BINET-LIKE FORMULAS FROM A SIMPLE EXPANSION

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Abstract. In this article we consider sequences arising from the expansions of certain simple expressions involving the golden ratio. The nth terms of these sequences are given by Binet-like formulas, and indeed Binet's formula for the Fibonacci numbers appears as a special case. We study here, via our general formulas, the extent to which three well-known mathematical properties of the Fibonacci sequence are mirrored in our more general sequences.

1. Introduction

We consider here the expansion of the expression $h_n(p,q) = (p\phi + q)^n$ where $p, n \in \mathbb{N}$, q is a non-negative integer and ϕ is the golden ratio given by

$$\phi = \frac{1+\sqrt{5}}{2}.$$

It follows on using the well-known result [5, 7]

$$\phi^m = F_m\phi + F_{m-1} \tag{1.1}$$

along with the fact that ϕ is irrational, that there exists a unique pair of integers $a_n(p,q) \in \mathbb{N}$ and $b_n(p,q) \geq 0$ such that

$$\begin{aligned} h_n(p,q) &= (p\phi + q)^n \\ &= a_n(p,q)\phi + b_n(p,q). \end{aligned}$$

In this paper we study some of the properties of the sequences $(a_n(p,q))$, $(b_n(p,q))$ and

$$\left(\frac{a_n(p,q)}{b_n(p,q)}\right) \tag{1.2}$$

in relation to three well-known properties of the Fibonacci sequence, noting that in the special case $h_n(1,0) = \phi^n$ we have, on using (1.1), $a_n(1,0) = F_n$ and $b_n(1,0) = F_{n-1}$. It is worth pointing out here that since $b_1(p,0) = 0$, it does need to be borne in mind in what follows that some of the results associated with (1.2) are only generally applicable for $n \geq 2$.

2. Some Initial Results

In order to avoid making the notation too cumbersome we will use simply a_n and b_n for $a_n(p,q)$ and $b_n(p,q)$, respectively, when considering the expansion of the general expression $(p\phi + q)^n$. Let us start by obtaining formulas for a_n and b_n. To this end,

$$\begin{aligned}
a_{n+1}\phi + b_{n+1} &= (p\phi + q)^{n+1} \\
&= (p\phi + q)(a_n\phi + b_n) \\
&= pa_n\phi^2 + (pb_n + qa_n)\phi + qb_n \\
&= pa_n(\phi + 1) + (pb_n + qa_n)\phi + qb_n \\
&= ((p+q)a_n + pb_n)\phi + pa_n + qb_n.
\end{aligned}$$

Comparing coefficients of ϕ gives the double recurrence relation

$$a_{n+1} = (p+q)a_n + pb_n \tag{2.1}$$

and

$$b_{n+1} = pa_n + qb_n. \tag{2.2}$$

Rearranging (2.1) to give

$$b_n = \frac{1}{p}\left(a_{n+1} - (p+q)a_n\right), \tag{2.3}$$

and then substituting (2.3) into (2.2), leads to the following recurrence relation for the sequence (a_n):

$$a_{n+2} = (p+2q)a_{n+1} + \left(p^2 - pq - q^2\right)a_n. \tag{2.4}$$

A standard method [2] for solving relations such as (2.4) is to try a solution of the form $a_n = \alpha^n$ to give

$$\alpha^{n+2} = (p+2q)\alpha^{n+1} + \left(p^2 - pq - q^2\right)\alpha^n.$$

We are interested in non-zero solutions, so we need to solve

$$\alpha^2 - (p+2q)\alpha - \left(p^2 - pq - q^2\right) = 0.$$

The roots of this quadratic equation are

$$\alpha_1 = \frac{p + 2q + p\sqrt{5}}{2} = p\phi + q \qquad \text{and} \qquad \alpha_2 = \frac{p + 2q - p\sqrt{5}}{2} = p\hat{\phi} + q,$$

where

$$\hat{\phi} = \frac{1 - \sqrt{5}}{2} = -\frac{1}{\phi}.$$

We thus have a general solution of the form

$$a_n = c\,(p\phi + q)^n + d\left(p\hat{\phi} + q\right)^n,$$

for some constants c and d. Using the initial conditions $a_0 = 0$ and $a_1 = p$ gives

$$c = \frac{1}{\sqrt{5}} \qquad \text{and} \qquad d = -\frac{1}{\sqrt{5}},$$

leading to the Binet-like formula

$$a_n = \frac{1}{\sqrt{5}}\left((p\phi + q)^n - \left(p\hat{\phi} + q\right)^n\right). \tag{2.5}$$

It then follows from (2.3) and (2.5) that

$$\begin{aligned}
b_n &= \frac{1}{p}\left(a_{n+1}-(p+q)a_n\right)\\
&= \frac{1}{p\sqrt{5}}\left((p\phi+q)^{n+1}-\left(p\hat{\phi}+q\right)^{n+1}-(p+q)(p\phi+q)^n+(p+q)\left(p\hat{\phi}+q\right)^n\right)\\
&= \frac{1}{p\sqrt{5}}\left((p\phi+q)^n(p\phi+q-(p+q))-\left(p\hat{\phi}+q\right)^n\left(p\hat{\phi}+q-(p+q)\right)\right)\\
&= \frac{1}{\sqrt{5}}\left((p\phi+q)^n(\phi-1)-\left(p\hat{\phi}+q\right)^n\left(\hat{\phi}-1\right)\right)\\
&= \frac{1}{\sqrt{5}}\left(\frac{1}{\phi}(p\phi+q)^n+\phi\left(p\hat{\phi}+q\right)^n\right).
\end{aligned} \tag{2.6}$$

3. Binet's Formula

The special case $a_n(1,0) = F_n$ has already been noted, and indeed it can be seen that (2.5) specializes to Binet's formula [1, 2, 7] for the nth Fibonacci number:

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right). \tag{3.1}$$

Similarly, $b_n(1,0) = F_{n-1}$, and (2.6) specializes to Binet's formula for the $(n-1)$th Fibonacci number. Note that from (3.1) we are able to infer the following three well-known properties, P1, P2, and P3, of the Fibonacci sequence [1, 7]:

(P1) The ratio of successive terms of the Fibonacci sequence tends to ϕ as n tends to infinity:

$$\lim_{n\to\infty}\frac{F_n}{F_{n-1}} \to \phi.$$

(P2) F_n is the nearest integer to

$$\frac{\phi^n}{\sqrt{5}}.$$

(P3) The ratio of successive terms of the Fibonacci sequences tends to ϕ in an oscillating manner:

$$\frac{F_2}{F_1} < \frac{F_4}{F_3} < \frac{F_6}{F_5} < \cdots < \phi < \cdots < \frac{F_7}{F_6} < \frac{F_5}{F_4} < \frac{F_3}{F_2}.$$

Since the ratio F_n/F_{n-1} arises from the expansion of $h_n(p,q)$ as a specialization of a_n/b_n, we consider here the potential for the sequence (a_n/b_n) to exhibit behaviors similar to those in P1 and P3, and also look at the circumstances under which each of a_n and b_n possess a property corresponding to P2.

4. Property 1

This is a very straightforward matter to deal with. From the fact that

$$p\phi + q > \left|p\hat{\phi}+q\right|,$$

it follows from (2.5) and (2.6) that

$$a_n \sim \frac{1}{\sqrt{5}}(p\phi+q)^n \qquad \text{and} \qquad b_n \sim \frac{1}{\phi\sqrt{5}}(p\phi+q)^n,$$

respectively, from which we obtain the result

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \phi.$$

Thus P1 is a general property of the sequences we are considering.

5. Property 2

It is clear that in order for a_n and b_n to have any possibility of equaling, for each $n \in \mathbb{N}$, the nearest integer to

$$\frac{1}{\sqrt{5}} (p\phi + q)^n \qquad \text{and} \qquad \frac{1}{\phi\sqrt{5}} (p\phi + q)^n ,$$

respectively, it must be the case that $p \in \mathbb{N}$ and $q \geq 0$ satisfy

$$\left| p\hat{\phi} + q \right| < 1.$$

This inequality rearranges to

$$\frac{p}{\phi} - 1 < q < \frac{p}{\phi} + 1,$$

the two solutions of which are given by

$$q = \left\lfloor \frac{p}{\phi} \right\rfloor \qquad \text{and} \qquad q = \left\lfloor \frac{p}{\phi} \right\rfloor + 1, \tag{5.1}$$

where $\lfloor x \rfloor$ is the *floor* of x, and is defined to be the largest integer not exceeding x.

Next, since

$$\lim_{n\to\infty} \left(p\hat{\phi} + q \right)^n = 0$$

when either of the conditions on q given by (5.1) are satisfied, we know that in each of these cases there exists some $N \in \mathbb{N}$ such that a_n satisfies P2 for all $n \geq N$, and similarly for b_n. Let us investigate this further to see if a little more information is forthcoming in this regard. Note that because a_n is an integer and

$$\left| p\hat{\phi} + q \right| > \left| p\hat{\phi} + q \right|^n$$

for any $n \geq 2$ when one of the conditions in (5.1) holds, it is true that a_n satisfies P2 for all $n \in \mathbb{N}$ if

$$\left| \frac{1}{\sqrt{5}} \left(p\hat{\phi} + q \right) \right| < \frac{1}{2}.$$

This is certainly the case whenever q takes one of the values given in (5.1).

The situation for b_n, however, is not quite so straightforward since, even when q complies with one of the conditions in (5.1), it is not necessarily the case that

$$\left| \frac{\phi}{\sqrt{5}} \left(p\hat{\phi} + q \right) \right| < \frac{1}{2}. \tag{5.2}$$

If $q = \lfloor p/\phi \rfloor$ then

$$\begin{aligned} \frac{\phi}{\sqrt{5}} \left(p\hat{\phi} + q \right) &= \frac{\phi}{\sqrt{5}} \left(-\frac{p}{\phi} + \left\lfloor \frac{p}{\phi} \right\rfloor \right) \\ &= -\frac{\phi}{\sqrt{5}} \left[\frac{p}{\phi} \right], \end{aligned} \tag{5.3}$$

where $[x] = x - \lfloor x \rfloor$ is the non-negative real number denoting the *fractional part* of x. It follows from (5.2) and (5.3) that if b_n is to satisfy P2 for all $n \in \mathbb{N}$ for this value of q then we require

$$\left[\frac{p}{\phi}\right] < \frac{\sqrt{5}}{2\phi}. \tag{5.4}$$

Similarly, when $q = \lfloor p/\phi \rfloor + 1$ we would require

$$\left[\frac{p}{\phi}\right] > \frac{1}{2\phi}. \tag{5.5}$$

A result in [8] tells us that for any irrational number x the set $\{[nx] : n \in \mathbb{N}\}$ is uniformly distributed in the interval $[0, 1]$. By this we mean that for any $u, v \in \mathbb{R}$ such that $0 \leq u < v \leq 1$ it is true that

$$\lim_{k\to\infty} \frac{T(u, v, k)}{k} = v - u,$$

where $T(u, v, k)$ is the number of elements of the finite set $\{[nx] : n = 1, 2, 3, \ldots, k\}$ lying between u and v. In the case being considered here this may be interpreted as saying, via (5.4) and (5.5), that out of all the sequences (b_n) eventually satisfying P2 for all $n \geq N$ for some $N \in \mathbb{N}$, the proportion of them possessing this property for all $n \in \mathbb{N}$ is

$$\frac{1}{2}\left(\frac{\sqrt{5}}{2\phi} + \left(1 - \frac{1}{2\phi}\right)\right) = \frac{1}{\phi}.$$

It is actually possible to take this a little further by noting that since the set $\{[p/\phi] : p \in \mathbb{N}\}$ is uniformly distributed in the interval $[0, 1]$, we may, for any given $\epsilon > 0$, find some $p \in \mathbb{N}$ such that $1 - \epsilon < [p/\phi] < 1$. This implies that for any $N_1 \in \mathbb{N}$ we may find a pair (p, q) such that P2 is not satisfied by b_n for each $n \leq N_1$ but for which P2 is satisfied by b_n for each $n \geq N_2$ for some $N_2 \in \mathbb{N}$ with $N_2 > N_1$. To take an explicit example,

$$\lim_{k\to\infty} \left[\frac{F_{2k}}{\phi}\right] = 1,$$

and therefore when $q = \lfloor p/\phi \rfloor$ it is possible, by choosing k sufficiently large and then setting $p = F_{2k}$, to ensure both that

$$\left|p\hat{\phi} + q\right| < 1$$

and

$$\left|p\hat{\phi} + q\right|^n > \frac{\sqrt{5}}{2\phi}$$

for all $n \leq N_1$.

As a brief aside, we show that the ability of (a_n) and (b_n) to satisfy P2 is intimately connected to a mathematical object called the *golden string* $S =$ "1011010110110101101 01 . . . ". This is defined in [6] to be the infinite string of ones and zeros constructed recursively as follows. Let $S_1 =$ "0" and $S_2 =$ "1", and then, for $k \geq 3$, S_k is defined to be the concatenation of the strings S_{k-1} and S_{k-2}. This gives us

$$S_3 = S_2S_1 = \text{"10"},$$
$$S_4 = S_3S_2 = \text{"101"},$$
$$S_5 = S_4S_3 = \text{"10110"},$$

and so on. Note that some authors interchange the positions of the ones and zeros while others use letters such as a's and b's [3, 4, 7]. From [4] we know that

$$\left\lfloor \frac{m+1}{\phi} \right\rfloor$$

corresponds to the number of ones in the first m digits of S. It is thus the case that P2 is eventually satisfied by the terms of (a_n) and (b_n) either when q is equal to the number of ones in the first $p-1$ digits of S, or when q is equal to one more than this. For example, on considering the first few digits of S above we see, on setting $p = 12$, that both $(a_n(12, 7))$ and $(a_n(12, 8))$ satisfy P2. It may easily be checked that when $p = 12$ there are no further values of q that allow P2 to be satisfied.

6. Property 3

It has already been shown that (a_n/b_n) tends to ϕ as n tends to infinity, so let us next consider the manner in which it approaches this limit. The following theorem tells us precisely when (a_n/b_n) satisfies the property corresponding to P3.

Theorem 6.1. *The sequence (a_n/b_n) is, for $n \geq 2$, monotonic increasing if*

$$q \geq \left\lfloor \frac{p}{\phi} \right\rfloor + 1,$$

and oscillating otherwise.

Proof. From (2.5) and (2.6) we have, after some simplification,

$$\begin{aligned}\frac{a_n}{b_n} &= \frac{(p\phi + q)^n - \left(p\hat{\phi} + q\right)^n}{\frac{1}{\phi}(p\phi + q)^n + \phi\left(p\hat{\phi} + q\right)^n} \\ &= \frac{1 - t^n}{\phi t^n + \phi - 1},\end{aligned}$$

where

$$t = \frac{p\hat{\phi} + q}{p\phi + q}.$$

Then

$$\begin{aligned}\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} &= \frac{1 - t^{n+1}}{\phi t^{n+1} + \phi - 1} - \frac{1 - t^n}{\phi t^n + \phi - 1} \\ &= \frac{\left(1 - t^{n+1}\right)(\phi t^n + \phi - 1) - (1 - t^n)\left(\phi t^{n+1} + \phi - 1\right)}{(\phi t^{n+1} + \phi - 1)(\phi t^n + \phi - 1)} \\ &= \frac{t^n(1 - t)(2\phi - 1)}{(\phi t^{n+1} + \phi - 1)(\phi t^n + \phi - 1)}.\end{aligned} \tag{6.1}$$

Next, from the definition of t, it is the case that $0 < t < 1$ if and only if,

$$q \geq \left\lfloor \frac{p}{\phi} \right\rfloor + 1. \tag{6.2}$$

It then follows that both the numerator and the denominator of (6.1) are positive when (6.2) holds, and hence that

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} > 0$$

in this case. On the other hand, remembering that $p \in \mathbb{N}$ and that q is a non-negative integer, it follows that

$$-\frac{1}{\phi^2} \leq t < 0$$

when

$$q \leq \left\lfloor \frac{p}{\phi} \right\rfloor.$$

In this case it is clear that the numerator of (6.1) is positive or negative according to whether n is even or odd, respectively, and, as is straightforward to show, the denominator is always positive for $n \geq 2$. This completes the proof of the theorem. □

7. Summary

Let us now combine the results of the previous three sections to see when a_n and b_n both comply with P2 for all $n \in \mathbb{N}$ while (a_n/b_n) simultaneously satisfies P1 and P3. First, for any $p \in \mathbb{N}$ it is necessary, in order for there to exist the possibility that P2 may be satisfied, that $q = \lfloor p/\phi \rfloor$ or $q = \lfloor p/\phi \rfloor + 1$. However, only the former of these leads to the oscillating convergence of (a_n/b_n) to ϕ as required by P3. We also know that for any pair $(p, \lfloor p/\phi \rfloor)$ it is the case that a_n satisfies P2 for all $n \in \mathbb{N}$, while b_n will only do so for $\frac{100}{\phi}\%$ of such pairs.

Finally, since P1 is always satisfied, we are able to say that $\frac{100}{\phi}\%$ of the pairs $(p, \lfloor p/\phi \rfloor)$, with $p \in \mathbb{N}$, give rise to sequences satisfying each of P1, P2, and P3 in the sense that

$$\lim_{k \to \infty} \frac{|A|}{k} = \frac{1}{\phi},$$

where the set A is given by

$$\{p : b_n \text{ satisfies P2 for all } n \in \mathbb{N} \text{ for the pair } (p, \lfloor p/\phi \rfloor), p = 1, 2, \ldots, k\}.$$

Only in such cases can our sequences be fully said to mirror these three properties of the Fibonacci sequence.

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