A NEW SEQUENCE DERIVED FROM A COMBINATION OF CUBES WITH VOLUME F_n^3

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Abstract. An arrangement of cubes with volume F_n^3, similar to the arrangement made with squares developing the Fibonacci spiral is observed. When lining up the front faces of each cube, there are spaces missing from a complete rectangular polyhedron. The volume of each successive missing space gives birth to a new sequence, yielding interesting relationships to the Fibonacci Sequence.

1. Introduction

One of the most popular configurations created with the Fibonacci Sequence is the arrangement of squares with area F_n^2 so that a spiral can be drawn as demonstrated in Figure 1.

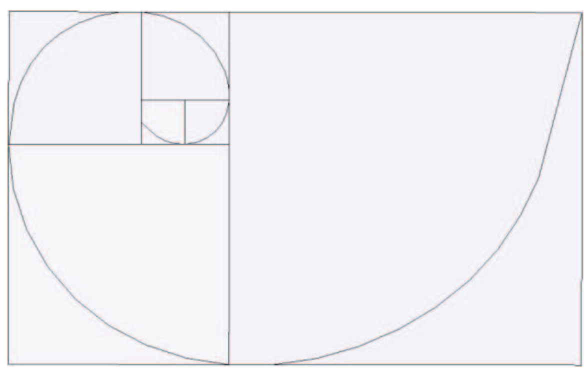

Figure 1. The Fibonacci spiral obtained from an arrangement of squares. The two smallest squares are of area 1×1, the one above those has an area of 2×2, to the left is a square of area 3×3, below that a square of area 5×5, and farthest to the right is one with area 8×8.

The arrangement of the squares is as follows:

(1) Begin with a square of area $0 \times 0 = 0$ (a point).
(2) Draw a square of area 1×1.
(3) To the right of that square, draw another square with area 1×1.
(4) Then on top of the first two squares of area 1×1, draw a square with area equal to 2×2.
(5) To the left, draw a square of area equal to 3×3 with its right edge perfectly aligned with the previously developed rectangle consisting of the first squares mentioned above.

(6) Below that, draw a square with area equal to 5×5 and with the top edge lined up with the bottom edge of the previously made rectangle.
(7) Finally, repeat this process–increasing the area of each new added square according to the square of the Fibonacci numbers.

The order in which these squares are arranged follows a sequence denoted as the "R-U-L-D rule," which stands for "Right, Up, Left, then Down." There are other patterns of arrangement that create the same image; however, RULD is what will be used for the following idea.

We begin by wondering what one could find by creating the same arrangement with cubes instead of squares. Using the RULD rule and observing the differences between the structure with cubes of volume equal to F_n^3 and the structure with squares of area equal to F_n^2, an interesting pattern emerges.

The combination of squares simply forms rectangles. However, when combining cubes in a similar fashion, rectangular polyhedrons are formed but with spaces missing as shown in Figure 2.

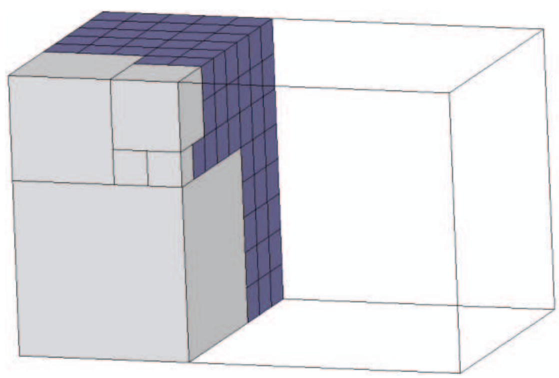

FIGURE 2. An arrangement of cubes with volume F_n^3 starting with a cube of volume $1\times1\times1$ and ending at a cube with volume $8\times8\times8$, which is transparent in order to show detail.

The darkest regions are where the missing spaces reside. These are the spaces left over when combining these cubes. Each cube "hangs" over the rest of the structure made before it.

For each rectangular polyhedron to be completed, the spaces created from the cubes "hanging over each other" must be filled in. Each "missing volume" follows the Fibonacci Sequence accordingly:

(1) The first cube will have a volume of $0 \times 0 \times 0$ which is simply a point. Anywhere touching that point is the next cube with a volume equal to $1 \times 1 \times 1$, as does the proceeding cube placed directly to the right. In between the first cube of zero volume and the second cube of volume 1, there is zero space between them. This is the same for the space between the two cubes of volume 1. However, when following the RULD rule and placing another cube of volume $2\times2\times2$ on top of the preceding aligned cubes, there is space missing from the entire rectangular polyhedron. This space is equal to $1 \times 1 \times 2$.

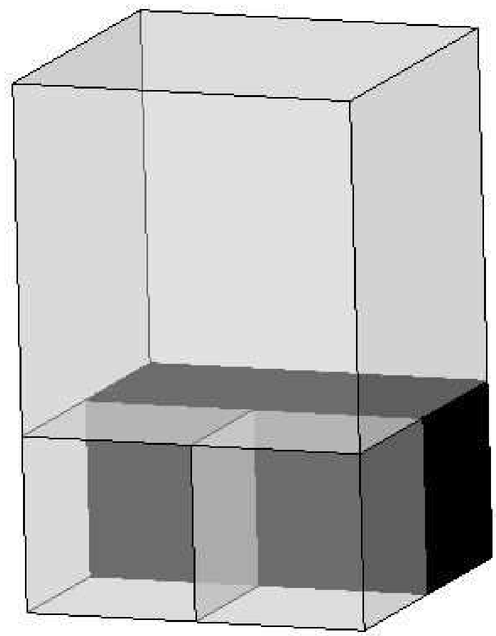

FIGURE 3. The translucent cubes are the first few from the Fibonacci Sequence. The missing volume from the completed rectangular polyhedron is shaded black and has a volume of $1 \times 1 \times 2$.

(2) After filling in this space and placing a cube of volume $3 \times 3 \times 3$ to the left of the entire polyhedron, another missing space is revealed with volume equal to $1 \times 2 \times 3$.

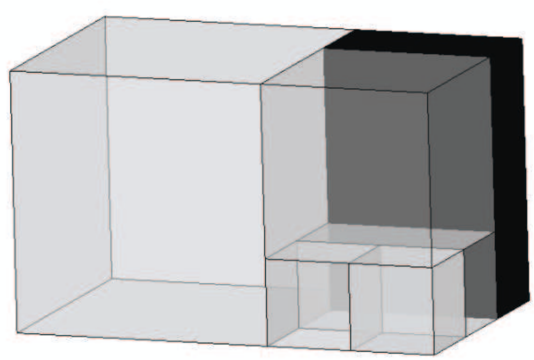

FIGURE 4. The previous missing volume is filled in and becomes translucent in the diagram. The new missing volume shaded in black has a volume of $1 \times 2 \times 3$.

(3) Again, upon filling in the missing space of volume $1 \times 2 \times 3$ and placing below the entire structure a cube with volume $5 \times 5 \times 5$, there is a missing space with a volume equal to $2 \times 3 \times 5$.

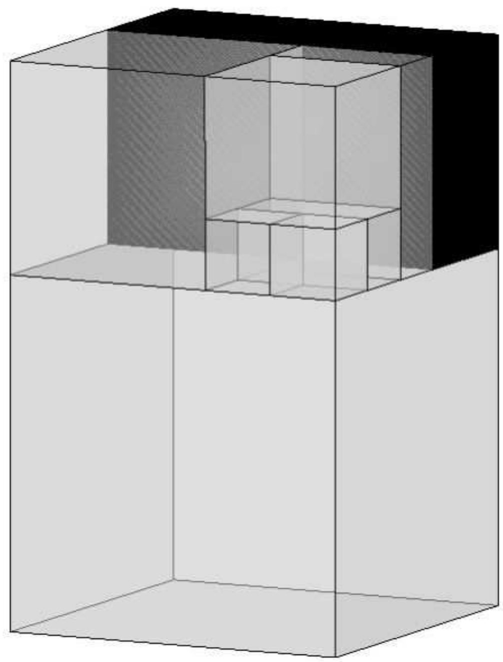

FIGURE 5. Again, the previous missing volume is filled in and becomes translucent in the diagram. The new missing volume shaded in black has a volume of $2 \times 3 \times 5$.

2. Notations and Results

As one continues to follow the RULD rule and observe the missing spaces from the completed rectangular polyhedron, the following sequence is obtained:

$$M = \{0 \times 1, 0 \times 1 \times 1, 1 \times 1 \times 2, 1 \times 2 \times 3, 2 \times 3 \times 5, \ldots, F_{n-1}F_nF_{n+1}, \ldots\} \tag{2.1}$$

where M stands for "missing."

From this relationship, several equations and other generalizations can be made. The Fibonacci Sequence, as is well-known,

$$F = \{F_n\}_{n=0}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots, F_{n-1}, F_n, F_{n+1}, \ldots\}, \tag{2.2}$$

for $n \in \mathbb{N}_0$, where as the Missing Volume Sequence is

$$M = \{M_n\}_{n=0}^{\infty} = \{0, 0, 2, 6, 30, 120, 520, 2148, \ldots, M_{n-1}, M_n, M_{n+1}, \ldots\} \tag{2.3}$$

for $n \in \mathbb{N}_0$. The first relation from (2.1) can be shown as

$$M_n = F_{n-1}F_nF_{n+1}. \tag{2.4}$$

Several formulas for certain volumes can also be given. The total volume of any set of n cubes is

$$V_F = \sum_{i=1}^{n} F_i^3 \tag{2.5}$$

beginning the index with $i = 1$ since the Fibonacci Sequence's base element is $F_0 = 0$. The volume of the missing spaces is as follows:

$$V_M = \sum_{i=1}^{n} M_{i-1} = \sum_{i=1}^{n-1} F_{i-1}F_iF_{i+1}. \tag{2.6}$$

The subscript for M starts with $i-1$ because the Missing Volume Sequence's base element M_0 and its first element M_1 are both equal to 0. This is also because the equation must be consistent with the values of F. The total volume of an entire rectangular polyhedron can be calculated by adding these two sums together; however, it can also be shown as one summation formula:

$$V_T = \sum_{i=1}^{n} F_i^3 + \sum_{i=1}^{n} M_{i-1} = \sum_{i=1}^{n}(F_i^3 + M_{i-1}). \tag{2.7}$$

Although these summation formulas appear to be correct, it would be undesirable to spend much time on summing all the terms from each sequence when calculating the volume of a solid polyhedron for a large combination of cubes. Observing the geometry of each succession of polyhedrons, it can later be proven that this simple formula for volume using only Fibonacci numbers can be used:

$$V_T = F_n^2F_{n+1}. \tag{2.8}$$

To find the nth term of the Missing Volume Sequence, Binet's Formula for the nth term of the Fibonacci Sequence [1] is used, which is as follows:

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \tag{2.9}$$

where $\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$ is the golden ratio. Since it is now known that $M_n = F_{n-1}F_nF_{n+1}$, we can substitute Binet's Formula to obtain the following formula:

$$M_n = \frac{\varphi^{n-1} - (-\varphi)^{-(n-1)}}{\sqrt{5}} \cdot \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \cdot \frac{\varphi^{n+1} - (-\varphi)^{-(n+1)}}{\sqrt{5}}. \tag{2.10}$$

This can then be rewritten as

$$M_n = \frac{\left[\varphi^{n-1} - (-\varphi)^{-(n-1)}\right]\left[\varphi^n - (-\varphi)^{-n}\right]\left[\varphi^{n+1} - (-\varphi)^{-(n+1)}\right]}{5\sqrt{5}}. \tag{2.11}$$

Finally, one last relationship was made between the Missing Volume Sequence and the Fibonacci Sequence:

$$M_n = F_n^3 + (-1)^n F_n. \tag{2.12}$$

This relationship is found by aligning the elements of F, F^3, and M accordingly.

n :	0	1	2	3	4	5	6	7
F_n :	0	1	1	2	3	5	8	13
F_n^3 :	0	1	1	8	27	125	512	2197
$(-1)^n F_n$:	0	-1	1	-2	3	-5	8	-13
M_n :	0	0	2	6	30	120	516	2184

The well-known Simson Identity [2], $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$, will later be shown to give more justice to the construction of the Missing Volume Sequence. From this and Binet's Formula, a smaller formula for M_n can be obtained:

$$M_n = \frac{\left[\varphi^n - (-\varphi)^{-n}\right]^3 + 5\,(-1)^n \left[\varphi^n - (-\varphi)^{-n}\right]}{5\sqrt{5}}. \tag{2.13}$$

From these equations and generalizations, several considerations must be addressed.

(1) One must note that the base element $M_0 = 0$ does not obtain its value from the formula $M_n = F_{n-1}F_nF_{n+1}$, since F_{n-1} would be equal to F_{-1}, an undefined element. M_0 simply denotes the space between the first cube having volume 0 (a point) and the next cube with a volume of 1. (However, the case where $F_{-1} \equiv -1$ would still allow for $M_0 = F_{-1}F_0F_1$.) As stated previously, the idea that there should be two initial zeros in the Missing Volume Sequence is given more support from Simson's identity, $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$.

(2) For each new cube of volume F_n^3 added to the combination, the corresponding missing volume actually has a volume equal to $F_{n-2}F_{n-1}F_n$. However, to keep the equations and the sequences consistent, we have $M_n = F_{n-1}F_nF_{n+1}$.

Qualitative characteristics also reside in the combination of the missing volumes. For example, the combination of M_1 and M_2 is merely just the missing volume M_2 since $M_1 = 0$. The combination of M_2 and M_3 forms a boot-like shape where M_2 is the "foot" and M_3 is the "leg." For M_3 and M_4, we have another structure that looks like a boot, except that it is rotated 90° counterclockwise relative to the previous boot. If one observes the missing volume M_2, it can also be seen as a boot in which the foot has zero volume and is imaginarily placed at the front left edge, making it consistent with the 90° counterclockwise rotation. (It is not entirely necessary to label each succeeding combination of missing volumes as a boot. One can just state that each missing volume is laying 90° counterclockwise relative to its predecessor. However, the boot nomenclature is used in order to describe diagrams which are useful for one to see the 90° difference.)

The reason for recognizing these characteristics is that it appears that a three-dimensional spiral can be developed with the combined shape of the missing volumes. A question that arises is if the missing spaces are used to develop the said three-dimensional spiral, or if the combination of the cubes themselves are used instead. It is possible that both cases are valid ways of doing this. However, this is still something that should later be investigated.

3. Proofs

We shall begin by proving that the two relationships between the Missing Volume Sequence and the Fibonacci Sequence are equal. That is,

$$F_{n-1}F_nF_{n+1} = F_n^3 + (-1)^n F_n.$$

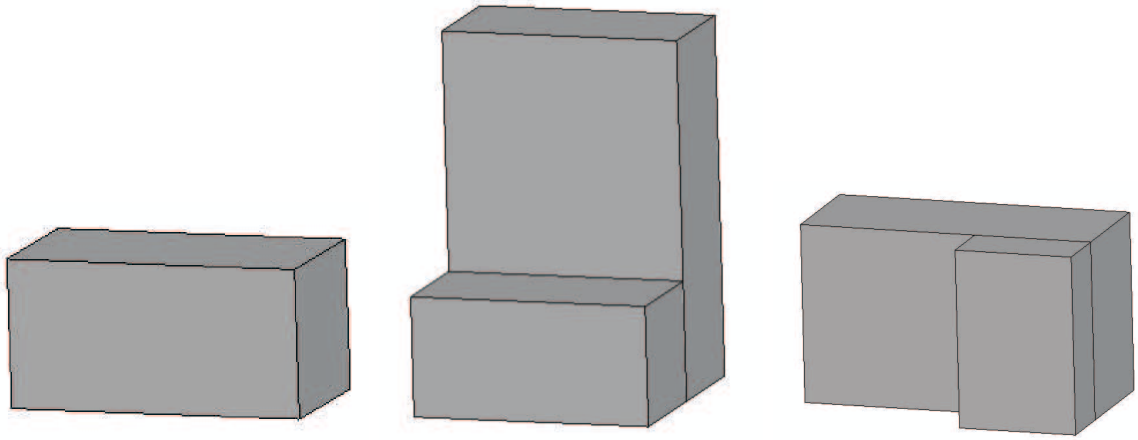

FIGURE 6. The first shape on the left is the combination of $M_1$and M_2, which simply is just M_2. In the middle is the combination of M_2 and M_3, and the last shape on the right is the combination of M_3 and M_4. These are the boots described earlier. Notice that M_2 is the foot and M_3 the leg of the middle boot, and that M_3 becomes the foot of the right boot as M_4 is the leg. One can see the 90° counterclockwise progression.

Factor out an F_n on each side of the above equation and we have

$$F_{n-1}F_{n+1} = F_n^2 + (-1)^n.$$

Subtracting F_n^2 from each side of the equation gives us

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

which simply is the Simson Identity. Now, for clarity, let us prove this identity to be true. Binet's Formula allows for

$$F_{n-1}F_{n+1} = \frac{\varphi^{n-1} - (-\varphi)^{-n+1}}{\sqrt{5}} \cdot \frac{\varphi^{n+1} - (-\varphi)^{-n-1}}{\sqrt{5}} = \frac{\varphi^{2n} + \varphi^{-2n} + (-1)^n(\varphi^2 + \varphi^{-2})}{5}.$$

Similarly, we have

$$F_n^2 = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \cdot \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} = \frac{\varphi^{2n} + \varphi^{-2n} + (-1)^n(-2)}{5}.$$

From the previous two equations, we get

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \frac{(\varphi^2 + \varphi^{-2} + 2)}{5} = (-1)^n \frac{(\varphi + \varphi^{-1})^2}{5}.$$

After plugging in the value for $\varphi = \frac{1+\sqrt{5}}{2}$, one can see that $\frac{(\varphi+\varphi^{-1})^2}{5} = 1$, and will receive the desired relation,

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

□

Using the identity just proven, we can now say

$$F_{n-1}F_nF_{n+1} - F_n^3 = (-1)^n F_n. \tag{3.1}$$

Next, using mathematical induction, we shall prove (2.8) for the volume of any complete rectangular polyhedron. Using (2.7) and plugging in (2.12) for M_i, along with changing the upper index of the summation for missing volumes, it must be proven that

$$\sum_{i=0}^{n} F_i^3 + \sum_{i=0}^{n-1} \left[F_i^3 + (-1)^i F_i\right] = F_n^2 F_{n+1}.$$

Let this be the statement $P(n)$, and let us first show that this is true for $n = 1$.

$$\sum_{i=0}^{1} F_i^3 + \sum_{i=0}^{1-1} \left[F_i^3 + (-1)^i F_i\right] = 1 + 0 = 1 = F_1^2 F_{1+1} = 1 \times 1 = 1.$$

Now, assume that $P(n)$ is true for some $n \in \mathbb{N}$. Then we must show that $P(n+1)$ also holds. This is equivalent to showing that

$$\sum_{i=0}^{n} F_i^3 + F_{n+1}^3 + \sum_{i=0}^{n-1} \left[F_i^3 + (-1)^i F_i\right] + F_n^3 + (-1)^n F_n = F_{n+1}^2 F_{n+2}.$$

Using that $P(n)$ is true and $F_n^3 + (-1)^n F_n = F_{n-1} F_n F_{n+1}$, we get

$$\sum_{i=0}^{n} F_i^3 + F_{n+1}^3 + \sum_{i=0}^{n-1} \left[F_i^3 + (-1)^i F_i\right] + F_n^3 + (-1)^n F_n = F_n^2 F_{n+1} + F_{n+1}^3 + F_{n-1} F_n F_{n+1}.$$

From $F_{n+1} = F_n + F_{n-1}$ and $F_{n+2} = F_n + F_{n+1}$, we can simplify the previous equation as,

$$\begin{aligned} F_n^2 F_{n+1} + F_{n+1}^3 + F_{n-1} F_n F_{n+1} &= F_{n+1}[F_n(F_n + F_{n-1}) + F_{n+1}^2] \\ &= F_{n+1}(F_n F_{n+1} + F_{n+1}^2) \\ &= F_{n+1}^2 F_{n+2}. \end{aligned} \tag{3.2}$$

Thus, the statement $P(n+1)$ is true, implying that $P(n)$ is true for all $n \in \mathbb{N}$. □

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[1] M. Livio, *The Golden Ratio: The Story of Phi, The World's Most Astonishing Number*, Broadway Books, New York, 2002.

[2] R. S. Melham, *A Fibonacci identity in the spirit of Simson and Gelin-Cesàro*, The Fibonacci Quarterly, **41.2** (2003), 142–143.

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