q-ANALOGS OF GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS

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ABSTRACT

The Fibonacci operator approach inspired by Andrews (2004) is explored to investigate q-analogs of the generalized Fibonacci and Lucas polynomials introduced by Chu and Vicenti (2003). Their generating functions are compactly expressed in terms of Fibonacci operator fractions. A determinant evaluation on q-binomial coefficients is also established which extends a recent result of Sun (2005).

1. INTRODUCTION

The generalized Fibonacci and Lucas polynomials are defined in [11] by

$$F_{n+1}(t) = F_n(t) + tF_{n-1}(t), \ n \geq 1 \tag{1}$$

with the initial conditions $F_0(t) = a$ and $F_1(t) = b$. When $t = 1$, they reduce, for $a = b = 1$ and $a = 2$ and $b = 1$, to Fibonacci and Lucas sequences, respectively, which have been extensively studied for their many beautiful and interesting combinatorial properties.

For the case $a = b = 1$, several slightly different q-analogs of $F_n(t)$ have been worked out by Carlitz [4], Cigler [7] and Schur [10]. On the related literature of recurrence relations and generating functions, refer to [1, 8] for the theory of orthogonal polynomials and [1, 2, 8, 9] for the Rogers-Ramanujan identities.

Differently from the works just mentioned, Andrews [3] recently introduced the Fibonacci operator η_x by $\eta_x f(x) = f(xq)$ for any given function $f(x)$. Then he obtained an unusual operator expression for the generating function of q-Fibonacci polynomials. Inspired by this operator approach, we shall study the full q-analog of $F_n(t)$ for a and b be arbitrary numbers and establish the corresponding generating functions in terms of η-operator fractions. Then we shall evaluate a determinant related q-binomial coefficients. Finally for some particular values of a and b, we shall give q-analogs of some generating functions established in [6], again in terms of η-operator fractions. We believe that these results on the q-incomplete Fibonacci and Lucas polynomials are new.

For two indeterminate x and q, the shifted factorial is defined by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = \prod_{k=0}^{n-1}(1-q^k x) \quad \text{with} \quad n = 1, 2, \cdots .$$

When $|q| < 1$, the infinite product

$$(x;q)_\infty = \prod_{n=0}^{\infty}(1-q^n x)$$

is well defined, which leads us to the following expression

$$(x;q)_n = \frac{(x;q)_\infty}{(q^n x;q)_\infty} \text{ for } n \in \mathbb{Z}.$$

The Gaussian q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}, & 0 \le m \le n, \\ 0, & otherwise. \end{cases}$$

2. q-ANALOGS OF THE GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS

The q-analogs of generalized Fibonacci and Lucas polynomials are introduced by [3]. Let us define a sequence of polynomial $S_n(t,q)$ by the recurrence relation

$$S_{n+1}(t,q) = S_n(t,q) + tq^{n-2}S_{n-1}(t,q),\ n \ge 1 \tag{2}$$

where $S_0(t,q) = a, S_1(t,q) = b$. It is obvious that $S_n(t,1) = F_n(t)$ with $F_n(t)$ being defined by (1).

Theorem 1: (The generating function defined by recurrence relation (2)).

$$\sum_{n=0}^{\infty} S_n(t,q)x^n = \frac{1}{1-x-tx^2\eta_x}\{a+(b-a)x\}.$$

Proof: Let $\sigma(x)$ stand for the expression on the left side of the equation in Theorem 1. To prove Theorem 1, we need to check the following equivalent relation:

$$(1-x-tx^2\eta_x)\sigma(x) = a+(b-a)x.$$

According to the definition of $\sigma(x)$, we have

$$\begin{aligned} & a+bx+\sum_{n\ge 2} S_n(t,q)x^n - \sum_{n\ge 0} S_n(t,q)x^{n+1} - t\sum_{n\ge 0} S_n(t,q)x^{n+2}q^n \\ =& a+bx-ax+\sum_{n\ge 2}\{S_n(t,q)-S_{n-1}(t,q)-tq^{n-2}S_{n-2}(t,q)\}x^n \end{aligned}$$

which reduces to $a+(b-a)x$ in view of recurrence relation (2). $\square$

In order to find explicit expression for $S_n(t,q)$, we will need the following lemma.

Lemma 2: (The Fibonacci operator composition)

$$(x+tx^2\eta_x)^n x^2 = x^{n+2}\sum_{j\geq 0} t^j x^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+1)}. \tag{3}$$

$$(x+tx^2\eta_x)^n x = x^{n+1}\sum_{j\geq 0} t^j x^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j^2}. \tag{4}$$

Proof: We can proceed with induction principle. For $n=0$, the first equation asserts $x^2=x^2$. Now suppose the first equation is true for n. Then we can verify it for $n+1$ as follows:

$$\begin{aligned}
(x+tx^2\eta_x)^{n+1}x^2 &= (x+tx^2\eta_x)x^{n+2}\sum_{j\geq 0} t^j x^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+1)} \\
=& x^{n+3}\sum_{j\geq 0} t^j x^j \left[\binom{n}{j}\right] q^{j(j+1)} + x^{n+4}\sum_{j\geq 0} t^{j+1} x^j \begin{bmatrix} n \\ j \end{bmatrix} q^{n+2+j(j+2)} \\
=& x^{n+3}\sum_{j\geq 0} t^j x^j \left[\binom{n}{j}\right] q^{j(j+1)} + x^{n+3}\sum_{j\geq 0} t^j x^j \begin{bmatrix} n \\ j-1 \end{bmatrix} q^{n+1+j^2} \\
=& x^{n+3}\sum_{j\geq 0} t^j x^j q^{j(j+1)} \left\{ \begin{bmatrix} n \\ j \end{bmatrix} + q^{n+1-j} \begin{bmatrix} n \\ j-1 \end{bmatrix} \right\} \\
=& x^{n+3}\sum_{j\geq 0} t^j x^j q^{j(j+1)} \begin{bmatrix} n+1 \\ j \end{bmatrix}
\end{aligned}$$

where the last line has been justified by q-binomial identity

$$\left[\binom{n+1}{j}\right] = \begin{bmatrix} n \\ j \end{bmatrix} + q^{n+1-j} \begin{bmatrix} n \\ j-1 \end{bmatrix}.$$

This proves the first equation. The equation (4) can be established similarly. □

Corollary 3: (Explicit expression for $S_n(t,q)$)

$$S_n(t,q) = a\sum_{j\geq 0} t^{j+1} \begin{bmatrix} n-2-j \\ j \end{bmatrix} q^{j(j+1)} + b\sum_{j\geq 0} t^j \begin{bmatrix} n-1-j \\ j \end{bmatrix} q^{j^2}.$$

Proof: According to the geometric series expansion, we have

$$\begin{aligned}
&\sum_{n\geq 0} S_n(t,q)x^n = \frac{1}{1-x-tx^2\eta_x}(a+(b-a)x)\\
&=\sum_{n\geq 0}(x+tx^2\eta_x)^n\{a+(b-a)x\}\\
&=\sum_{n\geq 0}(x+tx^2\eta_x)^n a+\sum_{n\geq 0}(x+tx^2\eta_x)^n(b-a)x\\
&=a\sum_{n\geq 0}(x+tx^2\eta_x)^{n-1}(x+tx^2)+(b-a)\sum_{n\geq 0}(x+tx^2\eta_x)^n x\\
&=a\sum_{n\geq 0}x^n\sum_{j\geq 0}x^jt^j\begin{bmatrix}n-1\\ j\end{bmatrix}q^{j^2}+at\sum_{n\geq 0}x^{n+1}\sum_{j\geq 0}x^jt^j\begin{bmatrix}n-1\\ j\end{bmatrix}q^{j(j+1)}\\
&\quad+(b-a)\sum_{n\geq 0}x^{n+1}\sum_{j\geq 0}x^jt^j\begin{bmatrix}n\\ j\end{bmatrix}q^{j^2}.
\end{aligned}$$

Extract the coefficient of x^n and we get Corollary 3. □

In view of Corollary 3, we can easily deduce that

$$\begin{aligned}
S_{n,k} &= [t^k]S_n(t,q)\\
&=[t^k]\left\{a\sum_{j\geq 0}t^{j+1}\begin{bmatrix}n-2-j\\ j\end{bmatrix}q^{j(j+1)}\right.\\
&\left.+b\sum_{j\geq 0}t^j\begin{bmatrix}n-1-j\\ j\end{bmatrix}q^{j^2}\right\}\\
&=a\begin{bmatrix}n-1-k\\ k-1\end{bmatrix}q^{k(k-1)}+b\begin{bmatrix}n-1-k\\ k\end{bmatrix}q^{k^2}.
\end{aligned}$$

For $B_{n,k}=S_{2n+1,n-k}$, it is trivial to see that

$$B_{n+k+i,k+i}=a\begin{bmatrix}n+2k+2i\\ n-1\end{bmatrix}q^{2\binom{n}{2}}+b\begin{bmatrix}n+2k+2i\\ n\end{bmatrix}q^{n^2},$$

then we have the following determinant evaluation.

Theorem 4: (determinant identity on q-binomial coefficients).

$$\det_{0\leq k,n\leq m}\left[B_{n+k+i,k+i}\right] = b^{m+1}q^{\frac{m(m+1)}{6}(1+5m+6i)}\prod_{n=0}^{m}\frac{(q^2;q^2)_n}{(q;q)_n}.$$

When $q\to 1$, we recover from this theorem a binomial determinant identity appeared in [11].

Proof: Note that $B_{n+k+i,k+i}$ is a polynomial of degree n in q^{2k} with the leading coefficient $\frac{b(-1)^n}{(q;q)_n} q^{\frac{n}{2}(1+3n+4i)}$. We can write $B_{n+k+i,k+i}$ formally as

$$B_{n+k+i,k+i} = \sum_{j=0}^{n} \lambda_j(n) q^{2kj} \quad \text{with} \quad \lambda_n(n) = \frac{b(-1)^n}{(q;q)_n} q^{\frac{n}{2}(1+3n+4i)}$$

where $\{\lambda_j(n)\}_{j=0}^{n}$ are constants independent of q^k.

For each n with $0 \leq n \leq m$, defining further

$$\lambda_j(n) = 0 \quad \text{if} \quad n < j \leq m$$

then we have the following determinant factorization

$$\det_{0\leq k,n\leq m} [B_{n+k+i,k+i}] \;=\; \det_{0\leq k,j\leq m} \left[q^{2kj}\right] \times \det_{0\leq j,n\leq m} [\lambda_j(n)] \,.$$

The former is the Vadermonde determinant whose evaluation reads as

$$\begin{aligned}\det_{0\leq k,j\leq m} \left[q^{2kj}\right] &= \prod_{0\leq \jmath<j\leq m} (q^{2j} - q^{2\jmath}) \\ &= (-1)^{\binom{1+m}{2}} \prod_{n=0}^{m} q^{2n(m-n)} (q^2;q^2)_n.\end{aligned}$$

The latter is the determinant of a diagonal matrix, which is evaluated by the product of the diagonal elements:

$$\det_{0\leq j,n\leq m} [\lambda_j(n)] = \prod_{n=0}^{m} \lambda_n(n) = b^{1+m}(-1)^{\binom{1+m}{2}} \prod_{n=0}^{m} \frac{q^{\frac{n}{2}(1+3n+4i)}}{(q;q)_n}.$$

Multiplying both evaluations just displayed and then simplifying the result, we get the determinant identity stated in the theorem. □

3. q-ANALOGS OF THE INCOMPLETE FIBONACCI AND LUCAS POLYNOMIALS

For the initial values $a = b = 1$, (1) reduces to the q-Fibonacci polynomial of Calitz [4]. Similarly for $a = 2, b = 1$, (1) reduces to the q-analog of the incomplete Lucas polynomial in [6].

For $a = b = 1$ and $a = 2, b = 1$ the generating function of $F_n(t,q)$ and $L_n(t,q)$ are given by Theorem 1 as follows:

$$\sum_{n=0}^{\infty} F_n(t,q) x^n = \frac{1}{1 - x - tx^2\eta_x} \tag{5}$$

$$\sum_{n=0}^{\infty} L_n(t,q) x^n = \frac{1}{1 - x - tx^2\eta_x}(2 - x). \tag{6}$$

where equation (5) has first been established by Andrews [3].

From them we can derive also the explicit generating functions.

Theorem 5: (Generating functions)

$$\sum_{n\geq 0} F_n(t,q)x^n = \sum_{j\geq 0} \frac{x^{2j}t^j q^{j(j-1)}}{(x;q)_{j+1}}. \tag{7}$$

$$\sum_{n\geq 0} L_n(t,q)x^n = \sum_{j\geq 0} \frac{x^{2j}t^j q^{j(j-1)}}{(x;q)_{j+1}} \left\{2 - xq^j\right\}. \tag{8}$$

Proof: By means of geometric series and equations (3)-(4), we can compute

$$\begin{aligned}\frac{1}{1-x-tx^2\eta_x} &= \sum_{n\geq 0}(x+tx^2\eta_x)^n 1 = \sum_{n\geq 0}(x+tx^2\eta_x)^{n-1}(x+tx^2)\\ &= \sum_{n\geq 0} x^n \sum_{j\geq 0} x^j t^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{j^2} + t\sum_{n\geq 0} x^{n+1} \sum_{j\geq 0} x^j t^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{j(j+1)}\\ &= \sum_{n\geq 0} x^n \sum_{j\geq 0} x^j t^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{j^2} + \sum_{n\geq 0} x^n \sum_{j\geq 0} x^j t^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{j(j-1)}\\ &= \sum_{n\geq 0} x^n \sum_{j\geq 0} x^j t^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)} = \sum_{n,j\geq 0} x^{n+2j} t^j \begin{bmatrix} n+j \\ j \end{bmatrix} q^{j(j-1)}.\end{aligned}$$

Recalling (5) and then applying the q-binomial formula

$$\sum_{j\geq 0} \begin{bmatrix} n+j \\ j \end{bmatrix} x^j = \frac{1}{(x;q)_{n+1}}$$

we get the generating function (7). Similarly, one can derive the generating function (8). □

Theorem 6: (Generating functions)

$$\sum_{k\geq n} F_k(t,q)x^k = x^n \frac{F_n(t,q) + xtq^{n-2}F_{n-1}(t,q)}{1-x-tx^2\eta_x} \tag{9}$$

$$\sum_{k\geq n} L_k(t,q)x^k = x^n \frac{L_n(t,q) + xtq^{n-2}L_{n-1}(t,q)}{1-x-tx^2\eta_x}. \tag{10}$$

Proof: Let us denote by $\delta(x)$ the expression on the left side of the equation in (9). We prove equivalently the relation:

$$(1-x-tx^2\eta_x)\delta(x) = \{F_n(t,q) + xtq^{n-2}F_{n-1}(t,q)\}x^n.$$

This can be accomplished as follows:

$$\begin{aligned}
&(1-x-tx^2\eta_x)\sum_{k\ge n}F_k(t,q)x^k\\
=&\sum_{k\ge n}F_k(t,q)x^k-\sum_{k\ge n}F_k(t,q)x^{k+1}-t\sum_{k\ge n}F_k(t,q)x^{k+2}q^k\\
=&F_n(t,q)x^n+F_{n+1}(t,q)x^{n+1}-F_n(t,q)x^{n+1}\\
+&\sum_{k\ge n}\left\{F_{k+2}(t,q)-F_{k+1}(t,q)-F_k(t,q)tq^{k-2}\right\}x^{k+2}\\
=&\left\{F_n(t,q)+xtq^{n-2}F_{n-1}(t,q)\right\}x^n.
\end{aligned}$$

Therefore (9) is valid. The equation (10) follows in the same way. □

Letting $a=b=1$ and $a=2,b=1$ in corollary 3, we have

$$\begin{aligned}
f_n(t,q)&=\sum_{j\ge 0}t^{j+1}\begin{bmatrix}n-2-j\\ j\end{bmatrix}q^{j(j+1)}+\sum_{j\ge 0}t^j\begin{bmatrix}n-1-j\\ j\end{bmatrix}q^{j^2}\\
&=\sum_{j\ge 0}t^j\begin{bmatrix}n-j\\ j\end{bmatrix}q^{j(j-1)}\\
l_n(t,q)&=2\sum_{j\ge 0}t^{j+1}\left[\binom{n-2-j}{j}\right]q^{j(j+1)}+\sum_{j\ge 0}t^j\left[\binom{n-1-j}{j}\right]q^{j^2}\\
&=\sum_{j\ge 0}t^j\left[\binom{n-j}{j}\right]q^{j(j-1)}+\sum_{j\ge 1}t^j\left[\binom{n-1-j}{j-1}\right]q^{j(j-1)}.
\end{aligned}$$

For two incomplete polynomial sequences defined by

$$F_{m,n}(t,q)=\sum_{j=0}^{m}t^j\begin{bmatrix}n-j\\ j\end{bmatrix}q^{j(j-1)}$$

and

$$L_{m,n}=\sum_{j=0}^{m}t^j\begin{bmatrix}n-j\\ j\end{bmatrix}q^{j(j-1)}+\sum_{j=1}^{m}t^j\begin{bmatrix}n-1-j\\ j-1\end{bmatrix}q^{j(j-1)}$$

Their generating functions defined respectively by

$$\Phi(x,y):=\sum_{m,n=0}^{\infty}F_{m,n}(t,q)x^my^n\quad\text{where}\quad 0\le m\le\frac{n}{2}$$

and

$$\Psi(x,y):=\sum_{m,n=0}^{\infty}L_{m,n}(t,q)x^my^n\quad\text{where}\quad 0\le m\le\frac{n}{2}$$

are given by the following:

Theorem 7: (Generating function)

$$\Phi(x,y) = \frac{1}{1-x} \cdot \frac{1}{1-y-txy^2\eta_y} \tag{11}$$

$$\Psi(x,y) = \frac{1}{1-x} \cdot \frac{1}{1-y-txy^2\eta_y}(2-y). \tag{12}$$

Proof: This generating function can be obtained through triple sum

$$\begin{aligned}\Phi(x,y) &= \sum_{m,n=0}^{\infty}\sum_{j=0}^{m} t^j \begin{bmatrix} n-j \\ j \end{bmatrix} q^{j(j-1)} x^m y^n \\ &= \sum_{0 \le j \le m < +\infty} x^m q^{j(j-1)} t^j \sum_{n=0}^{\infty} \begin{bmatrix} n-j \\ j \end{bmatrix} y^n.\end{aligned}$$

For the inner sum, changing the summation index by $n = i + 2j$ and then evaluating it as

$$y^{2j} \sum_{i=0}^{\infty} y^i \begin{bmatrix} i+j \\ j \end{bmatrix} = \frac{y^{2j}}{(y;q)_{j+1}}$$

we can simplify the double sum as follows:

$$\begin{aligned}\Phi(x,y) &= \sum_{0 \le j \le m < \infty} x^m t^j q^{j(j-1)} \frac{y^{2j}}{(y;q)_{j+1}} \\ &= \sum_{k=0}^{\infty} x^k \sum_{j=0}^{\infty} \frac{t^j x^j q^{j(j-1)} y^{2j}}{(y;q)_{j+1}} \\ &= \frac{1}{1-x} \cdot \frac{1}{1-y-txy^2\eta_y}\end{aligned}$$

where equation (5) and (7) have been combined for justifying the last step. This proves the identity (11). Similarly we can deduce the identity (12). □

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