FIBONACCI-RELATED SEQUENCES VIA ITERATED QRT MAPS

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Abstract. In this article we consider the sequences of rational functions that arise from the iterations of particular types of maps on the plane. We study a certain aspect of the behavior of these functions as the number of iterations increases. These types of behavior fall essentially into three categories, the intermediate one of which leads to polynomials with Fibonacci-related coefficients.

1. Introduction

In 1942, R. C. Lyness, a teacher of mathematics at Bristol Grammar School in the United Kingdom, wrote to *The Mathematical Gazette* [8] concerning the second-order nonlinear recurrence relation

$$u_{n+1} = \frac{pu_n + p^2}{u_{n-1}},$$

where $n \in \mathbb{N}$ and p is a non-zero constant. He pointed out that this sequence is cyclic with period 5 for all non-zero u_0 and u_1. For example, on setting $u_0 = x$, $u_1 = y$ and $p = 1$, we obtain the following sequence of rational functions in x and y:

$$x,\ y,\ \frac{y+1}{x},\ \frac{x+y+1}{xy},\ \frac{x+1}{y},\ x,\ y,\ \ldots. \tag{1.1}$$

This sequence may also be generated by iterating the map S on $\mathbb{R}^2$ given by

$$S : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ h_1(x,y) \end{pmatrix},$$

where $h_1(x, y) = \frac{y+1}{x}$. Indeed, this iteration gives rise to

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} y \\ \frac{y+1}{x} \end{pmatrix} \begin{pmatrix} \frac{y+1}{x} \\ \frac{x+y+1}{xy} \end{pmatrix} \begin{pmatrix} \frac{x+y+1}{xy} \\ \frac{x+1}{y} \end{pmatrix} \begin{pmatrix} \frac{x+1}{y} \\ x \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} y \\ \frac{y+1}{x} \end{pmatrix} \ldots,$$

which may be identified with a sequence of points in $\mathbb{R}^2$. As may be observed, the periodic sequence (1.1) appears in the upper entries and, shifted by one term, also in the lower entries. Note that we may write $S^5 = I$, where I is the identity mapping.

Similarly, if $h_1(x, y)$ is replaced in turn by $h_2(x, y) = \frac{1}{xy}$, $h_3(x, y) = \frac{1}{x}$ and $h_4(x, y) = \frac{y}{x}$, we obtain cyclic sequences of mappings with periods 3, 4 and 6, respectively. The periodic recurrences given by $u_{n+1} = h_k(u_{n-1}, u_n)$, $k = 1, 2, 3, 4$, are known as *Lyness cycles*. These are related to mathematical objects known as *QRT maps* [3, 9, 10], which form the focus of our investigation here and are introduced in Section 3.

In order to set the scene further, we consider the transformations

$$T_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{y+1}{x} \\ y \end{pmatrix} \quad \text{and} \quad T_2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ \frac{x+1}{y} \end{pmatrix}.$$

Note first that T_1 and T_2 are related to S, second that they are involutions (in other words, $T_1^2 = I$ and $T_2^2 = I$), and finally that they are related to each other by way of $T_2 = \theta T_1 \theta = \theta^{-1} T_1 \theta$, where θ is the involution given by

$$\theta : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}. \tag{1.2}$$

The map θ is, for obvious reasons, termed a *symmetry switch*. On applying T_1 and T_2 alternately we obtain what Fomin and Reading [4] call a 'moving window' for our cyclic sequence (1.1):

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} \frac{y+1}{x} \\ y \end{pmatrix} \begin{pmatrix} \frac{y+1}{x} \\ \frac{x+y+1}{xy} \end{pmatrix} \begin{pmatrix} \frac{x+1}{y} \\ \frac{x+y+1}{xy} \end{pmatrix} \begin{pmatrix} \frac{x+1}{y} \\ x \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} \begin{pmatrix} y \\ \frac{y+1}{x} \end{pmatrix} \begin{pmatrix} \frac{x+y+1}{xy} \\ \frac{y+1}{x} \end{pmatrix} \begin{pmatrix} \frac{x+y+1}{xy} \\ \frac{x+1}{y} \end{pmatrix} \begin{pmatrix} x \\ \frac{x+1}{y} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \cdots.$$

Indeed, we see that this may be visualized as a series of connected, alternating horizontal and vertical lines in the plane. This particular example has period 10. Note also that $(T_2T_1)^5 = (T_1T_2)^5 = I$ but T_1 and T_2 do not commute; in other words, $T_1T_2 \neq T_2T_1$. This suggests group-theoretic connections, something that is indeed discussed briefly in the final section.

The moving window above may be regarded as a special case of an iterated QRT map, the name of which, incidentally, is an acronym for the surnames of the mathematicians who introduced these maps in 1988: G. R. W. Quispel, J. A. G. Roberts and C. J. Thompson [9]. In this paper we are interested in an aspect of the long-term behavior of the rational functions appearing as entries in moving windows generated by QRT maps. After providing a brief background to these maps and their iterations, we go on to demonstrate three distinct types of behavior displayed by the resulting sequences of rational functions. We show that instances of what might be regarded as the intermediate category of behavior lead to polynomials whose coefficients are functions of Fibonacci numbers.

2. Notation

For the sake of clarity, we make the point here that throughout this article the convention of composing maps from the right is adopted. Thus, for example, BA denotes "apply A first, then B", so although in the above moving window the sequence of composed mappings is written from left to right (as is normal for any sequence), the sequence when written as a composition of mappings is given by $I, T_1, T_2T_1, T_1T_2T_1, \ldots$.

We also introduce here some notation in order to keep track of the sequences of rational functions arising by way of the moving windows. From the sequence

$$\begin{pmatrix} q_{1,0} \\ q_{2,0} \end{pmatrix} \begin{pmatrix} q_{1,1} \\ q_{2,1} \end{pmatrix} \begin{pmatrix} q_{1,2} \\ q_{2,2} \end{pmatrix} \begin{pmatrix} q_{1,3} \\ q_{2,3} \end{pmatrix} \begin{pmatrix} q_{1,4} \\ q_{2,4} \end{pmatrix} \cdots$$

we extract the sequence (u_n) given by $u_0 = q_{1,0}$, $u_1 = q_{2,1}$, $u_2 = q_{1,2}$, $u_3 = q_{2,3}$, and so on. This crisscrosses the moving window, starting at the top left and avoiding identical consecutive entries caused by horizontal or vertical switches in the mappings. For this reason we term (u_n) the *crisscross* sequence for the moving window. For the one given in the previous section the first four terms of this sequence are $u_0 = x$, $u_1 = y$, $u_2 = \frac{y+1}{x}$ and $u_3 = \frac{x+y+1}{xy}$. Note that although the moving window has period 10, this crisscross sequence has period 5.

3. QRT Maps

We provide here, for the interested reader, some of the technical background concerning QRT maps. Familiarity with every detail, however, is not essential for the understanding of this article. Indeed, the only point in this section that might be regarded as a prerequisite for what follows is the definition of the QRT map $Q = Q_\beta Q_\alpha$, where Q_α and Q_β are as given in (3.1) below.

Let $p(x, y)$ be a *biquadratic* polynomial. In other words, $p(x, y)$ is a function in two variables x and y that may be written in either of the forms $a_1(y)x^2+b_1(y)x+c_1(y)$ or $a_2(x)y^2+b_2(x)y+c_2(x)$; this implies, incidentally, that each of a_i, b_i and c_i, $i = 1, 2$, is of degree at most 2. The biquadratic curve defined by $p(x, y)$ is the set of points (x, y) in the plane for which $p(x, y) = 0$. From a geometrical point of view, a QRT map Q is a map on a biquadratic curve that sends each point on this curve to another by way of what are termed *horizontal* and *vertical switches*. It is shown in [3] that all QRT maps are of the form $Q = Q_\beta Q_\alpha$, where

$$Q_\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{f_1(y)-xf_2(y)}{f_2(y)-xf_3(y)} \\ y \end{pmatrix} \quad \text{and} \quad Q_\beta : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ \frac{g_1(x)-yg_2(x)}{g_2(x)-yg_3(x)} \end{pmatrix} \tag{3.1}$$

for some polynomials f_k and g_k, $k = 1, 2, 3$, each of degree no greater than 4.

We now explain how the horizontal and vertical switches are defined in terms of $p(x, y)$. Regarding $p(x, y)$ as a quadratic equation in x, we have $p(x, y) = a_1(y)x^2 + b_1(y)x + c_1(y)$. With x_1 and x_2 denoting the roots of this quadratic, their sum and product are given by

$$x_1 + x_2 = -\frac{b_1(y)}{a_1(y)} \quad \text{and} \quad x_1x_2 = \frac{c_1(y)}{a_1(y)},$$

respectively. Either of these formulas, or any linear combination of them for that matter, define the horizontal switch

$$\begin{pmatrix} x_1 \\ y \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ y \end{pmatrix}$$

on a fixed biquadratic curve $p(x, y) = 0$. The vertical switch is defined analogously by considering $p(x, y)$ written as a quadratic in y.

This map is thus far only defined on the fixed curve $p(x, y) = 0$. It is possible to show that this lifts to a map of the (x, y) plane that is defined on a one-parameter family of curves $p_c(x, y) = 0$ (labeled by the parameter c). We can then solve the equation $p_c(x, y) = 0$ for c in order to obtain a rational function $F(x, y)$ which is a ratio of biquadratic polynomials. Incidentally, it is clear from the definitions of the switches given above that any QRT map has the property that both Q_α and Q_β are involutions; after all, switching the roots of a quadratic is an involution.

Example 3.1. *The simplest example of a QRT map Q is that on the biquadratic curve $p(x, y) = 0$, where $p(x, y) = x^2 + y^2 - r^2$ for some $r \neq 0$. In other words, Q is a map on a circle, the center of which is at the origin. It is clear in this case that the horizontal and vertical switches are given by*

$$Q_\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x \\ y \end{pmatrix} \quad \text{and} \quad Q_\beta : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y \end{pmatrix},$$

respectively. The QRT map $Q = Q_\beta Q_\alpha$ is thus given by

$$Q : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x \\ -y \end{pmatrix}.$$

On comparing this with (3.1), we see that in this particularly simple case, $f_1(y) = f_3(y) = g_1(x) = g_3(x) = 0$ and $f_2(y) = g_2(x) = 1$. Note that $Q^2 = I$, so that Q is cyclic with period 2.

Furthermore, the moving window $I, Q_\alpha, Q_\beta Q_\alpha, Q_\alpha Q_\beta Q_\alpha, \ldots$ *given by*

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} -x \\ y \end{pmatrix} \begin{pmatrix} -x \\ -y \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \cdots$$

is cyclic with period 4, as is the crisscross sequence.

Example 3.2. *A somewhat more complicated geometric object is the Lyness curve* $p_c(x, y) = 0$ *where* $p_c(x, y) = (x + 1)(y + 1)(x + y + a) - cxy$ *for some parameters* a *and* c*. On setting* $a = 1$*, for example, it may be shown that the resulting QRT map* Q *is given by* $Q_\beta Q_\alpha$*, where*

$$Q_\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{y+1}{x} \\ y \end{pmatrix} \quad \textit{and} \quad Q_\beta : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ \frac{x+1}{y} \end{pmatrix}.$$

These are the maps T_1 *and* T_2 *given in Section 1. Here,* $f_1(y) = y + 1$*,* $g_1(x) = x + 1$*,* $f_2(y) = g_2(x) = 0$ *and* $f_3(y) = g_3(x) = -1$*. The QRT map induced by the Lyness curve for other values of* a *is considered in Section 5.*

The sequence of iterates $\left(Q^k : k \in \mathbb{N}\right)$ is known as the *discrete dynamical system* generated by the QRT map Q, where time is indexed by k. As a consequence of its definition, the one-parameter family of biquadratic curves $p_c(x, y) = 0$ is in fact a set of level curves for some rational function $F(x, y)$, which is in turn invariant under the map. In Example 3.2 this would be

$$F(x, y) = \frac{(x + 1)(y + 1)(x + y + a)}{xy}.$$

It follows that F is an integral for $\left(Q^k : k \in \mathbb{N}\right)$, and that this is thus an *integrable* discrete dynamical system. In this article we are essentially considering some simple aspects of the behavior of these systems as they evolve in discrete time.

Finally, we say a few words about symmetry and antisymmetry within the context of our work here. We term a QRT map *symmetric* when its associated one-parameter family of biquadratic polynomials is symmetric, or, in other words, when $p_c(x, y) = p_c(y, x)$. Such a map is in fact related to a 19th century construction due to Chasles [13]. On the other hand, a QRT map is known as *antisymmetric* if $p_c(x, y) = p_{-c}(y, x)$. See also [7, 14] for brief descriptions of these notions. The map given in Example 3.2 may be seen to be symmetric, but our two main results in Section 4 are based on antisymmetric QRT maps.

4. Coefficient Growth

From Examples 3.1 and 3.2, we know that it is possible for the sequence $\left(Q^k : k \in \mathbb{N}\right)$ to exhibit cyclic behavior. Not every QRT map behaves in this manner, however. We introduce here two QRT maps for which the form of the rational functions appearing in the moving window remains the same, but for which the moduli of the coefficients increase with each iteration. In both cases the coefficients are in fact functions of the Fibonacci numbers. We term this type of behavior *coefficient growth.*

In order to demonstrate this, let us first consider the following maps:

$$Q_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{-xy+2y^2-1}{y-x} \\ y \end{pmatrix} \quad \text{and} \quad Q_2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ \frac{-yx+2x^2-1}{x-y} \end{pmatrix}. \tag{4.1}$$

These arise by way of the one-parameter family of biquadratic polynomials $p_c(x, y) = x^2 + y^2 - 3xy + 1 + c(y - x)$, where c is the parameter (noting, as mentioned in Section 3, that this leads to an antisymmetric QRT map). Solving for c leads to the rational function

$$F(x,y) = \frac{x^2+y^2-3xy+1}{x-y},$$

which is a first integral (invariant) for both Q_1 and Q_2, and hence for the QRT map $Q = Q_2Q_1$. The entries in the corresponding moving window arise from I, Q_1, Q_2Q_1, $Q_1Q_2Q_1$, $Q_2Q_1Q_2Q_1$, ... The first few terms of the crisscross sequence (u_i) are

$$x, y, \frac{-xy+2y^2-1}{y-x}, \frac{x^2-5xy+6y^2-2}{y-x}, \frac{2x^2-11xy+15y^2-6}{y-x}, \frac{6x^2-31xy+40y^2-15}{y-x}.$$

We obtain here a general expression for the ith term of this sequence.

Theorem 4.1. *For $i \geq 0$, the terms of the crisscross sequence (u_i) for the map $Q = Q_2Q_1$ defined by (4.1) have the explicit form*

$$u_i = \frac{F_{i-2}F_{i-1}x^2 - \left(2F_{i-1}F_i + (-1)^{i-1}\right)xy + F_iF_{i+1}y^2 - F_{i-1}F_i}{y-x}.$$

Proof. First, we have

$$u_0 = \frac{F_{-2}F_{-1}x^2 - (2F_{-1}F_0 - 1)xy + F_0F_1y^2 - F_{-1}F_0}{y-x} = \frac{-x^2+xy}{y-x} = x$$

and

$$u_1 = \frac{F_{-1}F_0x^2 - (2F_0F_1 + 1)xy + F_1F_2y^2 - F_0F_1}{y-x} = \frac{-xy+y^2}{y-x} = y.$$

Suppose now that the statement is true for all i such that $0 \leq i \leq n+1$. On writing $u_n = \frac{A}{y-x}$ and $u_{n+1} = \frac{B}{y-x}$, and applying the mappings, we obtain

$$\begin{aligned} u_{n+2} &= \frac{\frac{2B^2}{(y-x)^2} - \frac{AB}{(y-x)^2} - 1}{\frac{B}{y-x} - \frac{A}{y-x}} \\ &= \frac{B(B-A) + B^2 - (y-x)^2}{(B-A)(y-x)} \\ &= \frac{B(B-A) + (B-y+x)(B-x+y)}{(B-A)(y-x)}. \end{aligned}$$

We will show that $B-y+x$, $B-x+y$ and $B-A$ may be expressed as $c_{n-1}d_n$, c_nd_{n-1} and $c_{n-1}d_{n-1}$, respectively, where

$$c_n = F_nx - F_{n+2}y - F_{n+1} \quad \text{and} \quad d_n = F_nx - F_{n+2}y + F_{n+1}.$$

This will in turn imply that $u_{n+2} = \frac{B+c_nd_n}{y-x}$.

We know, by our inductive hypothesis and *d'Ocagne's Identity* [15]

$$F_kF_{m+1} - F_{k+1}F_m = (-1)^mF_{k-m} \tag{4.2}$$

with $m = k-1$, that

$$\begin{aligned} B-y+x &= F_{n-1}F_nx^2 - (2F_nF_{n+1}+1)\,xy + F_{n+1}F_{n+2}y^2 - F_nF_{n+1} - y + x \\ &= (F_{n-1}x - F_{n+1}y - F_n)\,(F_nx - F_{n+2}y + F_{n+1}) \\ &= c_{n-1}d_n \end{aligned}$$

for even n. Similarly

$$B - x + y = (F_n x - F_{n+2} y - F_{n+1})(F_{n-1} x - F_{n+1} y + F_n) = c_n d_{n-1}.$$

We have corresponding factorizations when n is odd, giving $(B - y + x)(B - x + y) = c_{n-1} c_n d_{n-1} d_n$ for all values of n. It is also the case that

$$\begin{aligned} B - A &= F_{n-1}F_n x^2 - (2F_n F_{n+1} + (-1)^n)\, xy + F_{n+1}F_{n+2} y^2 - F_n F_{n+1} \\ &\quad - \left(F_{n-2}F_{n-1}x^2 - \left(2F_{n-1}F_n + (-1)^{n-1}\right) xy + F_n F_{n+1} y^2 - F_{n-1}F_n\right) \\ &= F_{n-1}^2 x^2 - (2F_n^2 + 2(-1)^n) xy + F_{n+1}^2 y^2 - F_n^2 \\ &= (F_{n-1}x - F_{n+1}y - F_n)(F_{n-1}x - F_{n+1}y + F_n) \\ &= c_{n-1} d_{n-1}. \end{aligned}$$

Finally,

$$\begin{aligned} B + c_n d_n &= F_{n-1}F_n x^2 - (2F_n F_{n+1} + (-1)^n)\, xy + F_{n+1}F_{n+2} y^2 - F_n F_{n+1} \\ &\quad + (F_n x - F_{n+2} y - F_{n+1})(F_n x - F_{n+2} y + F_{n+1}) \\ &= F_n F_{n+1} x^2 - \left(2F_{n+1}F_{n+2} + (-1)^{n+1}\right) xy + F_{n+2}F_{n+3} y^2 - F_{n+1}F_{n+2}, \end{aligned}$$

thereby proving the result by induction. □

From Theorem 4.1 we see that (u_i) does indeed undergo coefficient growth. Furthermore, on substituting $y = x + 1$ in u_n we obtain the polynomial

$$u_n(x) = F_{n-1}F_n x^2 + \left(2F_n^2 + (-1)^n\right) x + F_n^2.$$

The sequences of coefficients of x^2, x and x^0 in $u_n(x)$ appear in the *On-line Encyclopedia of Integer Sequences* [11] as A001654, A061646, and A007698, respectively, where various combinatorial interpretations of these numbers are given. Note also that $u_n(1) = F_{2n+1}$.

Next, consider the QRT map $Q = Q_4 Q_3$, where

$$Q_3 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{-y+2y^2-yx}{y-x} \\ y \end{pmatrix} \quad \text{and} \quad Q_4 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ \frac{-x+2x^2-xy}{x-y} \end{pmatrix}. \tag{4.3}$$

In this case the underlying one-parameter family of biquadratic polynomials is given by $p_c(x, y) = x^2 + y^2 - 3xy + \frac{x+y}{2} + c(y - x)$, and solving for c results in the following rational function:

$$F(x, y) = \frac{2x^2 + 2y^2 - 6xy + x + y}{2(x - y)}.$$

The entries in the corresponding moving window arise from I, Q_3, Q_4Q_3, $Q_3Q_4Q_3$, $Q_4Q_3Q_4Q_3$, and so on. This turns out to be slightly more complicated than the previous theorem in that we need different formulas for odd and even subscripts. Let (v_n) denote the crisscross sequence for this moving window.

Theorem 4.2. *For $i \geq 0$, the terms of the crisscross sequence (v_i) for the map $Q = Q_4Q_3$ defined by (4.3) are given by*

$$v_{2i} = \frac{\left(F_{2(i-1)}x - F_{2i}y\right)(F_{2i-1}x - F_{2i+1}y + F_{2i})}{y - x}$$

and

$$v_{2i+1} = \frac{\left(F_{2i}x - F_{2(i+1)}y\right)(F_{2i-1}x - F_{2i+1}y + F_{2i})}{y - x}.$$

Proof. We proceed by induction once more. It is straightforward to check that both these statements hold for $i = 0$. Assume therefore that they hold when $i = n$ for some non-negative integer n. For ease of notation, let

$$s_n = F_{n-1}x - F_{n+1}y \quad \text{and} \quad t_n = F_{n-1}x - F_{n+1}y + F_n,$$

noting that $s_{k+1} - s_k = s_{k-1}$ and $t_{k+1} - t_k = t_{k-1}$.

First, we have

$$\begin{aligned} v_{2(n+1)} &= -v_{2n+1}\left(\frac{1 - 2v_{2n+1} + v_{2n}}{v_{2n+1} - v_{2n}}\right) \\ &= -\frac{s_{2n+1}t_{2n}}{y - x}\left(\frac{y - x - 2s_{2n+1}t_{2n} + s_{2n-1}t_{2n}}{s_{2n+1}t_{2n} - s_{2n-1}t_{2n}}\right) \\ &= -\frac{s_{2n+1}}{y - x}\left(\frac{y - x - s_{2(n+1)}t_{2n}}{s_{2n}}\right). \end{aligned}$$

On using (4.2) with $m = k - 2$, a routine calculation shows that

$$s_{2(n+1)}t_{2n} - y + x = s_{2n}t_{2(n+1)},$$

and hence,

$$v_{2(n+1)} = \frac{s_{2n+1}t_{2(n+1)}}{y - x}.$$

Next,

$$\begin{aligned} v_{2n+3} &= -v_{2(n+1)}\left(\frac{1 - 2v_{2(n+1)} + v_{2n+1}}{v_{2(n+1)} - v_{2n+1}}\right) \\ &= -\frac{s_{2n+1}t_{2(n+1)}}{y - x}\left(\frac{y - x - 2s_{2n+1}t_{2(n+1)} + s_{2n+1}t_{2n}}{s_{2n+1}t_{2(n+1)} - s_{2n+1}t_{2n}}\right) \\ &= -\frac{t_{2(n+1)}}{y - x}\left(\frac{y - x - s_{2n+1}t_{2n+3}}{t_{2n+1}}\right). \end{aligned}$$

Employing (4.2) with $m = k - 2$ once more, it is straightforward to show that

$$s_{2n+1}t_{2n+3} - y + x = s_{2n+3}t_{2n+1}$$

and hence that

$$v_{2n+3} = \frac{s_{2n+3}t_{2(n+1)}}{y - x}$$

as required. $\square$

On setting $y = x + 1$ we obtain the quadratic polynomials

$$v_{2n}(x) = (F_{2n-1}x + F_{2n})(F_{2n}x + F_{2n-1})$$

and

$$v_{2n+1}(x) = \left(F_{2n+1}x + F_{2(n+1)}\right)(F_{2n}x + F_{2n-1}).$$

Notice that the roots of these polynomials comprise the negatives of the even-numbered convergents to the golden ratio ϕ, and their reciprocals. Indeed, for each polynomial it is the case that as $n \to \infty$, one root tends to $-\phi$ while the other tends to $-\frac{1}{\phi}$.

Some comments are in order here. A general biquadratic curve has total degree four (although clearly if the leading coefficient is 0 then the degree cannot be greater than three, and so on). Generically, assuming it has degree three or four, the curve has genus one, meaning that it is equivalent to an elliptic curve. However, both of the curves associated with the QRT maps in the current section have degree two. In other words, they are conics, having genus zero. It is thus the case that these may be regarded as highly degenerate examples of QRT maps. Furthermore, for real x and y, the conics considered here are each hyperbolas, and the asymptotes in both cases are a pair of lines with slopes equal to ϕ^2 and $\frac{1}{\phi^2}$, respectively. This might provide us with at least some indication as to why the coefficients of the rational functions are quadratic expressions in Fibonacci numbers.

5. Growth of Polynomials

Thus far we have encountered the relatively sedate behavior of sequences of rational functions that are either cyclic or retain their shape and exhibit coefficient growth. We now give an example in which this is not the case. On setting $a = \frac{1}{2}$ in Example 3.2 we obtain a Lyness curve that gives rise to the QRT map $Q = Q_6 Q_5$, where

$$Q_5 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{2y+1}{2x} \\ y \end{pmatrix} \quad \text{and} \quad Q_6 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ \frac{2x+1}{2y} \end{pmatrix}.$$

In this case $p_c(x, y) = (x + 1)(y + 1)(x + y + \frac{1}{2}) - cxy$ gives the one-parameter family of biquadratic polynomials, from which it follows that the integral is

$$F(x, y) = \frac{(x + 1)(y + 1)(2x + 2y + 1)}{2xy}.$$

With (w_i) denoting the crisscross sequence for the resultant moving window, it is found that the rational functions do not retain their shape, but instead have new terms introduced at each iteration; we call this behavior *polynomial growth*. The seventh term of (w_i), for example, is given by

$$w_7 = \frac{y\left(1 + 5x + 4y + 4x^2 + 4y^2 + 12xy + 4x^2y + 4xy^2\right)}{2(1 + x + 2y)(1 + x + 2y + xy)}.$$

Incidentally, for other meanings of the word "growth" (polynomial, exponential or otherwise) within the context of dynamics of maps, see [13]. It is also worth mentioning that generic rational maps exhibit "polynomial growth" in the sense used here. Thus behavior such as that observed for the QRT map $Q = Q_6 Q_5$ above is in fact the norm, while the main examples given by Theorems 4.1 and 4.2 are really rather rare.

6. Closing Comments

We highlight here several more points of interest in connection with the work carried out in this paper. First, from Example 3.2 and the discussion in Section 1, we know that the crisscross sequence that results on setting $a = 1$ in the Lyness curve will be cyclic. It is in fact globally cyclic in the sense that periodic behavior is exhibited from any starting point (x, y) on the curve. However, the fact that the terms in some crisscross sequence exhibit polynomial growth (as in Section 5) does not necessarily preclude them from displaying cyclic behavior, although it does of course show that any such behavior is not global. Indeed, for particular values of a and well-chosen starting points (x, y) on the resulting Lyness curves, cycles of period 9 may be observed [5].

On the other hand, it is shown in [12] that, under certain mild restrictions, if a QRT map is globally cyclic (as was the case with the map considered in Section 1), then its period must be either 2, 3, 4, 5 or 6. Note that this does not contradict the result mentioned in the previous paragraph since cycles of period 9 only arise from particular starting points.

Furthermore, it is possible for maps other than QRT maps to exhibit cyclic behavior. To take an example, consider

$$H_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\frac{(x+1)(y+1)}{y} \\ y \end{pmatrix} \quad \text{and} \quad H_2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -\frac{(y+1)(x+1)}{x} \end{pmatrix}.$$

A check shows that H_2H_1 is not a QRT map. Indeed, H_1 and H_2 are not even involutions. They do, however, give rise to the following moving window:

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} -\frac{(x+1)(y+1)}{y} \\ y \end{pmatrix} \begin{pmatrix} -\frac{(x+1)(y+1)}{y} \\ -\frac{xy+x+1}{x+1} \end{pmatrix} \begin{pmatrix} x \\ -\frac{xy+x+1}{x+1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \cdots.$$

It may be seen in this instance that the periods of both the moving window and the crisscross sequence are equal to 4.

Also, in a recent article appearing in *The Fibonacci Quarterly* [1], the author considers a certain aspect of the second-order nonlinear recurrence relation

$$x_{n+1}x_{n-1} = x_n^2 + A,$$

which may be recast as the QRT map Q_8Q_7, where

$$Q_7 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{y^2+A}{x} \\ y \end{pmatrix} \quad \text{and} \quad Q_8 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ \frac{x^2+A}{y} \end{pmatrix}.$$

This QRT map arises from the one-parameter family of biquadratic polynomials $p_c(x, y) = x^2 + y^2 + A - cxy$, and the corresponding rational function is given by

$$F(x, y) = \frac{x^2 + y^2 + A}{xy}.$$

In particular, the paper is concerned with the problem of determining, for a given value of A, which integer values of x_1 and x_2 give rise to sequences comprising purely integers.

The iterates of this QRT map are Laurent polynomials in the initial values x and y. They exhibit polynomial growth, but in a very controlled way (linear in n). There is in fact an analogue here to Theorems 4.1 and 4.2, namely that it is possible to give an explicit formula for the coefficients of the resultant Laurent polynomials in terms of binomial coefficients [2]. Within the context of integrability, the above recurrence is also discussed in [6], where the formula for the first integral corresponding to $p_c(x, y)$ is given for the more general situation in which there is an additional linear term on the right-hand side. To find integer sequences for this recurrence, as in [1], it suffices to look for integer solutions of the corresponding equation $p_c(x, y)$, which, for a conic, is a classical problem considered by Gauss (related to Pell's equation).

Finally, as was mentioned in the opening section, we demonstrate here a link with group theory. The dihedral group D_5 (sometimes written D_{10}) comprises the group of symmetries of a regular pentagon. This consists of 10 elements, 5 rotational symmetries and 5 reflection symmetries, and has the presentation

$$D_5 = \left\langle r, s | r^2 = s^2 = (rs)^5 = e \right\rangle,$$

where r and s are generators of the group and e denotes the identity. On considering the set of 10 mappings given in the moving window in Section 1, we see, from the way that they have been constructed and the relations $T_1^2 = T_2^2 = (T_2T_1)^5 = (T_1T_2)^5 = I$, that these maps form a group G with respect to composition of mappings. In fact, on setting $r = T_1$ and $s = T_2$, it becomes clear that G is isomorphic to D_5.

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References

[1] R. C. Alperin, *Integer sequences generated by* $x_{n+1} = \left(x_n^2 + A\right)/x_{n-1}$, The Fibonacci Quarterly, **49.4** (2011), 362–365.

[2] P. Caldero and A. Zelevinsky, *Laurent expansions in cluster algebras via quiver representations*, Moscow Mathematics Journal, **6** (2006), 411–429.

[3] J. J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*, Springer, 2010.

[4] S. Fomin and N. Reading, *Root systems and generalised associahedra*, arXiv:math/0505518v3, (2008). `http://arxiv.org/abs/math.CO/0505518`.

[5] A. Gasull, V. Manosa, and X. Xarles, *Rational periodic sequences for the Lyness recurrence*, Discrete and Continuous Dynamical Systems – Series A, **32** (2012), 587–604.

[6] A. N. W. Hone, *Singularity confinement for maps with the Laurent property*, Physics Letters A, **361** (2007), 341–345.

[7] P. Kassotakis and N. Joshi, *On non-QRT mappings of the plane*, Letters in Mathematical Physics, **91** (2010), 71–81.

[8] R. C. Lyness, *Notes 1581*, Mathematical Gazette, **26** (1942), 62.

[9] G. R. W. Quispel, J. A. G. Roberts, and C. J. Thompson, *Integrable mappings and soliton equations*, Physics Letters A, **126** (1988), 419–421.

[10] G. R. W. Quispel, J. A. G. Roberts, and C. J. Thompson, *Integrable mappings and soliton equations II*, Physica D, **34** (1989), 183–192.

[11] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org.`

[12] T. Tsuda, *Integrable mappings via rational elliptic surfaces*, Journal of Physics A, **37** (2004), 2721–2730.

[13] A. P. Veselov, *Growth and integrability in the dynamics of mappings*, Communications in Mathematical Physics, **145** (1992), 181–193.

[14] R. Willox, B. Grammaticos, and A. Ramani, *A study of the antisymmetric QRT mappings*, Journal of Physics A: Mathematical and General, **38** (2005), 5227–5236.

[15] E. W. Weisstein, *d'Ocagne's Identity*, From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/dOcagnesIdentity.html`

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