COMBINATORIAL PROOFS OF SOME FORMULAS FOR L_m^r

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Abstract. We provide combinatorial proofs of some formulas for the power of a Lucas number which as far as we know have only been proven using other methods. We achieve this by introducing a new kind of object we call a *bracket* into the usual square-and-domino tiling model.

1. Introduction

Let F_n and L_n denote the Fibonacci and Lucas numbers defined, respectively, by $F_0 = 0$, $F_1 = 1$ with $F_n = F_{n-1} + F_{n-2}$ if $n \geq 2$ and by $L_0 = 2$, $L_1 = 1$ with $L_n = L_{n-1} + L_{n-2}$ if $n \geq 2$. Recently, proofs which use tilings have explained a variety of identities involving Fibonacci and Lucas numbers, as popularized by Benjamin and Quinn in their text [3]. See for example, [1, 2, 5]. Here we add a certain feature to the usual square-and-domino tilings and find combinatorial proofs of the following four identities where $m \geq 1$, $n \geq 0$:

$$L_m^{2n+1} = \sum_{k=0}^{n} (-1)^{mk} \binom{2n+1}{k} L_{(2n+1-2k)m}, \tag{1.1}$$

$$L_m^{2n} = (-1)^{mn} \binom{2n}{n} + \sum_{k=0}^{n-1} (-1)^{mk} \binom{2n}{k} L_{(2n-2k)m}, \tag{1.2}$$

$$L_m^{2n+1} = \frac{1}{F_m} \sum_{k=0}^{n} (-1)^{m(n-k)} \frac{2k+2}{n+k+2} \binom{2n+1}{n-k} F_{(2k+2)m}, \tag{1.3}$$

and

$$L_m^{2n} = \frac{1}{F_m} \sum_{k=0}^{n} (-1)^{m(n-k)} \frac{2k+1}{n+k+1} \binom{2n}{n-k} F_{(2k+1)m}. \tag{1.4}$$

The first two identities above occur as V78 and V79 on pages 144–145 of *Proofs that Really Count* [3], where Benjamin and Quinn raise the question of finding their combinatorial proofs. See Vajda [6] for algebraic proofs. The latter two identities were recently derived by Ma and Zhang [4] using algebraic methods. In this note, we provide combinatorial proofs of Fibonacci and Lucas polynomial identities from which formulas (1.1)–(1.4) will follow as special cases upon using a new construction we term *bracketed tiling*.

2. Preliminaries

Consider a board of length n with cells labeled 1 to n. A tiling of this board (termed an *n-tiling*) is an arrangement of indistinguishable squares (pieces covering a single cell) and indistinguishable dominos (rectangular pieces covering two cells) which cover the board completely and no pieces overlap. Let $\mathfrak{F}_n$ denote the set of all (linear) n-tilings. When the board is circular, meaning that a domino may wrap around from cell n back to cell 1, we denote the set of all n-tilings by $\mathfrak{L}_n$. Circular tilings are also called *bracelets*. It is clear that $\mathfrak{F}_n \subseteq \mathfrak{L}_n$. Recall that

$$| \mathfrak{F}_n | = F_{n+1}, \qquad n \geq 1,$$

and

$$| \mathfrak{L}_n | = L_n, \qquad n \geq 1.$$

(If we let $\mathfrak{F}_0 = \{\varnothing\}$, the "empty tiling," and $\mathfrak{L}_0$ consist of two empty tilings of opposite orientation, then these relations hold for $n = 0$ as well.)

Now assign the weight x to every square in a tiling and the weight y to every domino. Given $T \in \mathfrak{F}_n$ (or $\mathfrak{L}_n$), define the weight $\omega(T)$ of the tiling to be the product of the weights of its tiles. The Fibonacci and Lucas polynomials (see, e.g., [3]) are given, respectively, as

$$F_n(x, y) := \sum_{T \in \mathfrak{F}_n} \omega(T)$$

and

$$L_n(x, y) := \sum_{T \in \mathfrak{L}_n} \omega(T).$$

As an example, we have $\mathfrak{L}_3 = \{sss,\ slr,\ lrs,\ rsl\}$ and $L_3(x, y) = x^3 + 3xy$, where s is a square, and l and r are the left and right halves of a domino. The $F_n(x, y)$ and $L_n(x, y)$ both satisfy a two-term recurrence of the form

$$a_{n+2} = xa_{n+1} + ya_n, \quad n \geq 1,$$

upon considering whether a tiling ends in a square or domino. By defining $F_0(x, y) = 1$ and $L_0(x, y) = 2$, the recurrence holds for $n = 0$ as well. Note that when $x = y = 1$, all tilings have unit weight, which implies that $F_n(1, 1) =| \mathfrak{F}_n |= F_{n+1}$ and $L_n(1, 1) =| \mathfrak{L}_n |= L_n$.

3. Bracketed Tilings

3.1. **Counting Bracketed Tilings.** A wide variety of combinatorial identities can be proven using constant-weight tilings, variable-weight tilings, or simply unit-weight tilings (see, e.g., [2, 3]). Here we take these methods in a different direction by introducing a new kind of object into our tilings. A *bracket* is an object that occupies a single cell, like a square. They come in two varieties, which we denote by $<$ and $>$, and must be placed according to the following criterion:

> *Every group of consecutive brackets must be properly paired and nested in a manner identical to parentheses. Such groups may occur even between the left and right halves of a domino.*

Bracketed tilings of length n with k bracket pairs, where $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, may be formed as follows. First select the k positions for the $<$ on a board of length n. This uniquely determines the positions of the $>$ since they must follow the $<$ without gaps. For once you have selected the slots to be occupied by left brackets, you fill in the gaps in such a way so that the first right bracket goes in the first available slot to the right of the first left bracket, the second right bracket goes in the first now available slot to the right of the second left bracket, and so on. If one runs out of spaces in which to place right brackets, then continue searching for spaces from left to right at the beginning of the tiling. Once they are placed, the left and right brackets are paired in a manner identical to parentheses. Below are two examples when $n = 8$ and $k = 3$:

$$\underline{\quad}\,\underline{\quad} \le \le \underline{\quad} \le \underline{\quad}\,\underline{\quad} \longrightarrow \underline{\quad}\,\underline{\quad} \le \le \ge \le \ge \ge$$

$$\underline{\quad}\,\underline{\quad} \le \underline{\quad}\,\underline{\quad} \le \le \underline{\quad} \longrightarrow \ge \underline{\quad} \le \ge \underline{\quad} \le \le \ge .$$

Note that in the second example, the $<$ in the sixth slot is paired with the $>$ in the first slot. We will say that a bracket pair $<>$ *wraps around* if the $>$ occurs to the left of the $<$. Once the positions of the k pairs of brackets are determined, cover the remaining $n-2k$ cells with a tiling of squares and dominos, where the left and the right halves of a domino may be separated by a group of consecutive brackets. This subtiling of squares and dominos may be a linear tiling or a bracelet containing a wraparound domino.

One might wonder, *a priori*, whether the bracket pairs $<>$ are uniquely determined once the brackets have been placed as described above. The proof of the following lemma shows that indeed they are.

Lemma 3.1. *For any $<$ within a bracketed tiling, the position of its matching bracket $>$ is uniquely determined.*

Proof. Selecting an arbitrary $<$, we can determine the position of its partner $>$ as follows. Start a count of 1 at the $<$. Move to the next cell, wrapping around if necessary. Either increment the counter if it contains a $<$ or decrement the counter if it contains a $>$ (note that these are the only two options regarding the next cell, since squares and dominos are not allowed between bracket pairs). Proceeding in this fashion, the matching $>$ is located where the counter reaches zero. □

We now differentiate two types of bracketed tilings. A *straight bracketed n-tiling* consists of k bracket pairs for some k, none of which wrap around, and whose subtiling of squares and dominos belongs to $\mathfrak{F}_{n-2k}$. A *bracketed n-bracelet* consists of k bracket pairs, some of which may wrap around, and whose subtiling of squares and dominos belongs to $\mathfrak{L}_{n-2k}$ (except when n is even and $k = \frac{n}{2}$, in which case there is just one possibility for the subtiling, not two). Let $\mathfrak{BF}_n$ denote the set of straight bracketed n-tilings and let $\mathfrak{BL}_n$ denote the bracketed n-bracelets. It is clear that $\mathfrak{BF}_n \subseteq \mathfrak{BL}_n$. Below are some examples:

$$\{l <><<>> r,\ slr <><> s,\ <<><<>>>\} \subset \mathfrak{BF}_8,$$

$$\{rl <><<>>,\ > rsslrl <,\ <>><<<>>\} \subset \mathfrak{BL}_8 - \mathfrak{BF}_8.$$

The following are not bracketed tilings at all

$$<<>,\ < lr >,\ < s >,\ ll <> rr,\ s <>><<> .$$

We extend the weights defined above on $\mathfrak{F}_n$ and $\mathfrak{L}_n$ to $\mathfrak{BF}_n$ and $\mathfrak{BL}_n$ by assigning every bracket pair the weight y and define the weight of a bracketed tiling to be the product of the weights of all of its tiles and brackets. Define the polynomials $F_n^*(x,y)$ and $L_n^*(x,y)$ by

$$F_n^*(x,y) := \sum_{T\in\mathfrak{BF}_n} \omega(T)$$

and

$$L_n^*(x,y) := \sum_{T\in\mathfrak{BL}_n} \omega(T).$$

When $n = 3$, for example, we have $\mathfrak{BL}_3 = \{sss,\ slr,\ lrs,\ rsl,\ s <>,\ <> s,\ > s <\}$ and $L_3^*(x,y) = x^3 + 6xy$.

Counting bracketed bracelets according to the number of $<>$ pairs yields the following proposition.

Proposition 3.2. *If $n \geq 0$, then*

$$L^*_{2n+1}(x,y) = \sum_{k=0}^{n} y^k \binom{2n+1}{k} L_{2n+1-2k}(x,y) \tag{3.1}$$

and

$$L^*_{2n}(x,y) = y^n \binom{2n}{n} + \sum_{k=0}^{n-1} y^k \binom{2n}{k} L_{2n-2k}(x,y). \tag{3.2}$$

Proof. Let k denote the number of $<>$ pairs in a bracketed m-bracelet, where $0 \leq k \leq \lfloor \frac{m}{2} \rfloor$. Select k cells on an m-board to be occupied by a $<$ in $\binom{m}{k}$ ways, which uniquely determines the positions of the $>$. These k bracket pairs contribute weight y^k. Cover the remaining $m-2k$ cells with a (weighted) member of $\mathfrak{L}_{m-2k}$. If $m = 2n$ is even, the cases when $k = n$ and $k < n$ must be differentiated. □

One may also count bracketed bracelets by filling in the gaps between sets of consecutive brackets with straight tilings instead of bracelets.

Proposition 3.3. *If $n \geq 0$, then*

$$L^*_{2n+1}(x,y) = \sum_{k=0}^{n} y^k \binom{2n+2}{k} F_{2n+1-2k}(x,y) \tag{3.3}$$

and

$$L^*_{2n}(x,y) = \sum_{k=0}^{n} y^k \binom{2n+1}{k} F_{2n-2k}(x,y). \tag{3.4}$$

Proof. Suppose the length of a bracketed bracelet is m. Choose an arbitrary subset S of $[m+1]$ of size k, where $0 \leq k \leq \lfloor \frac{m}{2} \rfloor$. If $m+1 \notin S$, then place a $<$ in the positions on an m-board corresponding to the elements of the subset, which uniquely determines the $>$. If $m+1 \in S$, place a $<$ in the $k-1$ positions on an m-board corresponding to the elements of $S - \{m+1\}$, which determines the $>$; then add the left half of a domino to the right-most cell not occupied by a bracket and the right half of a domino to the left-most unoccupied cell. In either case, fill in the remaining $m-2k$ cells with a (weighted) member of $\mathfrak{F}_{m-2k}$. If $m = 8$, for example, the bracketed tilings $> sl <> rs <$ and $>> rssl <<$ would be members of $\mathfrak{BL}_8$ which correspond, respectively, to the subsets $S = \{4, 8\}$ and $S = \{7,\ 8,\ 9\}$ of $[9]$. □

3.2. **A Sign-Changing Involution.** Define the sign of a member of $\mathfrak{BL}_m$ as $(-1)^k$, where k denotes the number of bracket pairs. We have the following.

Proposition 3.4. *If $n \geq 0$, then*

$$x^{2n+1} = \sum_{k=0}^{n} (-y)^k \binom{2n+1}{k} L_{2n+1-2k}(x,y) \tag{3.5}$$

and

$$x^{2n} = (-y)^n \binom{2n}{n} + \sum_{k=0}^{n-1} (-y)^k \binom{2n}{k} L_{2n-2k}(x,y). \tag{3.6}$$

Proof. The right sides of (3.5) and (3.6) give the total signed weight of all members of $\mathfrak{BL}_m$ for odd and even m, respectively. To complete the proof, we define a sign-reversing, weight-preserving involution of $\mathfrak{BL}_m$ whose sole survivor is the all-square tiling, which has weight x^m. Within a member of $\mathfrak{BL}_m$, call either a domino $l \cdots r$ or an un-nested bracket pair $< \cdots >$ an *outermost pair*. Define the *first* outermost pair to be the one whose left half is furthest to the left. Switching the first outermost pair to the opposite type produces the desired involution. For example, the tilings $> sl <> rs <$ and $>> rssl <<$ in $\mathfrak{BL}_8$ would be paired, respectively, with $> s <<>> s <$ and $>>> ss <<<$. □

Proposition 3.5. *If* $n \geq 0$, *then*

$$x^{2n+1} = \sum_{k=0}^{n} (-y)^k \left[\binom{2n+1}{k} - \binom{2n+1}{k-1} \right] F_{2n+1-2k}(x,y) \tag{3.7}$$

and

$$x^{2n} = \sum_{k=0}^{n} (-y)^k \left[\binom{2n}{k} - \binom{2n}{k-1} \right] F_{2n-2k}(x,y). \tag{3.8}$$

Proof. Let k denote the total number of bracket pairs plus the number of wraparound dominos (either 0 or 1) within a member of $\mathfrak{BL}_m$. Then the right sides of (3.7) and (3.8) give the total signed weight of all members of $\mathfrak{BL}_m$ for odd and even m according to the value of k; note that the sign is $(-1)^{k-1}$ if a wraparound domino occurs. Now apply the involution of the preceding proof. □

Remark 3.6. *Taking* $x = y = 1$ *in the above proofs, we get combinatorial explanations for such identities as*

$$1 = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} L_{2n+1-2k}$$

and

$$1 = \sum_{k=0}^{n} (-1)^k \left[\binom{2n+1}{k} - \binom{2n+1}{k-1} \right] F_{2n+2-2k}.$$

3.3. **Proofs of (1.1)–(1.4).** To complete the proofs of formulas (1.1)–(1.4), we will need the values of $F_n(x,y)$ and $L_n(x,y)$ when $x = L_m$ and $y = (-1)^{m+1}$. The following relations are equivalent to special cases of a more general result which was established in [2] and given both algebraic and combinatorial proofs.

Lemma 3.7. *If* $m, n \geq 1$, *then*

$$F_n\left(L_m, (-1)^{m+1}\right) = \frac{F_{(n+1)m}}{F_m} \tag{3.9}$$

and

$$L_n\left(L_m, (-1)^{m+1}\right) = L_{nm}. \tag{3.10}$$

Relations (3.9) and (3.10) also follow from substituting the Binet formula for L_m into those for $F_n(x,y)$ and $L_n(x,y)$ and simplifying.

We can now complete the proofs of identities (1.1)–(1.4).

Proof. Identities (1.1) and (1.2) follow, respectively, from (3.5) and (3.6) upon substituting $x = L_m$ and $y = (-1)^{m+1}$ and applying (3.10). Substituting $x = L_m$ and $y = (-1)^{m+1}$ into (3.7) and (3.8), and applying (3.9), yields identities (1.3) and (1.4), upon replacing k with $n-k$ and observing

$$\binom{r}{t} - \binom{r}{t-1} = \frac{r-2t+1}{r-t+1}\binom{r}{t}$$

for positive integers r and t. □

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