EXTENSION OF THE GCD STAR OF DAVID THEOREM TO MORE THAN TWO GCDS

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Abstract. The GCD Star of David Theorem and the numerous papers related to it have largely been devoted to showing the equality of the greatest common divisors of two sets of elements formed by partitioning various arrays of binomial coefficients for any location of these arrays in Pascal's triangle. In this paper, we extend the study to arrays divided into n subsets with n equal greatest common divisors for $n = 2, 3, 4, 5$, and ultimately, for arbitrary $n \geq 2$.

1. Introduction

The genesis of Gould's Star of David Conjecture [3], first proved by Hillman and Hoggatt [4], was a paper by Hoggatt and Hansell [5] showing that the product of the six binomial coefficients surrounding any given entry in Pascal's triangle is a perfect square. Equivalently, if one numbers consecutively the six entries forming a "hexagon" around any given entry, the product of the even numbered entries equals the product of the odd numbered entries. In addition to Gould's conjecture, Hoggatt and Hansell's paper sparked a number of papers [2, 7, 8, 10, 11] on perfect square and equal product patterns in Pascal's triangle.

The starting point from the present paper is the paper by Usiskin [11] in which he showed that the product of the a's equals the product of the b's equals the product of the c's for the three-diamond array of binomial coefficients shown in Figure 1 for any location of the array in Pascal's triangle.

$$\begin{array}{ccccccccccccccccc}
 & & a & & \bullet & & \bullet & & b & & \bullet & & \bullet & & c & & \\
 & b & & c & & \bullet & & c & & a & & \bullet & & a & & b & \\
c & & a & & b & & a & & b & & c & & b & & c & & a \\
 & b & & c & & \bullet & & c & & a & & \bullet & & a & & b & \\
 & & a & & \bullet & & \bullet & & b & & \bullet & & \bullet & & c & &
\end{array}$$

Figure 1.

Now, as Gould was led to conjecture the original GCD Star of David Theorem by considering the paper by Hoggatt and Hansell, we show here that $\gcd(S_a) = \gcd(S_b) = \gcd(S_c)$ where S_a, S_b, and S_c are the sets of a's, b's, and c's in Figure 1 and $\gcd(S_t)$ denotes the greatest common divisor of the elements in S_t for $t = a$, b, or c. In fact, similar results can be obtained for n n-by-n diamonds for $n = 2, 3, 4$, and 5 and we conjecture that similar results hold for all greater values of n. In the last section we show how to construct arbitrarily large triangular arrays that can be partitioned into n subsets with equal GCDs for arbitrary $n \geq 2$.

The proofs of all these results depend on the following easily proved lemmas [1, 9].

Lemma 1. *For the three arrays shown, if x, y, and z are adjacent coefficients anywhere in Pascal's triangle,*

$$\begin{array}{ccc} x & & y \\ & z & \end{array}\quad , \quad \begin{array}{ccc} x & & z \\ & y & \end{array}\quad , \quad \begin{array}{ccc} z & & x \\ & y & \end{array}$$

then $\gcd(x, y) = \gcd(x, y, z)$.

Lemma 2. *For the two arrays shown, if x, y, z, and w are coefficients anywhere in Pascal's triangle*

$$\begin{array}{ccc} & & y \\ x & w & \\ & & z \end{array}\quad , \quad \begin{array}{ccc} y & & \\ & w & x \\ z & & \end{array}$$

with x, y, and z adjacent to w, then $\gcd(x, y, z) = \gcd(x, y, z, w)$.

2. Results for Diamonds

We first consider Usiskin's array of Figure 1.

Theorem 1. *For any location of the array of coefficients in Figure 1 in Pascal's triangle,*

$$\gcd(S_a) = \gcd(S_b) = \gcd(S_c).$$

Proof. We first number the entries in Figure 1 as shown in Figure 2.

$$\begin{array}{ccccccccccccccccc}
 & & a_1 & & r & & s & & b_4 & & u & & v & & c_7 & & \\
 & b_1 & & c_1 & & t & & c_4 & & a_4 & & w & & a_7 & & b_7 & \\
c_2 & & a_2 & & b_2 & & a_5 & & b_5 & & c_5 & & b_8 & & c_8 & & a_8 \\
 & b_3 & & c_3 & & \bullet & & c_6 & & a_6 & & \bullet & & a_9 & & b_9 & \\
 & & a_3 & & \bullet & & \bullet & & b_6 & & \bullet & & \bullet & & c_9 & &
\end{array}$$

Figure 2.

Let $d = \gcd(S_a)$, $e = \gcd(S_b)$, and $f = \gcd(S_c)$. Since $d \mid a_i$ for $1 \le i \le 9$, repeated use of Lemma 1 and Lemma 2 show that $d \mid c_8$, $d \mid b_7$, $d \mid b_9$, $d \mid c_9$, $d \mid b_8$, $d \mid w$, $d \mid c_5$, $d \mid b_5$, $d \mid c_4$, $d \mid c_6$, $d \mid t$, $d \mid c_1$, $d \mid r$, $d \mid s$, $d \mid b_4$, $d \mid u$, $d \mid v$, $d \mid c_7$, $d \mid b_2$, $d \mid b_1$, $d \mid c_3$, $d \mid b_3$, $d \mid b_6$, and $d \mid c_2$. Therefore, $d \mid e$ and $d \mid f$. Similarly, one shows that $e \mid d$ and $e \mid f$ and also that $f \mid d$ and $f \mid e$. Thus, $d = e = f$ as claimed. $\square$

Results entirely analogous to Theorem 1 but for $n = 2$, 4, and 5 can all be proved using only Lemma 1 and Lemma 2. The challenge in each case, of course, is to partition the array under consideration into subsets for which the greatest common divisors are indeed equal. Also, we note that we were unable to get similar results for n diamonds with $n \ge 6$ though we have little doubt that such results are true for arbitrary n. Interested readers can contact the authors for suitable partitions for $n = 2$, 4, and 5.

3. Results for Triangles of Least Support

It is interesting to note that Lemma 1 and Lemma 2 immediately allow for the extension of Theorem 1 to what might be called the equilateral triangle of least support of an array; i.e.,

$$
\begin{array}{ccccccccc}
\bullet & & a & & \bullet & & b & & \bullet \\
 & b & & b & & a & & a & \\
 & & a & & \bullet & & b & & \\
 & & & \bullet & & \bullet & & & \\
 & & & & \bullet & & & &
\end{array}
$$

the smallest equilateral triangle containing the array in question where the top side is along a row of Pascal's triangle and whose other two sides are along main diagonals as shown for the two diamonds in the figure which is the two diamond analogue of Figure 1.

Also note that each dot in the figure can be replaced by either a or b without changing the final result. Further, we observe that Lemma 1 and Lemma 2 allow the extension of all theorems in the existing literature showing equal GCDs to be extended to solid triangles of least support for the arrays in question where, again, the added coefficients can be allocated to any of the subsets into which the array in question was originally divided without altering the result.

4. A General Construction for Arbitrarily Many GCDs

For arbitrary $n \geq 2$, we give a construction for arbitrarily large triangular sets of binomial coefficients that can be partitioned into subsets S_i, $1 \leq i \leq n$, with the property that $\gcd(S_i) = \gcd(S_j)$ for $1 \leq i < j \leq n$ for any positioning of the constructed set in Pascal's triangle. The construction is completely general and could be given a general proof. However, to conserve space, we consider only the case $n = 4$.

Consider the array of coefficients shown in Figure 3.

$$
\begin{array}{ccccccccccccccccc}
a_1 & & b_2 & & c_3 & & d_4 & & a_5 & & b_6 & & c_7 & & d_8 & & a_9 \\
 & d_2 & & a_3 & & b_4 & & c_5 & & d_6 & & a_7 & & b_8 & & c_9 & \\
 & & b_3 & & c_4 & & d_5 & & a_6 & & b_7 & & c_8 & & d_9 & & \\
 & & & a_4 & & b_5 & & c_6 & & d_7 & & a_8 & & b_9 & & & \\
 & & & & c'_5 & & d'_6 & & a'_7 & & b'_8 & & c'_9 & & & & \\
 & & & & & b'_6 & & c'_7 & & d'_8 & & a'_9 & & & & & \\
 & & & & & & d'_7 & & a'_8 & & b'_9 & & & & & & \\
 & & & & & & & c'_8 & & d'_9 & & & & & & & \\
 & & & & & & & & a'_9 & & & & & & & &
\end{array}
$$

Figure 3.

Theorem 2. *For the array of Figure 3, and any triangular array obtained from it by extending the pattern arbitrarily far to the right, located anywhere in Pascal's triangle,*

$$\gcd(S_a) = \gcd(S_b) = \gcd(S_c) = \gcd(S_d).$$

Proof. Let $d = \gcd(S_a)$, $e = \gcd(S_b)$, $f = \gcd(S_c)$, and $g = \gcd(S_d)$. First consider d. As argued earlier, Lemma 1 and Lemma 2 suffice to show that d divides all the elements in the equilateral triangle of coefficients of least support containing a_5, a_6, and a'_7. Moreover, it is clear that an element of S_a must appear at least once in the array among the top five elements of each diagonal in the array sloping downward and to the left from the fifth on. So, by Lemma 1, d divides every element in the top five rows of the entire array and hence in the entire array and any extension to the right, if any. Thus, $d \mid e$, $d \mid f$, and $d \mid g$. Similarly, e divides all coefficients in the triangle of least support containing b_2, b_3, and b_4. Hence, as for d, e divides every element in the array and $e \mid d$, $e \mid f$, and $e \mid g$. Since a similar argument holds for both f and g, it follows that $d = e = f = g$ as claimed. □

Observe that as noted in the proof of Theorem 2, the array in Figure 3 can be extended arbitrarily far to the right subject to the condition that an element of each of S_a, S_b, S_c, and S_d appear among the top four elements in each succeeding diagonal sloping downward and to the left. Moreover, except for c'_5, d'_6, and a'_7, the elements in the bottom five rows of the triangle can be completely arbitrarily chosen from among S_a, S_b, S_c, and S_d without affecting the validity of the result. Of course, a similar situation prevails for all other values of n as well.

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