which may be obtained without quadratic transformations. These six families are all related by linear hypergeometric transformations, and four of them yield p-adically continuous representations of the Fibonacci sequence. We also show that in fact infinitely many different rational arguments occur in hypergeometric representations of Fibonacci numbers, answering a question posed in [3]. All these identities are also expressed as combinatorial sums, as in [3].

2. NOTATIONS AND PRELIMINARIES

In this paper p will always denote a prime number, and $\mathbb{Z}_p$ and $\mathbb{Q}_p$ the ring of p-adic integers and the field of p-adic numbers, respectively. If x is a nonzero rational number we can write $x = p^k r/s$ where $k, r, s \in \mathbb{Z}$ and $(r, p) = (s, p) = 1$. The integer k is called the *p-adic ordinal of x* and denoted $k = \mathrm{ord}_p x$, and the *p-adic absolute value of x* is then defined by $|x|_p = p^{-k}$. We define $\mathrm{ord}_p 0 = +\infty$ and $|0|_p = 0$. With this definition $|\cdot|_p$ is a (non-archimedean) metric on $\mathbb{Q}$ and $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to this metric. The ring $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \le 1\} = \{x \in \mathbb{Q}_p : \mathrm{ord}_p x \ge 0\}$ may be viewed as the "unit disk" in $\mathbb{Q}_p$, or as the completion of the ring $\mathbb{Z}$ of integers with respect to the metric $|\cdot|_p$.

Because $\mathbb{Z}_p$ is a compact metric space, any continuous function on $\mathbb{Z}_p$ is uniformly continuous. For integers x, y it is clear that $|x-y|_p \le p^{-k}$ if and only if $x \equiv y \pmod{p^k}$. Therefore we have the following proposition (cf. [5]):

Proposition 2.1: *The integer sequence $\{a_n\}$ is p-adically interpolatable if and only if for every $M > 0$ there exists $N \ge 0$ such that $a_m \equiv a_n \pmod{p^M}$ whenever $m \equiv n \pmod{p^N}$.*

This condition is equivalent to the condition that $\{a_n\}$ be purely periodic modulo p^M for every M, with period equal to a power of p. As an example, for an integer a the sequence $\{a^n\}$ can be p-adically interpolated only if $a \equiv 1 \pmod{p}$, since this sequence is not purely periodic modulo p if $p|a$, and any period modulo p is a multiple of the order of a in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ otherwise; the period modulo p can be a power of p only if the period is 1. Consequently for $a \in \mathbb{Q}$ the sequence $\{a^n\}$ can be p-adically interpolated only if $a \equiv 1 \pmod{p\,\mathbb{Z}_p}$, that is, $a = b/c$ with $b \equiv c \not\equiv 0 \pmod{p}$. This fact is reflected in the example in the introduction (where we noted that $\{2^n F_n\}$ is 5-adically interpolatable but $\{F_n\}$ is not), and should be compared to the situation in $\mathbb{R}$, where the function $f(x) = a^x$ is defined (and continuous) for all real x only when $a > 0$.

For a prime number p we denote by t_p the minimal period of $\{F_n\}$ modulo p. The *rank of apparition of p in $\{F_n\}$*, denoted r_p, is the least positive integer r such that $p|F_r$. It is known ([9], §2) that r_p always exists and is equal to the minimal *restricted period* of $\{F_n\}$ modulo p, which is to say the least positive integer r such that $F_{k+r} \equiv sF_k \pmod{p}$ for some integer s and all integers k. It is also easily seen that $s \equiv F_{r_p+1} \pmod{p}$, that $p|F_m$ if and only if $r_p|m$, and that $t_p = a_p r_p$, where a_p is the order of the integer s in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$. The *rank of apparition of* p *in* $\{L_n\}$, denoted r'_p, is the least positive integer r', if one exists, such that $p|L_{r'}$. It is known ([9], §2) that r'_p exists if and only if r_p is even, and in this case $r_p = 2r'_p$. For $p \neq 5$ these invariants may be expressed in terms of the p-adic interpolatability of the Fibonacci sequence, as follows.

Theorem 2.2: *Suppose the sequence* $\{s^n F_{mn+k}\}_{n=0}^{\infty}$ *is* p*-adically interpolatable for some prime* p*, nonzero rational number* s*, and integers* m*,* k*. If* $p \neq 5$*, then* $r_p|m$ *and* $sF_{m+1} \equiv 1 \pmod{p\mathbb{Z}_p}$.

Proof: If a, b are any integers then the sequence $\{F_{mn+b}\}$ is a rational linear combination of $\{F_{mn+a}\}$ and $\{F_{m(n+1)+a}\}$. It follows that if $\{s^n F_{mn+k}\}$ is p-adically interpolatable for some integer k then it is for every k. Since $\lim_{j\to\infty} 1 + p^j = 1$ in $\mathbb{Z}_p$, if there is a continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ such that $f(n) = s^n F_{mn+1}$ for positive integers n then we must have $\lim_{j\to\infty} s^{1+p^j} F_{m(1+p^j)+1} = sF_{m+1}$ in $\mathbb{Q}_p$. However if $\mathrm{ord}_p s > 0$ then this limit is zero, and if $\mathrm{ord}_p s < 0$ then the p-adic ordinal of $s^{1+p^j} F_{m(1+p^j)+1}$ is unbounded below and thus the limit does not exist in $\mathbb{Q}_p$. Therefore we conclude that $\mathrm{ord}_p s = 0$.

Since $\mathrm{ord}_p s = 0$, we have $s^n F_{mn} \equiv 0 \pmod{p\mathbb{Z}_p}$ if and only if $r_p/(m, r_p)$ divides n, and thus the period of $\{s^n F_{mn}\}$ modulo p is a multiple of $r_p/(m, r_p)$. By Proposition 2.1, if $\{s^n F_{mn}\}$ is to be p-adically interpolatable $r_p/(m, r_p)$ must be a power of p. However, if $p \neq 5$ then $(p, r_p) = 1$ ([9], §2), which demands that $r_p/(m, r_p) = 1$, so $r_p|m$. Modulo p the sequence $\{s^n\}$ has period dividing $p-1$ and, if $p \neq 5$, the sequence $\{F_{mn+1}\}$ has period dividing $p^2 - 1$ ([9], §2). The period modulo p of $\{s^n F_{mn+1}\}$ thus divides $p^2 - 1$, and so if this period is a power of p then it must equal 1. Therefore $sF_{m+1} \equiv 1 \pmod{p\mathbb{Z}_p}$ if $p \neq 5$.

The *Gauss hypergeometric series* $F(a, b; c; z)$ is defined by

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \tag{2.1}$$

where the *Pochhammer symbol* $(a)_k$ is defined by $(a)_0 = 1$ and $(a)_k = \prod_{i=0}^{k-1}(a+i)$ for $k \geq 1$ (cf. [2], [3]). In general a, b, c, z may take any real or complex (or p-adic) value, although in all our identities they will be rational numbers. The series is undefined if $c \in \{0, -1, -2, -3, ...\}$, unless either a or b is a larger element of $\{0, -1, -2, -3, ...\}$. If the series is defined and either a or b lies in $\{0, -1, -2, -3, ...\}$, then the series represents a polynomial in z and is therefore defined for all (real, complex, or p-adic) z. Otherwise the series converges for real or complex z with $|z| < 1$ and diverges for $|z| > 1$. Furthermore, for $|z| < 1$ the function defined by (2.1) is analytic in a, b, and c on all of $\mathbb{R}$ (or $\mathbb{C}$) with the exception of simple poles at $c = 0$, $c = -1$, $c = -2$, $c = -3$, etc.

By a disk S in $\mathbb{Q}_p$ we mean a set of the form $\{x \in \mathbb{Q}_p : |x - x_0|_p \leq r\}$ or of the form $\{x \in \mathbb{Q}_p : |x - x_0|_p < r\}$, for some $x_0 \in \mathbb{Q}_p$ and some $r > 0$. Any disk in $\mathbb{Q}_p$ can be expressed in either of these forms, is both open and closed, and is compact. A function $f : S \to \mathbb{Q}_p$ defined on a disk S in $\mathbb{Q}_p$ is *analytic* if it can be represented by a power series which converges on S. Clearly $F(a, b; c; z)$ is analytic in z on any disk S on which it converges. The following theorem indicates that for fixed c and z, $F(a, b; c; z)$ is also continuous in a and b on any disk where it converges. (Since the set $\{0, -1, -2, -3, ...\}$ of c-values for which (2.1) is (in general) undefined is dense in $\mathbb{Z}_p$, the series is far from continuous in c on $\mathbb{Z}_p$. However, in the event that for fixed a, b, z there is a disk S in $\mathbb{Q}_p$ disjoint from $\mathbb{Z}_p$ for which $F(a, b; c; z)$ is convergent for $c \in S$, then $F(a, b; c; z)$ would be continuous in c on S).

Theorem 2.3: *Suppose $c \in \mathbb{Q}$ with $c \notin \{0, -1, -2, -3, ...\}$ and $a, b \in \mathbb{Q}_p$. Then the series $F(a, b; c; z)$ converges in $\mathbb{Q}_p$ for $\mathrm{ord}_p z > g(c) + g(1) - g(a) - g(b)$ and, if $a, b \in \mathbb{Q} \setminus \{0, -1, -2, -3, ...\}$, diverges for $\mathrm{ord}_p z \leq g(c) + g(1) - g(a) - g(b)$, where*

$$g(x) = \begin{cases} \mathrm{ord}_p x, & \text{if } x \notin \mathbb{Z}_p, \\ 1/(p-1), & \text{if } x \in \mathbb{Z}_p. \end{cases}$$

Furthermore, if such $b, c, z \in \mathbb{Q}_p$ are fixed so that $F(a, b; c; z)$ converges for $a \in S$, where S is a disk in $\mathbb{Q}_p$, then $F(a, b; c; z)$ represents a continuous function of a on S.

Remarks: The theorem remains valid if c is any element of $\mathbb{Q}_p \setminus \mathbb{Z}_p$, or if z is any element of the completion of an algebraic closure of $\mathbb{Q}_p$. Furthermore if $F(a, b; c; z)$ converges for $a \in S$, where S is some disk in $\mathbb{Q}_p$, then that disk may be assumed to contain $\mathbb{Z}_p$. In all the identities in this paper the parameters a, b, c are all rational, and all lie in $\mathbb{Z}_p$ except for $p = 2$. When $a, b, c \in \mathbb{Z}_p$ the series $F(a, b; c; z)$ therefore converges when $\mathrm{ord}_p z > 0$ and, for fixed z with $\mathrm{ord}_p z > 0$, represents a continuous function of a (and b) on $\mathbb{Z}_p$.

Proof: A series $\sum A_n$ in $\mathbb{Q}_p$ converges if and only if $|A_n|_p \to 0$, or equivalently, if and only if $\mathrm{ord}_p A_n \to +\infty$. Therefore $F(a, b; c; z)$ converges if and only if

$$\lim_{k \to \infty} \mathrm{ord}_p(a)_k + \mathrm{ord}_p(b)_k - \mathrm{ord}_p(c)_k - \mathrm{ord}_p(k!) + k \cdot \mathrm{ord}_p z = +\infty. \tag{2.2}$$

Since $|\cdot|_p$ is non-archimedean it follows readily that $\mathrm{ord}_p(x)_k = k \cdot \mathrm{ord}_p(x)$ when $x \in \mathbb{Q}_p \setminus \mathbb{Z}_p$. It is also well known that $\mathrm{ord}_p(k!) = \mathrm{ord}_p(1)_k = (k - S(k))/(p-1)$, where $S(k)$ denotes the sum of the digits in the base p expansion of k. We also have ([4], eq. (21.2.1)) the uniform estimates $\mathrm{ord}_p(x)_k \geq \mathrm{ord}_p(k!)$ for any $x \in \mathbb{Z}_p$, and $\mathrm{ord}_p(x)_k \leq \mathrm{ord}_p(k!) + \log_p(d) + \log_p(k + |x|)$ for a rational number $x \in \mathbb{Z}_p \setminus \{0, -1, -2, -3, ...\}$ with denominator d, where $\log_p$ is the (real) base p logarithm and $|\cdot|$ is the real absolute value. Noting that $S(k)/(p-1) \leq \log_p(k+1)$ for all positive integers k, we see that for $x \in \mathbb{Q}_p$ we have $\mathrm{ord}_p(x)_k \geq k \cdot g(x) - \log_p(k+1)$, and for $x \in \mathbb{Q} \setminus \{0, -1, -2, -3, ...\}$ we have $\mathrm{ord}_p(x)_k \leq k \cdot g(x) + O(\log_p k)$. The statement on convergence then follows from (2.2).

By considering the various cases (whether b and/or c lies in $\mathbb{Z}_p$ or not) and using (2.2), one observes that for fixed b, c, z the series converges precisely on a set $S \cup \{0, -1, -2, -3, ...\}$ where S is a disk in $\mathbb{Q}_p$ of the form $\{a \in \mathbb{Q}_p : \mathrm{ord}_p a > C\}$ for some constant C, and that the convergence is uniform in a on this set. Each term in the series (2.1) is a polynomial in a, and is therefore continuous in a. As a uniformly convergent sum on S of continuous functions in a on S, $F(a, b; c; z)$ is therefore a continuous function of a on S.

3. FINITE SUM IDENTITIES

In this section we give some identities for Fibonacci and Lucas numbers in terms of $F(a, b; c; z)$ with either a or b in $\{0, -1, -2, -3, ...\}$. Since the series represents a polynomial in this case, these are identities in $\mathbb{Q}$ and are thus valid independent of the metric (real or p-adic) on $\mathbb{Q}$. For each identity, however, we will indicate conditions under which the given hypergeometric function interpolates the given sequence of values in $\mathbb{Q}_p$ and in $\mathbb{R}$. Our treatment is similar to section 4 of [3]. We begin with the following fundamental identity.

Theorem 3.1: *For all integers m, n with $n > 0$, we have*

$$F_{mn} = nF_m \left(\frac{L_m}{2}\right)^{n-1} F\left(\frac{1-n}{2}, \frac{2-n}{2}; \frac{3}{2}; \frac{5F_m^2}{L_m^2}\right), \tag{3.1}$$

$$L_{mn} = L_m \left(\frac{L_m}{2}\right)^{n-1} F\left(\frac{-n}{2}, \frac{1-n}{2}; \frac{1}{2}; \frac{5F_m^2}{L_m^2}\right) \tag{3.2}$$

as identities in $\mathbb{Q}$. Therefore the functions $f_m, l_m : \mathbb{Z}_p \to \mathbb{Z}_p$ defined by

$$f_m(x) = \frac{2F_m}{L_m} x F\left(\frac{1-x}{2}, \frac{2-x}{2}; \frac{3}{2}; \frac{5F_m^2}{L_m^2}\right), \qquad \text{and}$$

$$l_m(x) = 2F\left(\frac{-x}{2}, \frac{1-x}{2}; \frac{1}{2}; \frac{5F_m^2}{L_m^2}\right)$$

are continuous on $\mathbb{Z}_p$ and satisfy $f_m(n) = (2/L_m)^n F_{mn}$ and $l_m(n) = (2/L_m)^n L_{mn}$ for positive integers n when $p = 5$; or when p is odd and $p|F_m$; or when $p = 2$ and $4|F_m$.

Proof: We substitute $z = \sqrt{5}F_m/L_m$ into the identity

$$F\left(a, \frac{1}{2} + a; \frac{3}{2}; z^2\right) = \frac{1}{2z(1-2a)} \left[(1+z)^{1-2a} - (1-z)^{1-2a}\right] \tag{3.3}$$

([3], eq. (3.2)) with $a = (1-n)/2$, and into

$$F\left(a, \frac{1}{2} + a; \frac{1}{2}; z^2\right) = \frac{1}{2} \left[(1+z)^{-2a} + (1-z)^{-2a}\right] \tag{3.4}$$

([3], eq. (10.15)) with $a = -n/2$, and compare these results with the Binet forms $F_{mn} = (\alpha^{mn} - \beta^{mn})/\sqrt{5}$ and $L_{mn} = \alpha^{mn} + \beta^{mn}$, where $\{\alpha^m, \beta^m\} = \{(L_m \pm F_m\sqrt{5})/2\}$. These identities are valid because the series terminate, and give the identities of the Theorem. When $p > 2$ the parameters a, b, c all lie in $\mathbb{Z}_p$ and therefore by Theorem 2.3 the series represent continuous functions of n on $\mathbb{Z}_p$ when $\mathrm{ord}_p(5F_m^2/L_m^2) > 0$, which is precisely when $p = 5$ or when $p|F_m$. If $p = 2$ then in both identities $\mathrm{ord}_2 c = -1$, one of a, b lies in $\mathbb{Z}_2$ and the other has 2-adic ordinal equal to -1; thus by Theorem 2.3 the series represent continuous functions of n on $\mathbb{Z}_2$ when $\mathrm{ord}_2(5F_m^2/L_m^2) > 0$, which is precisely when $4|F_m$.

Remarks: Dilcher's identities (1.1) and ([3], eq. (10.16)) may be obtained by taking $m = 1$ in this theorem, and ([3], eq. (4.23)) is obtained by taking $m = 2$. In $\mathbb{R}$ we have $|5F_m^2/L_m^2| < 1$ precisely when m is even, so the functions $f_m(x)$ and $l_m(x)$ also define analytic functions on all of $\mathbb{R}$ (or $\mathbb{C}$) for even m. Taking $m = 1$ in the theorem shows that $\{2^n F_n\}$ and $\{2^n L_n\}$ are 5-adically interpolatable. Therefore the fourth powers $\{16^n F_n^4\}$ and $\{16^n L_n^4\}$ are also 5-adically interpolatable, and since $16 \equiv 1 \pmod 5$, the sequence $\{16^{-n}\}$ is 5-adically interpolatable. It follows that both $\{F_n^4\}$ and $\{L_n^4\}$ are 5-adically interpolatable.

Since $L_m = F_{m-1} + F_{m+1}$, if an integer q divides F_m then $L_m \equiv 2F_{m+1} \pmod q$. So if $m = t_p$ for an odd prime p then $L_m/2 \equiv 1 \pmod{p\mathbb{Z}_p}$ and $\{(L_m/2)^n\}$ is p-adically interpolatable, which implies that $\{F_{t_p n}\}$ and $\{L_{t_p n}\}$ are p-adically interpolatable. Similarly with $q = 4$ and $m = 6$ we find that $\{F_{6n}\}$ and $\{L_{6n}\}$ are 2-adically interpolatable. If $m = r_p$ for an odd prime p then since $L_m \equiv 2F_{m+1} \pmod{p}$, $L_m/2$ reduces modulo p to $s = F_{m+1}$, which is an element of $(\mathbb{Z}/p\mathbb{Z})^\times$ of multiplicative order $a_p = t_p/r_p$. It follows that the sequences $\{s^n F_{r_p n}\}$ and $\{s^n L_{r_p n}\}$ can be p-adically interpolated for any integer s such that $sF_{r_p+1} \equiv 1 \pmod{p}$, and that $\{F_{r_p n}^{a_p}\}$ and $\{L_{r_p n}^{a_p}\}$ can be p-adically interpolated. We remark that Somer has shown ([9], Theorem 13) that $a_p \in \{1, 2, 4\}$ for every prime p. This means that $\{F_{r_p n}\}$ and $\{L_{r_p n}\}$ are (at worst) fourth roots of continuous functions of n on $\mathbb{Z}_p$ for odd primes p.

The identities of this theorem may be put in combinatorial form, giving

$$F_{mn} = F_m \left(\frac{L_m}{2}\right)^{n-1} \cdot \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \left(\frac{5F_m^2}{L_m^2}\right)^k, \tag{3.5}$$

$$L_{mn} = L_m \left(\frac{L_m}{2}\right)^{n-1} \cdot \sum_{k=0}^{[n/2]} \binom{n}{2k} \left(\frac{5F_m^2}{L_m^2}\right)^k. \tag{3.6}$$

These may be found in ([7], eq. (15)), and may be used to show that our functions f_m and l_m are in fact analytic functions on $\mathbb{Z}_p$, as follows.

Corollary 3.2: *The functions $f_m(x)$, $l_m(x)$ of Theorem 3.1 are analytic functions of x on $\mathbb{Z}_p$ when they are continuous.*

Proof: A theorem of Mahler ([8], Theorem 51.1) states that any continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ can be expressed uniquely in the form

$$f(x) = \sum_{k=0}^{\infty} a_k \binom{x}{k} \tag{3.7}$$

for some $a_k \in \mathbb{Q}_p$ (called the *Mahler coefficients of* f) such that $a_k \to 0$ in $\mathbb{Q}_p$, and f is analytic if and only if $a_k/k! \to 0$ in $\mathbb{Q}_p$ ([8], Theorem 54.4). The combinatorial forms (3.5), (3.6) show that

$$f_m(x) = \sum_{k=0}^{\infty} \frac{2F_m}{L_m} \left(\frac{5F_m^2}{L_m^2}\right)^k \binom{x}{2k+1}, \tag{3.8}$$

$$l_m(x) = \sum_{k=0}^{\infty} 2 \left(\frac{5F_m^2}{L_m^2}\right)^k \binom{x}{2k}, \tag{3.9}$$

from which we observe that their Mahler coefficients a_k satisfy $\mathrm{ord}_p(a_k/k!) \to +\infty$ when $\mathrm{ord}_p(5F_m^2/L_m^2) > 0$.

We obtain our next set of identities by transforming (3.1) by the linear transformation

$$F(a, b; c; z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right) \tag{3.10}$$

([3], eq. (4.1)) with n odd, which gives

$$F_{(2n+1)m} = (2n+1)F_m(-1)^{mn} F\left(-n, n+1; \frac{3}{2}; \frac{5F_m^2}{4(-1)^{m+1}}\right), \tag{3.11}$$

and transforming (3.2) with n even by the same formula, giving

$$L_{2mn} = 2(-1)^{mn} F\left(-n, n; \frac{1}{2}; \frac{5F_m^2}{4(-1)^{m+1}}\right). \tag{3.12}$$

Similarly, applying

$$F(a, b; c; z) = (1-z)^{-b} F\left(b, c-a; c; \frac{z}{z-1}\right) \tag{3.13}$$

([3], eq. (4.2)) to (3.1) with n even gives

$$F_{2mn} = nF_{2m}(-1)^{m(n-1)} F\left(1-n, n+1; \frac{3}{2}; \frac{5F_m^2}{4(-1)^{m+1}}\right), \tag{3.14}$$

and substituting (3.13) into (3.2) with n odd gives

$$L_{(2n+1)m} = L_m(-1)^{mn} F\left(-n, n+1; \frac{1}{2}; \frac{5F_m^2}{4(-1)^{m+1}}\right). \tag{3.15}$$

These are valid identites in $\mathbb{Q}$ for any integers m, n with $n > 0$. When $m = 1$ (3.11) and (3.14) become identities (4.4) and (4.5) of [3], although the factor of $(-1)^n$ in ([3], eq. (4.5)) should be $(-1)^{n-1}$; when $m = 2$ they become (4.39) and (4.40) of [3]. Since the argument always has absolute value larger than 1 these hypergeometric functions are not analytic functions of n on $\mathbb{R}$ or $\mathbb{C}$. However, by Theorem 2.3 these hypergeometric functions are continuous in n on $\mathbb{Z}_p$ when $p|F_m$ or when $p = 5$. It follows that $\{(-1)^n F_{2n}\}$, $\{(-1)^n L_{2n}\}$, $\{(-1)^n F_{2n+1}\}$, and $\{(-1)^n L_{2n+1}\}$ are 5-adically interpolatable, and that $\{(-1)^n F_{2r_p n}\}$, $\{(-1)^n L_{2r_p n}\}$, $\{(-1)^n F_{(2n+1)r_p}\}$, and $\{(-1)^n L_{(2n+1)r_p}\}$ are p-adically interpolatable for any prime p. The corresponding combinatorial forms are

$$F_{(2n+1)m} = (2n+1)F_m\,(-1)^{mn} \sum_{k=0}^{n} \binom{n+k}{2k} \frac{((-1)^m 5F_m^2)^k}{2k+1}, \tag{3.16}$$

$$F_{2mn} = F_{2m}(-1)^{m(n-1)} \sum_{k=0}^{n-1} \binom{n+k}{2k+1} \left((-1)^m 5F_m^2\right)^k, \tag{3.17}$$

$$L_{2mn} = (-1)^{mn} \left(2 + \sum_{k=1}^{n} \frac{n}{k} \binom{n+k-1}{2k-1} \left((-1)^m 5F_m^2\right)^k\right), \tag{3.18}$$

$$L_{(2n+1)m} = (-1)^{mn} L_m \sum_{k=0}^{n} \binom{n+k}{2k} \left((-1)^m 5F_m^2\right)^k. \tag{3.19}$$

By applying the linear transformation

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)(-z)^{-a}}{\Gamma(b)\Gamma(c-a)} F\left(a, 1-c+a; 1-b+a; \frac{1}{z}\right) + \frac{\Gamma(c)\Gamma(a-b)(-z)^{-b}}{\Gamma(a)\Gamma(c-b)} F\left(b, 1-c+b; 1-a+b; \frac{1}{z}\right) \tag{3.20}$$

([3], eq. (4.9)) to (3.1) and evaluating the gamma factors as in [3], we get the pair of formulas

$$F_{(2n+1)m} = F_m \left(\frac{5F_m^2}{4}\right)^n F\left(-n, -n-\frac{1}{2}; \frac{1}{2}; \frac{L_m^2}{5F_m^2}\right), \tag{3.21}$$

$$F_{2mn} = nF_{2m} \left(\frac{5F_m^2}{4}\right)^{n-1} F\left(1-n, \frac{1}{2}-n; \frac{3}{2}; \frac{L_m^2}{5F_m^2}\right). \tag{3.22}$$

By applying this same transformation to (3.2) we get the pair

$$L_{(2n+1)m} = (2n+1)L_m \left(\frac{5F_m^2}{4}\right)^n F\left(-n, \frac{1}{2}-n; \frac{3}{2}; \frac{L_m^2}{5F_m^2}\right), \tag{3.23}$$

$$L_{2mn} = 2\left(\frac{5F_m^2}{4}\right)^n F\left(-n, \frac{1}{2}-n; \frac{1}{2}; \frac{L_m^2}{5F_m^2}\right). \tag{3.24}$$

These are valid identites in $\mathbb{Q}$ for any integers m, n with $n > 0$. When $m = 1$ one obtains identities (4.10) and (4.11) of [3] from (3.21), (3.22); when $m = 2$ we get (4.35) and (4.36) of [3]. The hypergeometric functions are analytic in n on $\mathbb{R}$ or $\mathbb{C}$ when m is odd, and continuous in n on $\mathbb{Z}_p$ when p is odd and $p|L_m$, or when $p = 2$ and $4|L_m$. From the identity $L_m^2 - 5F_m^2 = 4(-1)^m$ we see that if p is odd and $p|L_m$ then $5F_m^2/4 \equiv (-1)^{m+1} \pmod{p\mathbb{Z}_p}$, which shows that $\{(-1)^{(r'_p+1)n}F_{(2n+1)r'_p}\}$ and $\{(-1)^{(r'_p+1)n}L_{(2n+1)r'_p}\}$ are p-adically interpolatable for those odd primes p for which r'_p exists. Similarly we see that $\{F_{6n+3}\}$ and $\{L_{6n+3}\}$ are 2-adically interpolatable. The combinatorial forms are

$$F_{(2n+1)m} = F_m \left(\frac{5F_m^2}{4}\right)^n \cdot \sum_{k=0}^{n} \binom{2n+1}{2k} \left(\frac{L_m^2}{5F_m^2}\right)^k, \tag{3.25}$$

$$F_{2mn} = \frac{F_{2m}}{2} \left(\frac{5F_m^2}{4}\right)^{n-1} \cdot \sum_{k=0}^{n-1} \binom{2n}{2k+1} \left(\frac{L_m^2}{5F_m^2}\right)^k, \tag{3.26}$$

$$L_{(2n+1)m} = L_m \left(\frac{5F_m^2}{4}\right)^n \cdot \sum_{k=0}^{n} \binom{2n+1}{2k+1} \left(\frac{L_m^2}{5F_m^2}\right)^k, \tag{3.27}$$

$$L_{2mn} = 2\left(\frac{5F_m^2}{4}\right)^n \cdot \sum_{k=0}^{n} \binom{2n}{2k} \left(\frac{L_m^2}{5F_m^2}\right)^k. \tag{3.28}$$

Our next set of identities is obtained by applying the transformation formula

$$\begin{aligned} F(a,b;c;z) &= (1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} F\left(a, c-b; a-b+1; \frac{1}{1-z}\right) \\ &\quad + (1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} F\left(b, c-a; b-a+1; \frac{1}{1-z}\right) \end{aligned} \tag{3.29}$$

([3], eq. (4.16)) to (3.1), which gives

$$F_{(2n+1)m} = F_m(-1)^{(m+1)n} F\left(-n, n+1; \frac{1}{2}; \frac{L_m^2}{4(-1)^m}\right), \tag{3.30}$$

$$F_{2mn} = nF_{2m}(-1)^{(m+1)(n+1)} F\left(1-n, n+1; \frac{3}{2}; \frac{L_m^2}{4(-1)^m}\right). \tag{3.31}$$

By applying this same transformation to (3.2), we get

$$L_{(2n+1)m} = L_m(-1)^{(m+1)n}(2n+1) F\left(-n, n+1; \frac{3}{2}; \frac{L_m^2}{4(-1)^m}\right), \tag{3.32}$$

$$L_{2mn} = 2(-1)^{(m+1)n} F\left(-n, n; \frac{1}{2}; \frac{L_m^2}{4(-1)^m}\right). \tag{3.33}$$

These are valid identites in $\mathbb{Q}$ for any integers m, n with $n > 0$. When $m = 1$ one obtains identities (4.17) and (4.18) of [3]; when $m = 2$ we get (4.33) and (4.34) of [3]. The hypergeometric functions are analytic in n on $\mathbb{R}$ or $\mathbb{C}$ only when $m = 1$, and continuous in n on $\mathbb{Z}_p$ whenever $p|L_m$. The combinatorial forms are

$$F_{(2n+1)m} = F_m(-1)^{(m+1)n} \sum_{k=0}^{n} \binom{n+k}{2k} \left((-1)^{m-1}L_m^2\right)^k, \tag{3.34}$$

$$F_{2mn} = F_m L_m(-1)^{(m+1)(n+1)} \sum_{k=0}^{n-1} \binom{n+k}{2k+1} \left((-1)^{m-1}L_m^2\right)^k, \tag{3.35}$$

$$L_{(2n+1)m} = (-1)^{(m+1)n}(2n+1)L_m \sum_{k=0}^{n} \binom{n+k}{2k} \frac{((-1)^{m-1}L_m^2)^k}{2k+1}, \tag{3.36}$$

$$L_{2mn} = (-1)^{(m+1)n}\left(2 + \sum_{k=1}^{n} \frac{n}{k}\binom{n+k-1}{2k-1}\left((-1)^{m-1}L_m^2\right)^k\right). \tag{3.37}$$

For sake of completeness we give the identities

$$F_{mn} = F_m(L_m)^{n-1}F\left(\frac{1-n}{2}, \frac{2-n}{2}; 1-n; \frac{4(-1)^m}{L_m^2}\right) \qquad (n > 1), \tag{3.38}$$

$$L_{mn} = L_m^n F\left(\frac{-n}{2}, \frac{1-n}{2}; 1-n; \frac{4(-1)^m}{L_m^2}\right) \qquad (n > 2), \tag{3.39}$$

which are obtained by applying the transformation

$$F\left(a, b; a+b+m; z\right) = \frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)}F(a, b; 1-m; 1-z) \tag{3.40}$$

([3], eq. (4.6)) to (3.1) and (3.2) respectively. By applying this same transformation to (3.21)-(3.24) we get

$$F_{m(2n+1)} = F_m(5F_m^2)^n F\left(-n, -\frac{1}{2} - n; -2n; \frac{4(-1)^{m+1}}{5F_m^2}\right), \tag{3.41}$$

$$L_{m(2n+1)} = L_m(5F_m^2)^n F\left(-n, \frac{1}{2} - n; -2n; \frac{4(-1)^{m+1}}{5F_m^2}\right), \tag{3.42}$$

$$F_{2mn} = \frac{L_m(5F_m^2)^n}{5F_m} F\left(1-n, \frac{1}{2} - n; 1-2n; \frac{4(-1)^{m+1}}{5F_m^2}\right), \tag{3.43}$$

$$L_{2mn} = (5F_m^2)^n F\left(-n, \frac{1}{2} - n; 1-2n; \frac{4(-1)^{m+1}}{5F_m^2}\right) \qquad (n > 1). \tag{3.44}$$

These are valid in $\mathbb{Q}$ for any integers m, n with $n > 0$, except as noted above. When $m = 1$ (3.38), (3.41), and (3.43) become (4.8), (4.14), and (4.15) of [3]; when $m = 2$ (3.38) becomes (4.27) of [3]. The hypergeometric functions here are in general not continuous in n on $\mathbb{Z}_p$, as the denominator parameters c are not constant and the arguments $z = 4(-1)^m/L_m^2$ and $z = 4(-1)^{m+1}/5F_m^2$ never satisfy the conditions of Theorem 2.2 even if c were constant.

4. INFINITE SERIES IDENTITIES

In this section we illustrate a few infinite series identities which may be obtained from hypergeometric transformations of identities from the previous section. The primary tool will be Euler's identity

$$F(a, b; c; z) = (1-z)^{c-a-b}F(c-a, c-b; c; z) \tag{4.1}$$

([3], eq. (4.3)), which is a formal power series identity whenever c is not a nonpositive integer. Under this condition (4.1) is valid (in $\mathbb{R}$ or $\mathbb{Q}_p$) whenever both sides are convergent. When this identity is applied to (3.1) and (3.2), we obtain

$$F_{mn} = nF_m(-1)^{mn}\left(\frac{2}{L_m}\right)^{n+1} F\left(\frac{n+2}{2}, \frac{n+1}{2}; \frac{3}{2}; \frac{5F_m^2}{L_m^2}\right), \tag{4.2}$$

$$L_{mn} = L_m(-1)^{mn}\left(\frac{2}{L_m}\right)^{n+1} F\left(\frac{n+1}{2}, \frac{n}{2}; \frac{1}{2}; \frac{5F_m^2}{L_m^2}\right), \tag{4.3}$$

which are precisely (3.1), (3.2) with n replaced by $-n$, since $F_{-n} = (-1)^{n-1}F_n$ and $L_{-n} = (-1)^n L_n$. For $n > 0$ these are valid in $\mathbb{R}$ when m is even; when $m = 2$ one obtains ([3], eq. (4.25)). Furthermore they are valid in $\mathbb{Z}_p$ when $p = 5$, when p is odd and $p|F_m$, or when $p = 2$ and $4|F_m$. The corresponding combinatorial forms are

$$F_{mn} = (-1)^{mn}F_m\left(\frac{2}{L_m}\right)^{n+1}\sum_{k=0}^{\infty}\binom{n+2k}{2k+1}\left(\frac{5F_m^2}{L_m^2}\right)^k, \tag{4.4}$$

$$L_{mn} = (-1)^{mn}L_m\left(\frac{2}{L_m}\right)^{n+1}\sum_{k=0}^{\infty}\binom{n+2k-1}{2k}\left(\frac{5F_m^2}{L_m^2}\right)^k. \tag{4.5}$$

We remark that when m is even these two series of rational numbers

$$\sum_{k=0}^{\infty}\binom{n+2k}{2k+1}\left(\frac{5F_m^2}{L_m^2}\right)^k, \qquad \sum_{k=0}^{\infty}\binom{n+2k-1}{2k}\left(\frac{5F_m^2}{L_m^2}\right)^k \tag{4.6}$$

have the interesting property that they converge both in $\mathbb{R}$, and in $\mathbb{Z}_p$ for all primes p dividing $5F_m$, to the *same* rational numbers, namely $(-1)^{mn}(2/L_m)^{n+1}F_{mn}/F_m$ and $(-1)^{mn}(2/L_m)^{n+1}L_{mn}/L_m$.

Applying (4.1) to the finite-sum identities (3.21)-(3.24) likewise results in those same identities with n replaced by $-n$; the resulting infinite series are valid in $\mathbb{R}$ when m is odd, and valid in $\mathbb{Z}_p$ when p is odd and $p|L_m$, or when $p = 2$ and $4|L_m$. When $m = 1$ the series resulting from transforming (3.21), (3.22) by (4.1) are precisely identities (4.12), (4.13) of [3]. The combinatorial forms can in fact be obtained by replacing n by $-n$ in (3.25)-(3.28), and read

$$F_{(2n+1)m} = (-1)^{m+1}F_m\left(\frac{4}{5F_m^2}\right)^{n+1}\cdot\sum_{k=0}^{\infty}\binom{2n+2k}{2k}\left(\frac{L_m^2}{5F_m^2}\right)^k, \tag{4.7}$$

$$F_{2mn} = \frac{1}{2}F_{2m}\left(\frac{4}{5F_m^2}\right)^{n+1}\cdot\sum_{k=0}^{\infty}\binom{2n+2k}{2k+1}\left(\frac{L_m^2}{5F_m^2}\right)^k, \tag{4.8}$$

$$L_{(2n+1)m} = (-1)^{m+1} L_m \left(\frac{4}{5F_m^2}\right)^{n+1} \cdot \sum_{k=0}^{\infty} \binom{2n+2k+1}{2k+1} \left(\frac{L_m^2}{5F_m^2}\right)^k, \tag{4.9}$$

$$L_{2mn} = 2\left(\frac{4}{5F_m^2}\right)^n \cdot \sum_{k=0}^{\infty} \binom{2n+2k-1}{2k} \left(\frac{L_m^2}{5F_m^2}\right)^k. \tag{4.10}$$

Again these series have the property that when m is odd they converge in $\mathbb{R}$, and in $\mathbb{Z}_p$ for all primes p dividing L_m, to the same rational numbers.

When (4.1) is applied to (3.11), (3.12), (3.14), and (3.15), however, the results are not just the same identities with n replaced by $-n$. For example, (4.1) transforms (3.11) and (3.14) into

$$F_{(2n+1)m} = (2n+1)\frac{F_{2m}}{2}(-1)^{mn}\sqrt{(-1)^m}F\left(n+\frac{3}{2}, n-\frac{1}{2}; \frac{3}{2}; \frac{5F_m^2}{4(-1)^{m+1}}\right), \tag{4.11}$$

$$F_{2mn} = 2nF_m(-1)^{mn}\sqrt{(-1)^m}F\left(n+\frac{1}{2}, \frac{1}{2}-n; \frac{3}{2}; \frac{5F_m^2}{4(-1)^{m+1}}\right). \tag{4.12}$$

While these series do not converge in $\mathbb{R}$ or $\mathbb{C}$, they are valid in $\mathbb{Z}_p$ for an appropriate choice of $\sqrt{(-1)^m}$ when $p|F_m$ or when $p=5$. We remark that if k is an integer then $\sqrt{k} \in \mathbb{Z}_p$ if and only if $\mathrm{ord}_p k$ is even and $k = p^{2e}k'$ with the Legendre symbol $(k'|p) = 1$. However it is easy to see from the identity $L_m^2 - 5F_m^2 = 4(-1)^m$ that $((-1)^m|p) = 1$ when $p|F_m$ or $p = 5$, and thus $\sqrt{(-1)^m} \in \mathbb{Z}_p$.

Similarly, applying (4.1) to (3.30), (3.31) gives

$$F_{(2n+1)m} = (-1)^{(m+1)n}\frac{2}{\sqrt{(-1)^{m+1}5}}F\left(n+\frac{1}{2}, -\frac{1}{2}-n; \frac{1}{2}; \frac{L_m^2}{4(-1)^m}\right), \tag{4.13}$$

$$F_{2mn} = (-1)^{(m+1)n}\frac{2nL_m}{\sqrt{(-1)^{m+1}5}}F\left(n+\frac{1}{2}, \frac{1}{2}-n; \frac{3}{2}; \frac{L_m^2}{4(-1)^m}\right), \tag{4.14}$$

which are valid in $\mathbb{R}$ only when $m = 1$, and are found in ([3], eq. (4.19), (4.20)). These are also valid in $\mathbb{Z}_p$ when p is odd and $p|L_m$, which is precisely the condition for $\sqrt{(-1)^{m+1}5} \in \mathbb{Z}_p$.

We conclude this section with an example involving the quadratic transformation

$$F(a, b; a-b+1; z) = (1+z)^{-a}F\left(\frac{a}{2}, \frac{a+1}{2}; a-b+1; \frac{4z}{(1+z)^2}\right) \tag{4.15}$$

([3], eq. (4.21)). Applied to (3.21), this gives

$$F_{(2n+1)m} = \sqrt{\frac{2L_{2m}}{5}}\left(\frac{L_{2m}}{2}\right)^n F\left(\frac{-2n-1}{4}, \frac{1-2n}{4}; \frac{1}{2}; \frac{5F_{2m}^2}{L_{2m}^2}\right). \tag{4.16}$$

This identity is valid in $\mathbb{R}$ for all positive integers m, n, and valid in $\mathbb{Z}_p$ when p is odd and $p|L_m$ or when $p = 2$ and $4|L_m$. Again the square root factor lies in $\mathbb{Z}_p$ under these conditions. For $m = 1$ it appears as equation (4.22) of [3]. However, if one applies (4.15) to (3.22) one merely obtains the even m case of (3.1). If (4.16) is transformed by (4.1), we obtain

$$F_{(2n+1)m} = \sqrt{\frac{8}{5L_{2m}}}\left(\frac{2}{L_{2m}}\right)^n F\left(\frac{2n+3}{4}, \frac{2n+1}{4}; \frac{1}{2}; \frac{5F_{2m}^2}{L_{2m}^2}\right), \tag{4.17}$$

valid in $\mathbb{R}$ for any positive integers m, n, and valid in $\mathbb{Z}_p$ when p is odd and $p|L_m$ or when $p = 2$ and $4|L_m$. When $m = 1$ this is identity (4.24) of [3]. We caution that while the hypergeometric series in (4.16) and (4.17) converge in $\mathbb{Z}_5$, the identities (4.16), (4.17) are not valid in $\mathbb{Z}_5$, because they arise from a transformation of (3.21), which is not convergent in $\mathbb{Z}_5$. To see that these identities are not valid in $\mathbb{Z}_5$, observe that the square root factors do not lie in $\mathbb{Z}_5$, while all other factors lie in $\mathbb{Z}_5$. The hypergeometric series in (4.16), (4.17) thus have the curious property of converging in $\mathbb{R}$, and in $\mathbb{Z}_p$ whenever p is odd and $p|L_m$ or when $p = 2$ and $4|L_m$, to the same *irrational* sums. We have not determined the sums of these hypergeometric series in $\mathbb{Z}_5$.

5. CONCLUDING REMARKS

These identities may all be generalized to general Lucas sequences $\{U_n\}$, $\{V_n\}$ of the first and second kinds defined by $U_n = PU_{n-1} - QU_{n-2}$, $V_n = PV_{n-1} - QV_{n-2}$, $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, $V_1 = P$. Just substitute $z = \sqrt{D}U_m/V_m$ into (3.3) and (3.4), where $D = P^2 - 4Q$ is the discriminant of the characteristic polynomial $1 - PT + QT^2$ of the recurrence. Most of the properties remain unchanged; however, the sequences are not purely periodic modulo p if $p|Q$ and it is no longer true that $a_p \in \{1, 2, 4\}$ in general.

We have shown that if $z = 5F_m^2/L_m^2$ then there is a representation of a subsequence of the Fibonacci and Lucas sequences in terms of Gauss hypergeometric functions with argument $z, 1-z, \frac{1}{z}, 1-\frac{1}{z}, \frac{1}{1-z}$, and $\frac{z}{z-1}$. Is this a complete list of possible rational arguments? Each of the twelve rational arguments listed in Table 1 of [3] occurs as the $m = 1$ or $m = 2$ case of one of these. Although quadratic transformations such as (4.15) may lead to new identities such as (4.16), we do not know whether a quadratic transformation can lead to a representation with a different rational argument than listed here; in particular none of the quadratic transformations employed in [3] yields a new rational argument.

¿From our collection of identities we learn that their p-adic continuity properties exert strong influence over which arguments can occur. For example, if one had an identity of the form

$$F_{mn+k} = C_m \cdot P_m(n) s_m^n F(a(n), b(n); c; z_m) \tag{5.1}$$

where $P_m(n), a(n), b(n)$ are polynomials in $\mathbb{Z}[n]$ and C_m, s_m, c, z_m are rational numbers, then every prime dividing the numerator of z_m must also divide $5F_m$. (By Theorem 2.3 if $\mathrm{ord}_p z_m > 0$ then $F(a(n), b(n); c; z_m)$ would be p-adically continuous as a function of n and therefore the sequence $s_m^{-n} F_{mn+k}$ would be p-adically interpolatable. By Theorem 2.2 if $p \neq 5$ then $r_p|m$ and thus $p|F_m$). All the identities of section 3 are of this form with the exception that $a(n), b(n)$ are in general polynomials in $\mathbb{Z}[\frac{1}{2}][n]$. The above argument remains valid except for $p = 2$, that is, every odd prime dividing the numerator of z_m in such an identity must also divide $5F_m$. So the set of possible rational arguments is clearly quite limited by these p-adic considerations.

REFERENCES

[1] R. André-Jeannin. "Generalized Complex Fibonacci and Lucas Functions." *The Fibonacci Quarterly* **29.1** (1991): 13-18.

[2] H. Bateman. *Higher Transcendental Functions.* A. Erdélyi ed., Vol. 1, McGraw-Hill, New York, 1953.

[3] K. Dilcher. "Hypergeometric Functions and Fibonacci Numbers." *The Fibonacci Quarterly* **38.4** (2000): 342-363.

[4] B. Dwork. *Lectures on p-adic Differential Equations.* Springer-Verlag, New York, 1982.

[5] F. Gouvea. *p-adic Numbers. An Introduction.* Second edition, Springer, Berlin, 1998.

[6] A. F. Horadam and A. G. Shannon. "Fibonacci and Lucas Curves." *The Fibonacci Quarterly* **26.1** (1988): 3-13.

[7] S. Rabinowitz. "Algorithmic Manipulation of Fibonacci Identities." *Applications of Fibonacci Numbers.* Vol. 6 (1996), G. E. Bergum, ed., Kluwer Academic Publishers, 389-408.

[8] W. H. Schikhof. *Ultrametric Calculus. An Introduction to p-adic Analysis.* Cambridge University Press, London, 1984.

[9] L. Somer. "The Divisibility of Primary Lucas Recurrences With Respect to Primes." *The Fibonacci Quarterly* **18.4** (1980): 316-334.

[10] L. Somer. "Divisibility of Terms in Lucas Sequences of the Second Kind by Their Subscripts." *Applications of Fibonacci Numbers.* Vol. 6 (1996), G. E. Bergum, ed., Kluwer Academic Publishers, 473-486.

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