SHARPER UPPER BOUNDS FOR THE ORDER OF APPEARANCE IN THE FIBONACCI SEQUENCE

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ABSTRACT. Let F_n be the nth Fibonacci number. The order of appearance $z(n)$ of a natural number n is defined as the smallest natural number k such that n divides F_k. In 1975, J. Sallé proved that $z(n) \leq 2n$, for all positive integers n. In this paper, we shall provide sharper upper bounds for $z(n)$ which are substantially smaller than $2n$ for some values of n. Moreover, we shall prove that

$$\liminf_{n\to\infty} \frac{z(n)}{n} = 0.$$

1. INTRODUCTION

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [6] together with its very extensive annotated bibliography for additional references and history).

The study of the divisibility properties of Fibonacci numbers has always been a popular area of research. For example, it is still an open problem to decide if there are infinitely many primes in that sequence. Let n be a positive integer, the *order (or rank) of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is defined as the smallest positive integer k, such that $n \mid F_k$ (some authors also call it *order of apparition*, or *Fibonacci entry point*). This function can be implemented in `Mathematica` [22] as

```
z[n_]:=Catch[Do[i;If[Mod[Fibonacci[i],n]==0,Throw[i]],{i,2*n}]]
```

There are several results about $z(n)$ in the literature. For instance, in 1878, E. Lucas showed that $z(n) < \infty$ for all $n \geq 1$. The proof of this fact is an immediate consequence of the Théorème Fondamental of Section XXVI in [8, p. 300]. We remark that there is no known closed formula for $z(n)$ and so Diophantine equations related to $z(n)$ are one of the best tools for understanding its behavior. This function gained great attention in 1992, when Z. H. Sun and Z. W. Sun [20] proved that all solutions of the Diophantine equation $z(n) = z(n^2)$ are composite numbers, would imply the Fermat's Last Theorem. It is known that there are no prime solutions when $n < 3.23 \cdot 10^{15}$ (PrimeGrid Project, May 2012).

Recently, the author wrote a series of papers related to $z(n)$. We refer the reader to [9, 10, 11, 12, 14] for explicit formulas for the order of appearance of integers related to Fibonacci and Lucas numbers, such as $F_m \pm 1, F_nF_{n+1}F_{n+2}, F_n^k$ and $L_nL_{n+1}L_{n+2}$. Also, solutions for the Diophantine equation $z(n) = n + \ell$, with $\ell \geq 0$, were studied in [13, 15, 16]. For instance, for $\ell = 0$, the solutions are of the form 5^k or $12 \cdot 5^k$ ($k \geq 0$), for $\ell = 1$, the solutions are prime numbers and for $\ell = 2$, the only solution is $n = 4$.

Concerning upper bounds for $z(n)$, one can apply Dirichlet's Box Principle to get the bound $z(n) \leq (n-1)^2 + 1$ (see [21, Theorem, p. 52]). In the case of a prime number p, one has the better bound $z(p) \leq p + 1$.

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In 1975, J. Sallé [19] proved that $z(n) \leq 2n$, for all positive integers n. This is the sharpest upper bound for $z(n)$, since for example, $z(6) = 12$ and $z(30) = 60$. Actually, proceeding along the same lines as the proof of Theorem 1.1 of [13], one obtains that

$$z(n) = 2n \text{ if and only if } n = 6 \cdot 5^k, \text{ for } k \geq 0. \tag{1.1}$$

However, apart from these cases this upper bound is very weak. For instance, $z(3731) = 280 < 0.08 \cdot 3731$. We still remark that Sallé's proof depends, strongly, on a result due to Carmichael [2, Theorem XIII].

In this paper, we shall combine a refinement of the method in [19] together with the author's approach [13] (using formulas for the p-adic order of Fibonacci numbers) to get substantially better upper bounds for $z(n)$, when $n \neq 6 \cdot 5^k$ is a composite number. This proof does not depend on the Carmichael result. Moreover, the improvement depends on the number of distinct prime factors of n, denoted by $\omega(n)$. Our main results are the following.

Theorem 1.1. *We have*

(i) $z(2^k) = 3 \cdot 2^{k-2}$ *(for $k \geq 3$)*, $z(3^k) = 4 \cdot 3^{k-1}$ *(for $k \geq 1$) and* $z(5^k) = 5^k$ *(for $k \geq 0$).*

(ii) *If $p > 5$ is a prime, then*

$$z(p^k) \leq \left(p - \left(\frac{5}{p}\right)\right) p^{k-1}, \text{ for } k \geq 1,$$

where, as usual, $(\frac{a}{q})$ denotes the Legendre symbol of a with respect to a prime $q > 2$.

For the cases when $\omega(n) \geq 2$, we proved the following theorem.

Theorem 1.2. *Let n be an odd integer with $\omega(n) \geq 2$, then*

$$z(n) \leq 2 \cdot \left(\frac{2}{3}\right)^{\omega(n)-\delta_n} n,$$

where

$$\delta_n = \begin{cases} 0, & \text{if } 5 \nmid n; \\ 1, & \text{if } 5 \mid n. \end{cases}$$

Theorem 1.3. *Let n be an even integer with $\omega(n) \geq 2$, we have that*

(i) *If $\nu_2(n) \geq 4$, then*

$$z(n) \leq \frac{3}{4} \cdot \left(\frac{2}{3}\right)^{\omega(n)-\delta_n-1} n.$$

(ii) *If $\nu_2(n) = 1$, then*

$$z(n) \leq \begin{cases} 3n/2, & \text{if } \omega(n) = 2 \text{ and } 5 \mid n; \\ 2n, & \text{if } \omega(n) = 2 \text{ and } 5 \nmid n; \\ 3 \cdot (2/3)^{\omega(n)-\delta_n-1} n, & \text{if } \omega(n) > 2. \end{cases}$$

(iii) *If $\nu_2(n) \in \{2, 3\}$, then*

$$z(n) \leq \begin{cases} 3n/2, & \text{if } \omega(n) = 2 \text{ and } 5 \mid n; \\ n, & \text{if } \omega(n) = 2 \text{ and } 5 \nmid n; \\ (2/3)^{\omega(n)-\delta_n-2} n, & \text{if } \omega(n) > 2, \end{cases}$$

where $\nu_2(n)$ is the 2-adic valuation.

We organize the paper as follows. In Section 2, we will recall some useful properties of Fibonacci numbers such as a result concerning the p-adic order of F_n. Section 3 will be devoted to the proof of theorems. In the last section, we shall discuss the behavior of $z(n)/n$ when n increases.

2. Auxiliary Results

Before proceeding further, we state some facts on Fibonacci numbers for the convenience of the reader.

Lemma 2.1. *We have*

(a) $n \mid m$ *if and only if* $F_n \mid F_m$.
(b) $F_{p-(\frac{5}{p})} \equiv 0 \pmod p$, *for all primes* p.

Proofs of these assertions can be found in [6]. We refer the reader to [1, 5, 18] for more details and additional bibliography.

The second lemma is a consequence of the previous one.

Lemma 2.2. (*Cf. Lemma 2.2 (c) of* [10]) *If* $n \mid F_m$, *then* $z(n) \mid m$.

Note that Lemma 2.1 (b) and Lemma 2.2 implies that $z(p)$ divides $p-(5/p)$, for all primes p. In particular, $z(p) \le p+1$ for all primes p.

The p-adic order (or valuation) of r, $\nu_p(r)$, is the exponent of the highest power of a prime p which divides r. Throughout the paper, we shall use the known facts that $\nu_p(ab^\epsilon) = \nu_p(a) + \epsilon\nu_p(b)$, for $\epsilon \in \{-1, 1\}$, and that $a \mid b$ if and only if $\nu_p(a) \le \nu_p(b)$, for all primes p.

We remark that the p-adic order of Fibonacci numbers was completely characterized, see [4, 7]. For instance, from the main results of Lengyel [7], we extract the following result.

Lemma 2.3. *For* $n \ge 1$, *we have*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3; \\ 1, & \text{if } n \equiv 3 \pmod 6; \\ 3, & \text{if } n \equiv 6 \pmod{12}; \\ \nu_2(n)+2, & \text{if } n \equiv 0 \pmod{12}, \end{cases}$$

$\nu_5(F_n) = \nu_5(n)$, *and if* p *is prime* $\ne 2$ *or* 5, *then*

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{if } n \not\equiv 0 \pmod{z(p)}. \end{cases}$$

A proof for this result can be found in [7].

As usual, from now on we use the well-known notation $[a, b] = \{a, a+1, \dots, b\}$, for integers $a < b$.

Now we are ready to deal with the proofs of our results.

3. The Proofs

3.0.1. *Proof of Theorem 1.1. (i)* By Theorem 1.1 of [12], we have that $z(F_n^k) = nF_n^{k-1}/2$, for $k \ge 3$ and $n \equiv 3 \pmod 6$, and also $z(F_n^{k+1}) = nF_n^k$, for $k \ge 0$ and $n \not\equiv 3 \pmod 6$. Since $2 = F_3$, $3 = F_4$ and $5 = F_5$, one can easily use the previous formulas to get the desired result.

(ii) By Lemma 2.2, it suffices to prove that $p^k \mid F_{(p-(5/p))p^{k-1}}$. This follows from the fact that

$$\begin{aligned} \nu_p(F_{(p-(5/p))p^{k-1}}) &= \nu_p((p-(5/p))p^{k-1}) + \nu_p(F_{z(p)}) \\ &= k - 1 + \nu_p(F_{z(p)}) \ge k = \nu_p(p^k) \end{aligned}$$

where we used Lemma 2.3 together with $\nu_p(F_{z(p)}) \geq 1$. □

3.0.2. *Proof of Theorem 1.2.* Write $n = 5^a p_1^{a_1} \cdots p_k^{a_k}$, where $p_i \notin \{2, 5\}$ is prime ($p_i \neq p_j$, if $i \neq j$) and $a_i \geq 1$, for all $i \in [1, k]$. Setting $z_i = p_i - (5/p_i)$, we have that $z(p_i) \mid z_i$, for $i \in [1, k]$. We claim that $n \mid F_m$, where

$$m := 2 \cdot 5^a \left(\frac{z_1 p_1^{a_1 - 1}}{2}\right) \cdots \left(\frac{z_k p_1^{a_k - 1}}{2}\right).$$

Note that m is well defined, because z_i is even for all $i \in [1, k]$. Since $5^a, p_1^{a_1}, \ldots, p_k^{a_k}$ are pairwise coprime and $5^a \mid F_m$ (keep in mind that $\nu_5(F_m) = \nu_5(m) = a$), it suffices to prove that $p_i \mid F_m$, or equivalently, $\nu_{p_i}(F_m) \geq a_i$, for all $i \in [1, k]$. First, observe that $z(p_i) \mid m$, because m can be written as

$$5^a \left(\frac{z_1 p_1^{a_1 - 1}}{2}\right) \cdots (z_i p_i^{a_i - 1}) \cdots \left(\frac{z_k p_1^{a_k - 1}}{2}\right).$$

Therefore, Lemma 2.3 gives

$$\nu_{p_i}(F_m) = \nu_{p_i}(m) + \nu_{p_i}(F_{z(p_i)}) = a_i - 1 + \sum_{j=1}^{k} \nu_{p_i}(z_j) + \nu_{p_i}(F_{z(p_i)}) \geq a_i.$$

Thus, $n \mid F_m$ and so $z(n) \leq m$ (by Lemma 2.2). However,

$$\frac{m}{n} = 2 \cdot \left(\frac{z_1}{2p_1}\right) \cdots \left(\frac{z_k}{2p_k}\right) \leq 2 \cdot \left(\frac{p_1 + 1}{2p_1}\right) \cdots \left(\frac{p_k + 1}{2p_k}\right) \leq 2 \cdot \left(\frac{2}{3}\right)^k,$$

where we used that $z(p_i) \leq p_i + 1$ and that $(p_i + 1)/2p_i \leq 2/3$ (since $p_i \geq 3$). The previous inequality together with the fact that $\omega(n) = k + \delta_n$ yields

$$z(n) \leq m \leq 2 \cdot \left(\frac{2}{3}\right)^{\omega(n) - \delta_n} n.$$

□

3.0.3. *Proof of Theorem 1.3.* *(i)* If $n = 2^a 5^b p_1^{a_1} \cdots p_k^{a_k}$, we choose

$$m := 3 \cdot 2^{a-2} \cdot 5^b \left(\frac{z_1 p_1^{a_1 - 1}}{2}\right) \cdots \left(\frac{z_k p_1^{a_k - 1}}{2}\right).$$

Proceeding as in the proof of Theorem 1.2, one has that $5^b p_1^{a_1} \cdots p_k^{a_k} \mid F_m$. In order to prove that $n \mid F_m$, it is therefore enough to show that 2^a divides F_m. Indeed, since $a \geq 4$, then $12 \mid m$ and Lemma 2.3 yields

$$\nu_2(F_m) = \nu_2(m) + 2 = a + \sum_{j=1}^{k} (\nu_2(z_j) - 1) \geq a = \nu_2(2^a),$$

where we used that $\nu_2(z_i) \geq 1$, for all $i \in [1, k]$. As in the previous section, we get

$$\frac{m}{n} \leq \frac{3}{4} \cdot \left(\frac{2}{3}\right)^k.$$

The result follows because $z(n) \leq m$ and $\omega(n) = k + 1 + \delta_n$.

(ii) and (iii) These items can be proved similarly, by suitable choices of m in each case. The only case requiring further analysis occurs when $\nu_2(n) \in \{2, 3\}$ and $\omega(n) > 2$. For that, we have $n = 2^a 5^b p_1^{a_1} \cdots p_k^{a_k}$ ($a \in \{2, 3\}$) while m can be chosen as

$$3 \cdot 2^{a-1} \cdot 5^b \left(\frac{z_1 p_1^{a_1-1}}{2}\right) \cdots \left(\frac{z_k p_1^{a_k-1}}{2}\right).$$

Since $a - 1 \geq 1$, then $6 \mid n$ and Lemma 2.3 gives

$$\nu_2(F_m) \geq 3 \geq a = \nu_2(n).$$

The upper bound for m/n is $(3/2) \cdot (2/3)^k = (2/3)^{k-1}$ and the result follows since $\omega(n) = k + 1 + \delta_n$.

The proof of Theorem 1.3 is then complete. □

4. On the Behavior of $z(n)/n$

In this section, we shall discuss the quotient $z(n)/n$, for $n \geq 1$. A few approximated values of this sequence are

$$1, 1.5, 1.333, 1.5, 1, 2, 1.142, 0.75, 1.333, 1.5, 0.909, 1, 0.538, 1.714, 1.333, \ldots.$$

Clearly this sequence is not convergent (since $z(2^k)/2^k = 3/4$ and $z(3^k)/3^k = 4/3$, for all $k \geq 3$). However, using the equivalence in (1.1) together with the fact that $z(n)/n \leq 2$, for all n, we deduce that

$$\limsup_{n\to\infty} \frac{z(n)}{n} = 2.$$

But what is the value of $\liminf z(n)/n$? Our final result provides an answer to this question.

Proposition 4.1. *We have that*

$$\liminf_{n\to\infty} \frac{z(n)}{n} = 0.$$

Before the proof, we recall that $p_n\#$ denotes the nth *primorial* number which is defined as the product of the first n prime numbers. For instance, the values of $p_n\#$ for $n \in [1, 10]$ are

$$2, 6, 30, 210, 2310, 30030, 510510, 9699690, 223092870, 6469693230, \ldots$$

which is the OEIS [17] sequence A002110. To the best of our knowledge, the name primorial was coined in 1987 by Dubner [3].

Proof. Since $z(n)/n \geq 0$, then it suffices to prove that $\lim_{n\to\infty} z(p_n\#)/p_n\# = 0$. For that, note that Theorem 1.3 (ii) implies that $z(p_n\#) \leq 3 \cdot (2/3)^{n-2} p_n\#$, for all $n > 2$ and therefore,

$$0 \leq \frac{z(p_n\#)}{p_n\#} \leq 3 \cdot \left(\frac{2}{3}\right)^{n-2}$$

holds for all $n > 2$. Since $\lim_{n\to\infty}(2/3)^{n-2} = 0$, the Squeeze Theorem gives

$$\lim_{n\to\infty} \frac{z(p_n\#)}{p_n\#} = 0$$

and the proof is complete. □

For example, $z(n) < n/2013$ if

$$n = p_{24} = 23768741896345550770650537601358310.$$

This shows that our bounds are effectively much better than $2n$, mainly when $\omega(n)$ is large.

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