COMBINATORICS OF TWO-TONED TILINGS

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ABSTRACT. We introduce the function $a(r, n)$ which counts tilings of length $n + r$ that utilize white tiles (whose lengths can vary between 1 and n) and r identical red squares. These tilings are called two-toned tilings. We provide combinatorial proofs of several identities satisfied by $a(r, n)$ and its generalizations, including one that produces kth order Fibonacci numbers. Applications to integer partitions are also provided.

1. INTRODUCTION

For nonnegative integers r and n, we define $a(r, n)$ to count the ways to tile a strip of length $n + r$ using white tiles of any length (with total length n) and r identical red squares (where each square has length one). We call such a tiling a *two-toned tiling* of length $n + r$ or simply an $(n + r)$-tiling. For example, $a(1, 3) = 12$ counts the 4-tilings: $R111$, $R12$, $R21$, $R3$, $1R11$, $1R2$, $11R1$, $2R1$, $111R$, $12R$, $21R$, $3R$, where R denotes a red square and a number denotes the length of a white tile. For example, $1R2$ indicates a white square followed by a red square followed by a white tile of length 2 (a domino). Likewise, $a(3, 1) = 4$ counts the 4-tilings $RRR1$, $RR1R$, $R1RR$, $1RRR$, and in general, $a(r, 1) = r + 1$.

The initial motivation for exploring the $a(r, n)$ function was to count compositions of integers (i.e., ordered partitions) with various restrictions, as described in [2]. Indeed, we'll see how the enumeration of certain two-toned tilings provides formulas for the number of compositions of an integer that contain at least (or exactly) p instances of a specified summand. We'll also see some surprising connections with Fibonacci numbers and their generalizations.

One can compute $a(r, n)$ recursively as follows.

Identity 1. *For $r \geq 0$, $a(r, 0) = 1$. For $n \geq 1$, $a(0, n) = 2^{n-1}$. For $n, r \geq 1$,*
$$a(r, n) = a(r - 1, n) + 2a(r, n - 1) - a(r - 1, n - 1).$$

Proof. With no white tiles, $a(r, 0) = 1$ counts the r-tiling consisting of r red squares. With no red squares, $a(0, n) = 2^{n-1}$ counts the tilings of an n-strip using only white tiles (equivalently, compositions of the integer n); to construct such a tiling, we can decide for each cell $1 \leq i \leq n - 1$ whether or not to end a tile at that cell. When n and r are both positive, we establish the recurrence by considering the three ways that an $(n + r)$-tiling can end. Clearly, there are $a(r - 1, n)$ tilings that end with a red square, and there are $a(r, n - 1)$ tilings that end with a white square. To create a tiling that ends with a longer white tile, we take a tiling of length $n + r - 1$ with r red squares that does not end with a red square (there are $a(r, n-1) - a(r-1, n-1)$ of these) and lengthen the terminating white tile by one unit. Hence the number of $(n + r)$-tilings is $a(r - 1, n) + 2a(r, n - 1) - a(r - 1, n - 1)$, as desired. $\square$

Using this recurrence, small values of $a(r, n)$ are displayed in Table 1.

n	0	1	2	3	4	5
r						
0	1	1	2	4	8	16
1	1	2	5	12	28	64
2	1	3	9	25	66	168
3	1	4	14	44	129	360
4	1	5	20	70	225	681
5	1	6	27	104	363	1182

TABLE 1. A table of $a(r,n)$ for $0 \leq r, n \leq 5$.

2. COMBINATORIAL IDENTITIES

In this section, we will present several ways to express $a(r,n)$ as sums of products of binomial coefficients. But before we do that, we can't resist explaining the sums of the diagonals of Table 1, reminiscent of a famous pattern from Pascal's triangle.

Identity 2. *For $n \geq 0$,*

$$\sum_{r=0}^{n} a(r, n-r) = F_{2n+1}.$$

Proof. Combinatorially, the left side of the identity counts two-toned n-tilings with r red squares, for $0 \leq r \leq n$. As is well-known [1], F_{2n+1} counts tilings of length $2n$ using only white tiles of length 1 or 2 (i.e., squares and dominoes). The bijection appears in [1, page 14, exercise 8] but we can describe it in one sentence. Reading the two toned n-tiling from left to right, we replace each red square with a domino, each white square with two consecutive squares, and more generally, each white tile of length k becomes a square followed by $k-1$ dominoes followed by a square. $\square$

By partitioning the two-toned $(n+r)$-tilings in various ways, one can express $a(r,n)$ in terms of binomial coefficients. For example, by considering the number of white tiles, one obtains the following identity.

Identity 3. *For $r \geq 0$ and $n \geq 1$,*

$$a(r,n) = \sum_{j=1}^{n} \binom{n-1}{j-1} \binom{r+j}{r}.$$

Proof. We show that, for $1 \leq j \leq n$, the summand enumerates two-toned $(n+r)$-tilings that use exactly j white tiles (along with r red squares). To see this, we first create an n-tiling that uses exactly j white tiles and no red squares. This can be achieved $\binom{n-1}{j-1}$ ways by deciding, for each of the cells 1 through $n-1$, which $j-1$ of them will contain the end of a tile. (Cell n must always contain the end of the last tile.) Labeling the white tiles $w_1, w_2, \ldots, w_j$ the number of ways to interleave them with r red squares is $\binom{r+j}{r}$ since among the $r+j$ tiles, we choose which tiles will be the red squares. Then the remaining tiles will be white tiles in their given order. For example, if $j = 6$ and $r = 4$ and we decide that tiles 2, 3, 5, and 8 will be red squares, then we must have the tiling $w_1 RRw_2 Rw_3 w_4 Rw_5 w_6$. $\square$

We note that another way to obtain the second binomial coefficient in the previous identity is to use "multisets." The notation $\left(\!\binom{n}{k}\!\right)$ (sometimes pronounced "n multi-choose k") counts the size k multisubsets of $\{1,\ldots,n\}$, which are size k subsets where repetition of elements is allowed. For example, $\left(\!\binom{9}{7}\!\right)$ would count size 7 multisubsets of $\{1,\ldots,9\}$, like $\{2,2,2,3,5,8,8\}$. It is well-known that $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$; it also counts the ways to express the integer k as an ordered sum of n nonnegative integers. (For example, the multisubset $\{2,2,2,3,5,8,8\}$ would correspond to expressing 7 as the sum of the nine integers $7 = 0+3+1+0+1+0+0+2+0$, where the ith summand denotes the number of times that i appears in the multisubset. Thus, after we have created a length n tiling with j white tiles, the r red squares can be interleaved by choosing, among $j+1$ locations (before and after each tile) how many red squares to insert. This can be done $\left(\!\binom{j+1}{r}\!\right) = \binom{r+j}{r}$ ways.

The last identity considered how the red squares were interleaved among the white tiles. The next identity takes the reverse approach.

Identity 4. *For $r \geq 0$, $n \geq 1$,*

$$a(r,n) = \sum_{j=1}^{r+1} \binom{r+1}{j}\binom{n-1}{j-1}2^{n-j}.$$

Proof. Starting with r identical red squares, there are $r+1$ potential regions (before and after each red square) where white tiles can potentially go. We show that the summand counts those two-toned tilings with exactly j non-empty white regions. Our construction will first insert n white squares into these j regions, then convert some of the adjacent white squares into longer tiles. To create such a tiling, first choose j regions that will be required to have at least one white tile. This can be done in $\binom{r+1}{j}$ ways. Next, for each subset $\{x_1, x_2, \ldots, x_{j-1}\}$ of $\{1, 2, \ldots, n-1\}$, which can be chosen in $\binom{n-1}{j-1}$ ways, we have $1 \leq x_1 < x_2 < \cdots < x_{j-1} < n$, and we put x_1 white squares in region 1, $x_2 - x_1$ white squares in region 2, $x_3 - x_2$ white squares in region 3, and so on, and $n - x_{j-1}$ white squares in region j. For each white square, except for the leftmost square in each of the j regions, we can freely decide whether or not to attach it to the white tile on its left. This can occur in 2^{n-j} ways. Hence the number of tilings with exactly j white regions is $\binom{r+1}{j}\binom{n-1}{j-1}2^{n-j}$, as desired. $\square$

The next identity is simpler, since the power of 2 term is entirely outside the sum. The proof is trickier, as is to be expected, since the power of 2 could possibly be negative. An intriguing aspect of the following proof is that it crucially depends on the number of adjacent red squares in three different tiling problems, yet that quantity does not appear in the sum.

Identity 5. *For $r \geq 0$, $n \geq 1$,*

$$a(r,n) = 2^{n-r-1}\sum_{j=0}^{r}\binom{r+1}{j}\binom{n+r-j}{n}.$$

Proof. Let T denote the set of two-toned $(n+r)$-tilings, with r red squares and white tiles with total length n. By definition, $|T| = a(r,n)$. Next, let S denote the set of square-only tilings with r red squares and n white squares. Clearly $|S| = \binom{n+r}{r}$. Finally, we define D to be the set of "divided" square-only tilings with r red squares, n white squares, and where we are allowed to place a red vertical divider in the boundary between any consecutive red squares or to the left of a red square on cell 1 or to the right of a red square on cell $n+r$. (You can think of the left and right "walls" of the two-toned $(n+r)$-tiling colored red.) For example,

with $r = 10$ and $n = 4$, the tiling $|RRR|RwRwwR|RwRR|R|$ is a valid divided tiling where we have inserted dividers to the left of the first, fourth, seventh, and tenth red squares, as well as at the right wall. The divided tiling $R|R|RRwRwwR|RwR|R|R$ has tiles in the same order, but with dividers to the left of the second, third, seventh, ninth, and tenth red squares. We claim that for $0 \leq j \leq r$, the summand $\binom{r+1}{j}\binom{n+r-j}{n}$ enumerates those tilings that have exactly j dividers. To see this, we begin with r consecutive red squares and insert dividers in j of the $r + 1$ potential boundaries (which can be done in $\binom{r+1}{j}$ ways). Next we allocate n white squares among the $r + 1 - j$ undivided regions in $\left(\binom{r+1-j}{n}\right) = \binom{n+r-j}{n}$ ways. Consequently,

$$|D| = \sum_{j=0}^{r} \binom{r+1}{j}\binom{n+r-j}{n}.$$

To complete the proof, we need to prove that $a(n, r) = 2^{n-r-1}|D|$, which we accomplish by means of the intermediate set S.

For $k \geq 0$, we define S_k to be the set of tilings in S containing exactly k red/red boundaries (including a boundary before an initial red square or after a terminating red square). For example, when $r = 10$ and $n = 4$, the tiling $RRRRwRwwRRwRRR$ has $k = 8$, whereas $wwwwRRRRRRRRRR$ has $k = 10$. Analogously, we can partition T and D by defining T_k and D_k as the set of tilings in T or D, respectively, with exactly k red/red boundaries, as previously described. For any tiling in S_k, there are exactly 2^k ways that we can insert dividers to create a tiling in D_k. Consequently,

$$|D_k| = 2^k|S_k|.$$

Likewise, for a fixed k and each tiling in S_k, we can create 2^v tilings in T_k where v, a function of k, is the number of white/white boundaries (since such boundaries can be joined together to create non-square tiles). For example, the tiling $RRRRwRwwRRwRRR$ has $v = 1$ and creates 2 tilings in T_8 since the ww squares in the middle can be left alone or turned into a domino. A tiling with r red squares and k red/red boundaries will necessarily have $r + 1 - k$ red/white boundaries (as read from left to right) since after the left wall and after each of the red squares, we must have a red/red or red/white boundary. Moreover, because the tiling begins and ends with a red wall, there must be an equal number of red/white boundaries and white/red boundaries, thus there are also $r + 1 - k$ white/red boundaries. For example, the tiling $wRRwwRw$ has $r = 3$ and $k = 1$ has $r + 1 - k = 3$ red/white boundaries (before the first, second and fourth white square) and has 3 white/red boundaries (after the first, third and fourth white square). Since the total number of boundaries in an $(n + r)$-tiling is $n + r + 1$, it follows that $v = (n + r + 1) - k - 2(r + 1 - k) = n - r + k - 1$. Therefore, $|T_k| = 2^{n-r+k-1}|S_k| = 2^{n-r+k-1}|D_k|/2^k = 2^{n-r-1}|D_k|$. Consequently,

$$a(n, r) = |T| = \sum_{k \geq 0} |T_k| = \sum_{k \geq 0} 2^{n-r-1}|D_k| = 2^{n-r-1}|D|$$

as desired. $\qquad\square$

3. Generalizations

For $s \geq 0$, we combinatorially define $a_s(r, n)$ to be the number of two-toned tilings of a strip of length $r + n + s$ with r red squares, with the restriction that the last s tiles must be white. Note that $a_0(r, n) = a(r, n)$. The motivation for this definition comes from the following identity.

Identity 6. *For $s \geq 1$, $r, n \geq 0$,*

$$a_s(r, n) = \sum_{j=0}^{n} a_{s-1}(r, j).$$

Proof. The left side counts two-toned tilings of length $r + n + s$ with r red squares and the restriction that the last s tiles are white. If j denotes the length of the last white tile, then $1 \leq j \leq n + 1$ and the preceding tiling can occur in $a_{s-1}(r, n + 1 - j)$ ways. (Note that the length of a tiling counted by this quantity is $r + (n + 1 - j) + (s - 1) = r + n + s - j$.) Consequently, $a_s(r, n) = \sum_{j=1}^{n+1} a_{s-1}(r, n + 1 - j) = \sum_{j=0}^{n} a_{s-1}(r, j)$. $\qquad\square$

Identity 7. *For $r \geq 1$, $s, n \geq 0$,*

$$a_s(r, n) = \sum_{j=0}^{n} a_{s+j}(r - 1, n - j).$$

Proof. The left side counts the same quantity as in Identity 6. The last red square of such a tiling must be followed by at least s (and at most $s + n$) white tiles. For $0 \leq j \leq n$, the number of tilings that have exactly $s + j$ white tiles after the last red square is equal to the number of $(n + r - 1 + s)$-tilings with $r - 1$ red squares where the last $s + j$ tiles must be white, which is $a_{s+j}(r - 1, n - j)$, and the identity follows. $\qquad\square$

We can generalize the statement and proof of Identity 3 to get the following.

Identity 8. *For $r, s \geq 0$, $n \geq 1$,*

$$a_s(r, n) = \sum_{j=0}^{n} \binom{n - 1 + s}{j - 1 + s} \binom{r + j}{r}.$$

Proof. This time we create an $(n + r + s)$-tiling by first creating, for each $0 \leq j \leq n$ an $(n + s)$-tiling with exactly $s + j$ white tiles, which can be done in $\binom{n+s-1}{s+j-1}$ ways. Next we interlace the r red squares with the first j white tiles, which can be done in $\binom{r+j}{r}$ ways, producing an $(n + r + s)$-tiling with r red squares that ends with at least s white tiles. Summing over all values of j from 0 to n produces the identity. $\qquad\square$

The situation where $s = r$ has an especially nice closed form.

Identity 9. *For $r \geq 0$ and $n \geq 1$,*

$$a_r(r, n) = \binom{n + r}{r} \frac{n + 2r}{n + r} 2^{n-1}.$$

Proof. We first algebraically simplify the expression on the right. Since $\binom{n+r}{r}\frac{n+2r}{n+r} = \binom{n+r}{r} + \frac{r}{n+r}\binom{n+r}{r} = \binom{n+r}{r} + \binom{n+r-1}{r-1}$, we equivalently prove that

$$a_r(r, n) = \left[\binom{n + r}{r} + \binom{n + r - 1}{r - 1} \right] 2^{n-1}.$$

The left side counts two-toned $(n + 2r)$-tilings with r red squares, where the last r tiles are required to be white. We now replace the r red squares with the last r white tiles (say one at a time, from left to right) and distinguish these r tiles in some way (say by turning them

pink). Since this process is reversible, it follows that $a(n, r)$ also counts the $(n+r)$-tilings with white and pink tiles (of any length) where there are exactly r pink tiles.

Next we show that this same quantity is counted by the right side of the identity. Let X be a size r subset of $\{1, \ldots, n+r\}$ denoting the r cells of the $(n+r)$-tiling where the pink tiles will begin. Naturally X can be chosen in $\binom{n+r}{r}$ ways. Next, for $2 \leq j \leq n+r$, if cell j is not in X we can freely decide whether or not a tile will start at cell j. This can be done in 2^n or 2^{n-1} ways, depending on whether or not cell 1 is in X. Since there are $\binom{n+r-1}{r-1}$ subsets where X contains 1, the number of tilings generated is $\binom{n+r-1}{r-1}2^n + \left[\binom{n+r}{r} - \binom{n+r-1}{r-1}\right]2^{n-1} = \left[\binom{n+r}{r} + \binom{n+r-1}{r-1}\right]2^{n-1}$, as desired. $\square$

We define the kth order Fibonacci number $F_n^{(k)}$ with initial conditions $F_n^{(k)} = 0$ for $n \leq 0$, $F_1^{(k)} = 1$, and for $n \geq 2$, $F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}$.

Amusingly, $F_n^{(k)}$ can be expressed in terms of two-toned tilings.

Identity 10. *For $n, k \geq 1$,*

$$F_{n+1}^{(k)} = \sum_{r=0}^{\lfloor \frac{n}{k+1} \rfloor} (-1)^r a_r(r, n - r(k+1)).$$

Proof. It is easy to see (as in [1]) that $F_{n+1}^{(k)}$ counts n-tilings with white tiles of length k or smaller. We count these using the principle of inclusion-exclusion, by looking at all n-tilings and subtracting off those with "long" tiles (those with tiles of length greater than k). Notice that when $r = 0$, the initial summand $a_0(0, n) = 2^{n-1}$ counts all unrestricted white tilings of length n. From this we subtract those n-tilings where the first tile is long, and subtract those where the second tile is long, and subtract those where the third tile is long, et cetera. (Naturally some tilings may be subtracted multiple times.) Then for every $1 \leq x < y$ we add back those tilings where tiles x and y are among the long tiles. Then for every $1 \leq x < y < z$, we subtract those tilings where tiles x, y, z are long, and so on. In general, for each $r \geq 0$, we will add or subtract (depending on $(-1)^r$), the sum, over every r-tuple $1 \leq x_1 < x_2 < \cdots < x_r$, of the number of n-tilings where tiles $x_1, x_2 \ldots, x_r$ (and possibly others) are long. We claim that this sum is equal to $a_r(r, n - r(k+1))$.

To see this, notice that $a_r(r, n - r(k+1))$ counts two-toned $(n - r(k-1))$-tilings with r red squares, with the restriction that the last r tiles are white. We now lengthen each of the last r tiles by k and replace each red square, from left to right with these r (necessarily long) tiles. Thus we have created a tiling of length $n - r(k-1) - r + rk = n$ with the property that if tiles $x_1, \ldots, x_r$ were initially red squares, then the tiles that replaced them are guaranteed to be long.

Note that an all-white n-tiling with m long tiles will generate $\binom{m}{r}$ r-tuples of long tiles, and therefore be counted $(-1)^r \binom{m}{r}$ times in the sum. Thus, a tiling with no long tiles will only be counted once (in the initial summand) and a tiling with $m > 0$ long tiles will be net counted $\sum_{r \geq 0} \binom{m}{r}(-1)^r = 0$ times, as desired. $\square$

By Identity 9, we have an immediate corollary. For $n, k \geq 1$,

$$F_{n+1}^{(k)} = \sum_{r=0}^{\lfloor \frac{n}{k+1} \rfloor} (-1)^r \binom{n-rk}{r} \frac{n-rk+r}{n-rk} 2^{n-rk-r-1}.$$

4. APPLICATION TO COMPOSITIONS

Two-toned tilings can be exploited to count compositions with certain properties. Compositions of the integer n can be viewed as (uncolored) n-tilings. For example, there are 2^8 compositions of the integer 9, and the composition 5121 is represented by the 9-tiling consisting of a tile of length 5, followed by a square, then a domino, and then a square. The following theorems enumerate compositions of the integer n that contain at least (or exactly) p instances of the summand k. We define $L(k, n)$ to be the number of compositions of n that contain at least one instance of the summand k, and for $p \geq 1$, let $L_p(k, n)$ denote the number of compositions of n with at least p instances of the summand k.

Identity 11. *For $n, k \geq 1$,*

$$L(k, n) = \sum_{j \geq 1} (-1)^{j-1} a(j, n - jk).$$

Proof. The summand $a(j, n - jk)$ counts two-toned $(n + j - jk)$-tilings that contain exactly j red squares. If we replace each red square by a pink tile of length k, we obtain an n-tiling with white and pink tiles where there are exactly j pink tiles, all of which have length k. Thus, for example, $a(1, n - k)$ counts compositions of n (i.e., n-tilings) with first summand equal to k plus those with second summand k plus those with third summand k, and so on. Then $a(2, n - k)$ counts those compositions of n with first and second summand equal to k plus those with first and third summand equal to k, and so on. The identity follows from the principle of inclusion-exclusion. $\square$

Note that the above sum is finite, since $a(j, n - jk) = 0$ when $jk > n$. The principle of inclusion-exclusion can be modified to count the number of ways that a property occurs at least p times ([1, 3, 4]) by taking each summand (unsigned) with index $j \geq p$ in the usual inclusion-exclusion sum, and multiplying it by $(-1)^{j-p} \binom{j-1}{p-1}$. Consequently, we have the following generalization.

Identity 12. *For $n, k, p \geq 1$,*

$$L_p(k, n) = \sum_{j \geq p} (-1)^{j-p} \binom{j-1}{p-1} a(j, n - jk).$$

Finally, let $E_p(k, n)$ denote the number of compositions of n where the number k appears exactly p times. Since $E_p(k, n) = L_p(k, n) - L_{p+1}(k, n)$ and $\binom{j-1}{p-1} + \binom{j-1}{p} = \binom{j}{p}$, Identity 12 gives us the following identity.

Identity 13. *For $n, k, p \geq 1$,*

$$E_p(k, n) = \sum_{j \geq p} (-1)^{j-p} \binom{j}{p} a(j, n - jk).$$

5. ACKNOWLEDGMENT

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