ON AN 'UNCOUNTED' FIBONACCI IDENTITY AND ITS q-ANALOGUE

NEAL O. SMITH

ABSTRACT. In [1], Benjamin and Quinn offer several nontrivial Fibonacci identities which are, in their words, 'in need of combinatorial proof' . In this article, we give a combinatorial proof of one of these identities. Our combinatorial proof offers insight as to how to produce a q-Fibonacci generalization of this result as well.

1. INTRODUCTION

It is well-known that the Fibonacci numbers enumerate an easily understood class of objects. To wit, $f_n = F_{n+1}$ counts the number of ways to tile a 1 by n chessboard with dominoes and monominoes.

Further, there is a well-known q-analogue of the Fibonacci numbers, given by the recurrence $F_{n+1}(q) = F_n(q) + q^{n-1}F_{n-1}(q)$ with the initial conditions $F_1(q) = F_2(q) = 1$. These polynomials in q are generalizations of the Fibonacci numbers in the sense that $\lim_{q\to 1^-} F_{n+1}(q) = f_n$. In [2], many elementary q-Fibonacci identities are enumerated. In principle, given any counting sequence one can find a suitable q-generalization of said sequence.

A useful concept in the analysis of such q-sequences is the notion of a *statistic* on a set. Given a set S, a statistic w on S is simply a function $w : S \to \{0, 1, 2, \dots\}$. As we will see in a moment, the q-Fibonacci sequence can be concretely understood as a generating function for a particular statistic on a set of tilings. See [4] and [5] for further discussion concerning generalized counting sequences and their related statistics.

In general, when a sequence has a counting interpretation, one hopes that its q-analogue admits a counting interpretation, typically as a generating function of a statistic on a suitable set. In the case of the q-Fibonacci sequence, the interpretation is as follows. Given a 1 by n chessboard, number the squares $1, 2, \dots, n$ from left to right. Given a tiling of this board, define the *weight* of this tiling to be the sum of all i so that a (single) domino covers squares i and $i + 1$.

Example 1.1. *Consider a tiling of a 1 by 7 board, containing exactly two dominoes, covering cells 3-4 and 5-6. Said tiling has weight 8.*

With the notion of the weight of a tiling in hand, it is then easy to speak of the q-Fibonacci sequence as a generating function of this statistic. Namely, let $g_{n+1}(q) = \sum_{\Delta\in T_n} q^{w(\Delta)}$, where T_n is the set of all tilings of a 1 by n board using dominoes and monominoes and $w(\Delta)$ denotes the weight of the tiling Δ. Of course, we use the convention that $q^0 = 1$ for all q. Then, one can easily verify that for all $n \geq 1$, $g_n(q) = F_n(q)$; one simply notes that the sequence $\{g_n(q)\}_{n=1,2,\dots}$ satisfies the q-Fibonacci recurrence with the appropriate initial conditions.

Example 1.2. *Note that there are precisely $f_3 = 3$ tilings of a 1 by 3 chessboard, as follows.*

(1) *three monominoes*
(2) *a monomino, followed by a domino*

(3) *a domino, followed by a monomino.*

Then, tiling (1) has weight 0, tiling (2) has weight 2, and tiling (3) has weight 1. Thus, $F_4(q) = 1 + q + q^2$.

To finish the preliminaries, it is sometimes useful to *break* a tiling. We say that a tiling can be broken at cell a if a (single) domino does not cover cells a and $a+1$ (some authors say there is a *fault* at cell a). In many of these instances, it is useful to tile the two halves of the board separately; this leads to the notion of a *shifted* tiling. Namely,

$$F_{n+1}^{(a)}(q) = F_n^{(a)}(q) + q^{n-1+a} F_{n-1}^{(a)}(q), \quad F_1^{(a)}(q) = F_2^{(a)}(q) = 1. \tag{1}$$

It is easy to verify that $F_{n+1}^{(a)}(q)$ is a generating function for the latter portions of the tilings of a 1 by $n+a-1$ chessboard that can be broken at cell a (that is, the portions occurring after cell a); one simply shows that said generating function satisfies the same recurrence as $F_{n+1}^{(a)}(q)$, with the same initial conditions (conditioning on whether the last cell is covered by a domino or a monomino).

For the sake of concreteness, let us consider an example.

Example 1.3. *Consider the possible tilings of a 1 by 5 board that can be broken at cell 2. The latter three squares of such a tiling must be covered by either three monominoes, a domino followed by a monomino, or a monomino followed by a domino. In such a tiling (of the 1 by 5 board), the weights of the tilings of the latter three squares are 0, 3, and 4, respectively. Thus,* $F_4^{(2)}(q) = 1 + q^3 + q^4$.

Next, we give an example that shows how tilings with faults can be enumerated using these shifted q-Fibonacci numbers.

Example 1.4. *Consider the set of tilings of a 1 by 5 board that can be broken at cell 2. One can easily verify that there are six such tilings, as follows.*

(1) *a tiling of 5 monominoes*
(2) *a tiling with exactly one domino covering cells 3 and 4*
(3) *a tiling with exactly one domino covering cells 4 and 5*
(4) *a tiling with exactly one domino covering cells 1 and 2*
(5) *a tiling with exactly two dominoes, covering cells 1-2 and 3-4.*
(6) *a tiling with exactly two dominoes, covering cells 1-2 and 4-5.*

Note these tilings contribute weights 0, 3, 4, 1, 4, and 5, respectively. Further note that the generating function for the set of such tilings is $1 + q + q^3 + 2q^4 + q^5 = (1+q)(1+q^3+q^4)$, *which in turn is simply* $F_3(q) \cdot F_4^{(2)}(q)$. *The point is that one could obtain this directly;* $F_3(q)$ *accounts for how to tile the first two squares of the board, while* $F_4^{(2)}$ *accounts for how to tile the latter three squares (taking into account the fault at cell two).*

2. A Combinatorial Proof of a Result from Benjamin and Quinn

We begin by stating a result from [1]. This result can be obtained by differentiating the Fibonacci generating function. Alas, this method offers no enumerative insight on why the identity holds. Combinatorial proofs of this result have been given by Wood [6] and by Herreshoff [3]. We will give a new combinatorial proof of said result which will make it easy to formulate a q-generalization of this theorem.

Theorem 2.1. *For $n \geq 1$,*

$$f_1 + 2f_2 + \cdots + nf_n = (n+1)f_{n+2} - f_{n+4} + 3. \tag{2}$$

Before we give a combinatorial proof of this result, we need to establish some notation.

Let $A_1^{(1)}$ denote the set of tilings of a 1 by 1 board. Let $A_2^{(1)}$ and $A_2^{(2)}$ be two copies of the set of tilings of a 1 by 2 board that are distinguishable in some way (perhaps using boards of different colors). And for $1 \leq j \leq n$, let $A_j^{(1)}, A_j^{(2)}, \ldots, A_j^{(j)}$ denote j distinct copies of the set of tilings of a 1 by j board. Finally, let $S = \cup_{1 \leq k \leq l \leq n} A_k^{(l)}$. Note that clearly, $|S| = f_1 + 2f_2 + \cdots + nf_n$.

Now, let $B_{n+2}^{(1)}, \ldots, B_{n+2}^{(n+1)}$ denote $n+1$ distinct copies (using colors $1, \ldots, n+1$, respectively) of the set of tilings of a 1 by $n+2$ board, and let T denote the union of these sets. Then it is clear that $|T| = (n+1)f_{n+2}$.

The basic idea of the proof will be as follows. We will set aside a subset $S' \subseteq S$ of cardinality 3 (namely $A_1^{(1)} \cup A_2^{(1)}$), and a subset $T' \subseteq T$ of cardinality f_{n+4} (which we will specify in a moment), and produce a bijection from $T - T'$ to $S - S'$.

Define T' by

$$T' = B_{n+2}^{(n-1)} \cup B_{n+2}^{(n)} \cup (B_{n+2}^{(n+1)} - C) \tag{3}$$

where C denotes the set of tilings of a 1 by $n+2$ board of color $n+1$ which have a domino covering squares $n+1$ and $n+2$. Note that T' can be thought of as two copies of a set of $n+2$)-tilings along with a set of $(n+2)$-tilings all of which end in a monomino.

We claim that $|T'| = f_{n+4}$. To prove this, one could simply invoke the well-known Fibonacci identity $f_{n+4} = 3f_{n+2} - f_n$. However, we will give a combinatorial proof of this fact, for it will prove useful later. An examination of this easy proof will yield a nontrivial q-Fibonacci identity (see Theorem 3.1).

Lemma 2.2. $|T'| = f_{n+4}$.

Proof. We produce a bijection from T' to a set of $(n+4)$-tilings. Given a tiling $\Delta \in B_{n+2}^{(n-1)}$, simply append two monominoes at the end to obtain an $(n+4)$-tiling. If $\Delta \in B_{n+2}^{(n)}$, append a domino to the end of the tiling.

Given a tiling in $B_{n+2}^{(n+1)} - C$ note that the tiling can be thought of as an $(n+1)$-tiling Δ' followed with a monomino. Produce an $(n+4)$-tiling by inserting a domino between Δ' and the trailing monomino.

Since any $(n+4)$-tiling ends with either two monominoes, a domino, or a domino followed by a monomino, the map described above is clearly a bijection. $\square$

With this result in hand, to prove Theorem 2.1 it will suffice to produce a bijection

$$\left(\bigcup_{1 \leq l \leq n-2} B_{n+2}^{(l)} \right) \bigcup C \to S - \left(A_1^{(1)} \bigcup A_2^{(1)} \right). \tag{4}$$

First, map C onto $A_n^{(n)}$ in the natural way. Note that C is a set of $(n+2)$-tilings that end with a domino and $A_n^{(n)}$ is a set of n-tilings. Thus, the map in question simply deletes the domino at the end of such a given $(n+2)$-tiling. This correspondence is obviously a bijection.

Having done this, it remains to produce a bijection from $\bigcup_{1 \leq l \leq n-2} B_{n+2}^{(l)}$ onto the remaining elements of $S - (A_1^{(1)} \bigcup A_2^{(1)})$.

We proceed by considering the terminal segment of a given $(n+2)$-tiling. Given an $(n+2)$-tiling of color k, we scan the tiling from right to left. Note that one could just as easily scan the tiling from left to right. However, when attempting to generalize Theorem 2.1, one would then have to consider the shifted q-Fibonacci sequence. We stop scanning when we encounter a domino, or when we have scanned a total of $k+1$ tiles (a tile being a monomino or a domino).

For instance, given an $(n+2)$-tiling $\Delta \in B_{n+2}^{(1)}$ (i.e. of color $k = 1$), if the last tile is a domino, then we stop scanning and delete said domino to obtain an n-tiling of color $n-1$. Otherwise, the last two tiles are covered by a monomino followed by a domino or a pair of monominoes. In the former case, delete these two tiles to obtain an $(n-1)$-tiling of color $n-1$. In the latter case (i.e. where the maximum possible number of monominoes has been scanned), delete all of these monominoes to obtain an n-tiling. In any case where the maximum number of monominoes has been scanned without finding a domino, we shall color this tiling using color 1.

In general, given an $(n+2)$-tiling Δ of color $1 \le k \le n-2$ (that is, $\Delta \in B_{n+2}^{(k)}$), there are $k+2$ cases to consider. If the last domino covers cells $j+1$ and $j+2$ (where $n-k \le j \le n$), then remove that domino and all monominoes to its right to obtain a j-tiling to which we assign color $n-k$. If the tiling ends with $k+1$ monominoes, then delete them all to obtain an $(n+1-k)$-tiling to which we assign color 1.

Summing over all $n-2$ colors of $(n+2)$-tilings, we have a proof of Theorem 2.1. We refer the reader to Figure 1 for a diagram illustrating the preimages of the various sets used in this proof.

$$
\begin{array}{lllll}
A_3^{(1)} & A_4^{(1)} & \cdots & A_n^{(1)} & \\
+(n-1) & +(n-2) & & +2 & \\
\text{monominoes} & \text{monominoes} & & \text{monominoes} & \\
A_2^{(2)} & & & & \\
+\text{ domino} & & & & \\
\text{in cells } 3,\, 4 & & & & \\
A_3^{(2)} & A_3^{(3)} & & & \\
+\text{ domino} & +\text{ domino} & & & \\
\text{in cells } 4,\, 5 & \text{in cells } 4,\, 5 & & & \\
\vdots & \vdots & \ddots & & \\
A_{n-1}^{(2)} & A_{n-1}^{(3)} & \cdots & A_{n-1}^{(n-1)} & \\
+\text{ domino} & +\text{ domino} & & +\text{ domino} & \\
\text{in cells} & \text{in cells} & & \text{in cells} & \\
n,\, n+1 & n,\, n+1 & & n,\, n+1 & \\
A_n^{(2)} & A_n^{(3)} & \cdots & A_n^{(n-1)} & A_n^{(n)} \\
+\text{ domino} & +\text{ domino} & & +\text{ domino} & +\text{ domino} \\
\text{in cells} & \text{in cells} & & \text{in cells} & \text{in cells} \\
n+1,\, n+2 & n+1,\, n+2 & & n+1,\, n+2 & n+1,\, n+2 \\
\downarrow & \downarrow & & \downarrow & \downarrow \\
B_{n+2}^{(n-2)} & B_{n+2}^{(n-1)} & \cdots & B_{n+2}^{(1)} & C
\end{array}
$$

Figure 1. An illustration of the bijection used in Theorem 2.1.

To read Figure 1, note that under the inverse of the bijection in Theorem 2.1, the preimage of C is $A_n^{(n)}$. In general, note that the preimage of $B_{n+2}^{(k)}$ (for $1 \le k \le n-2$) is the union of the sets in the corresponding column of the table. Also recall that the three-element set $A_1^{(1)} \bigcup A_2^{(1)}$ was set aside.

3. q-ANALOGUES OF THE PREVIOUS RESULTS

Aesthetic considerations aside, one of the benefits of combinatorial proof is that often a judiciously constructed combinatorial argument can offer insight as to how to produce a q-analogue of a given combinatorial identity. In this section we mention how the preceding argument gives rise to a q-analogue of Theorem 2.1.

We begin by stating a q-analogue of the identity $f_{n+4} = 3f_{n+2} - f_n$. The previous combinatorial argument will prove useful.

Theorem 3.1. *For $n \ge 1$,*

$$F_{n+5}(q) = F_{n+3}(q) + q^{n+3}F_{n+3}(q) + q^{n+2}(F_{n+3}(q) - q^{n+1}F_{n+1}(q)). \tag{5}$$

Proof. This non-obvious identity becomes transparent by careful observation of the earlier combinatorial argument. Further note that putting $q = 1$ recovers the Fibonacci result.

As before, we start with a set of cardinality $3f_{n+2} - f_n$, namely 2 sets of $(n+2)$-tilings (which are of course distinguishable in some way) together with a set of $(n+2)$-tilings which do not end in a domino (i.e. whose last square is covered by a monomino). Note that this latter set has cardinality $f_{n+2} - f_n$. Also, the q-contribution of this set is $F_{n+3}(q) - q^{n+1}F_{n+1}(q)$, for the former term is the generating function of all $(n+2)$-tilings and the latter term is the generating function (with respect to the weighting function that we have discussed) for those $(n+2)$-tilings which end in a domino (and such a tiling can be thought of an n-tiling followed by a domino covering squares $n+1$ and $n+2$).

Now, given a tiling $\Delta \in B_{n+2}^{(n-1)}$, simply append two monominoes at the end to obtain an $(n+4)$-tiling. Under this map, these tilings make a q-contribution of $F_{n+3}(q)$. If $\Delta \in B_{n+2}^{(n)}$, we appended a domino to the end of the tiling. Under this map, these tilings make a q-contribution of $q^{n+3}F_{n+3}(q)$.

Given a tiling in $B_{n+2}^{(n+1)} - C$ we noted that the tiling can be thought of as an $(n+1)$-tiling Δ' followed by a monomino. Produce an $(n+4)$-tiling by inserting a domino between Δ' and the trailing monomino. As mentioned above, the q-contribution of $B_{n+2}^{(n+1)} - C$ is $F_{n+3}(q) - q^{n+1}F_{n+1}(q)$. Inserting a domino into the tiling makes the q-contribution of this set of tilings $q^{n+2}(F_{n+3}(q) - q^{n+1}F_{n+1}(q))$.

Since this map is a bijection onto a (single) set of $(n+4)$-tilings and the q-contribution of such a set is $F_{n+5}(q)$, we have a proof of Theorem 3.1. □

With this result, we can now state a q-generalization of Theorem 2.1.

Theorem 3.2. *For $n \ge 1$,*

$$\sum_{k=2}^{n+1} F_k(q)(1 + (k-2)q^k) =$$

$$(n-1)F_{n+3}(q) + q^{n+2}F_{n+3}(q) + q^{n+3}F_{n+3}(q) - F_{n+5}(q) + q^{n+1}(1 - q^{n+2})F_{n+1}(q) + q + 2. \tag{6}$$

Again, note that substituting $q = 1$ in the above recovers Theorem 2.1.

Proof. To prove this identity, we simply examine the bijection (actually its inverse) that was used to prove Theorem 2.1. Figure 1 in the previous section is meant to illustrate this bijection.

To prove the theorem, first consider the left hand side of Theorem 3.2. Under the indicated map, note that this map simply appends monominoes to the end of all tilings in $A_j^{(1)}$ for each $3 \leq j \leq n$. Thus, the total q-contribution of the top row in the figure (together with $A_1^{(1)} \bigcup A_2^{(1)}$, which had been set aside) is simply $F_2(q) + F_3(q) + \cdots + F_{n+1}(q)$. Then, given a tiling in $A_j^{(k)}$ (where $j, k \geq 2$), the map in question simply appends a domino covering squares $j+1$ and $j+2$, along with an appropriate number of monominoes. Thus, the q-contribution of tilings in $A_j^{(k)}$ $(j, k \geq 2)$ is $q^{j+1}F_{j+1}(q)$. Summing over all tilings clearly yields the left hand side of Theorem 3.2.

On the right hand side, note that the three tilings in $A_1^{(1)} \bigcup A_2^{(1)}$ make a total q-contribution of $q + 2$. Now, recall that $(\bigcup_{1 \leq l \leq n-2} B_{n+2}^{(l)}) \bigcup C$ is mapped onto $S - (A_1^{(1)} \bigcup A_2^{(1)})$. Since $(\bigcup_{1 \leq l \leq n-2} B_{n+2}^{(l)}) \bigcup C$ is $n-2$ copies of a set of $(n+2)$-tilings together with a set of $(n+2)$-tilings ending in a domino, it is clear that the right hand side of Theorem 3.2 equals

$$(n-2)F_{n+3}(q) + q^{n+1}F_{n+1}(q) + q + 2. \tag{7}$$

This yields an interesting q-Fibonacci identity, but we would like an identity where putting $q = 1$ immediately precipitates Theorem 2.1.

Writing equation (7) as

$$(n-1)F_{n+3}(q) + q^{n+1}F_{n+1}(q) + q + 2 - F_{n+3}(q) \tag{8}$$

and applying Theorem 3.1 to rewrite $F_{n+3}(q)$ finishes the proof. □

Remark 3.3. *It is possible to obtain a q-analogue of Theorem 2.1 in terms of the shifted tilings discussed in Examples 1.3 and 1.4. To modify the arguments, one merely appends monominoes and dominoes to the beginning of a given tiling, rather than at the end of said tiling. We invite the reader to thus produce an alternate formulation of Theorem 3.2.*

References

[1] A. Benjamin and J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof,* The Mathematical Association of America, 2003.

[2] K. Garrett, *Weighted Tilings and q-Fibonacci Numbers,* preprint.

[3] M. Herreshoff, *A Combinatorial Proof of* $\sum_{k=0}^{n} kf_n$ *or Coins on a Fibonacci Tiling,* to appear in Applications of Fibonacci Numbers Volume 11 (William Webb, ed.), Kluwer Academic Publishers, 2008.

[4] M. Shattuck. *Bijective Proofs of Parity Theorems for Partition Statistics,* Journal of Integer Sequences, 8 (2005), Article 05.1.5.

[5] C. Wagner, *Partition Statistics and q-Bell Numbers* $(q = -1)$, Journal of Integer Sequences, 7 (2004), Article 04.1.1.

[6] P. Wood, *A Bijective Proof of* $f_1 + 2f_2 + \cdots + nf_n = (n+1)f_{n+2} - f_{n+4} + 3$, Integers: The Electronic Journal of Combinatorial Number Theory, (2006), Article 6:A2.

MSC2000: 05A19, 11B39

Department of Mathematics and Computer Science, Augusta State University, Augusta, GA 30904

E-mail address: nsmith12@aug.edu