GENERATING COMPOSITE SEQUENCES BY APPENDING DIGITS TO SPECIAL TYPES OF INTEGERS

LENNY JONES AND MARIA MARKOVICH

Abstract. We say that the positive integer k is *d-composite* if, when you append the digit d, any number of times on the right of k, the resulting integer is composite. Clearly, every positive integer is d-composite when $d \in \{2, 4, 5, 6, 8\}$. In addition, if $\gcd(k, d) > 1$, then k is d-composite. The first author has shown that, for any given fixed digit $d \in \{1, 3, 7, 9\}$, there exist infinitely many positive integers k with $\gcd(k, d) = 1$ that are d-composite. He also showed that 37 is the smallest 1-composite integer and that the pair $(37, 38)$ is the smallest pair of consecutive 1-composite integers. In this article, we prove similar results for special types of integers such as perfect powers, Sierpiński numbers, Riesel numbers, and Fibonacci numbers. For example, among our results, we show that the smallest Fibonacci number F_n, such that both F_n and F_n^2 are 1-composite, is $F_{21} = 10946$.

1. Introduction

In [18], the first author proved that for any given fixed digit $d \in \{1, 3, 5, 7\}$, there exist infinitely many positive integers k, such that $\gcd(k, d) = 1$ and every integer in the sequence

$$kd, \quad kdd, \quad kddd, \quad kdddd, \quad kddddd, \quad \ldots,$$

is composite. We refer here to such positive integers k as *d-composite*. The first author also shows in [18] that $k = 37$ is the smallest 1-composite integer, and that (37, 38) is the smallest pair of consecutive 1-composite integers. More recently, Grantham, Jarnicki, Rickert and Wagon [13] have shown that there exist infinitely many positive integers that are *pandigital-composite*; that is, integers k such that $\gcd(k, d) = 1$ and yield only composites when repeatedly appending *any* digit d. In this article, we present similar results when further restrictions are placed on the positive integers. In particular, we prove the following.

Theorem 1.1. *Given any fixed positive integer $r \not\equiv 0, 12, 18, 24 \pmod{36}$, there exist infinitely many Fibonacci numbers F_n such that F_n^r is 1-composite.*

Theorem 1.2.

1. *The smallest perfect square that is 1-composite is 65^2.*
2. *There are infinitely many positive integers k such that both k^2 and $(k+1)^2$ are 1-composite. The smallest such pair is $(65^2, 66^2)$.*
3. *The smallest perfect cube that is 1-composite is 26^3.*
4. *There are infinitely many positive integers k such that $k^2 + 1$ is 1-composite. The smallest such value of k is $k = 44$.*
5. *There exist infinitely many positive integers n such that both F_n and F_n^2 are 1-composite. The smallest such value of n is $n = 21$.*

Theorem 1.3. *There exist infinitely many positive integers k such that k^2 is simultaneously Riesel, Sierpiński, and pandigital-composite.*

Remark 1.4. According to a private communication, Bloome, Ferguson, Kozek, and Noorman have recently found infinitely many positive integers k that are simultaneously Riesel, Sierpiński, and pandigital-composite.

Although Theorem 1.1 and Theorem 1.2 deal only with 1-composite integers, analogous results can be established in many cases for the d-composite situation when $d \in \{3, 7, 9\}$. We give an example of such a result in Section 6. We have treated the pandigital-composite situation separately in Theorem 1.3.

2. Preliminaries

The following concept, due to Erdős [7], is crucial to the proofs of our results.

Definition 2.1. A *covering* of the integers is a system of congruences $x \equiv a_i \pmod{m_i}$ such that every integer satisfies at least one of the congruences. A covering is said to be a *finite covering* if the covering contains only finitely many congruences.

Remark 2.2. Since all coverings in this paper are finite coverings, we omit the word "finite".

Quite often when a covering is used to solve a problem, there is a set of prime numbers associated with the covering. In the situations occurring in this article, for each congruence $n \equiv z_i \pmod{m_i}$ in the covering, there exists a corresponding prime p_i, such that either $10^{m_i} \equiv 1 \pmod{p_i}$ or $2^{m_i} \equiv 1 \pmod{p_i}$ or $2^{m_i} \equiv -1 \pmod{p_i}$, and $p_j \neq p_i$ for all $j \neq i$. Because of this correspondence, we indicate the covering using a set $\mathbb{C}$ of ordered triples (z_i, m_i, p_i). We abuse the definition of a covering slightly by referring to the set $\mathbb{C}$ as a "covering".

In this article we focus on appending digits to certain special integers κ. We define s_n to be the integer resulting from appending the digit d to the integer κ exactly n times. Thus,

$$s_n := 10^n\kappa + d\left(10^{n-1} + 10^{n-2} + \cdots + 10 + 1\right) = 10^n\kappa + d\left(\frac{10^n - 1}{9}\right). \tag{2.1}$$

The special integers κ of interest here are perfect powers, Fibonacci numbers, Sierpiński numbers, Riesel numbers and integers that are contained in various intersections of these sets. To avoid trivial situations, we require that $d \in \{1, 3, 7, 9\}$ and that $\gcd(\kappa, d) = 1$.

The sequence of *Fibonacci numbers* $\{F_n\}$ is defined by the recursion $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. A *Riesel number* is an odd positive integer k such that $k \cdot 2^n - 1$ is composite for all integers $n \geq 1$, while a *Sierpiński number* is an odd positive integer k such that $k \cdot 2^n + 1$ is composite for all integers $n \geq 1$. In 1956, Riesel [22] proved that there are infinitely many Riesel numbers, and in 1960, Sierpiński [23] proved that there are infinitely many Sierpiński numbers. Since then, other authors have examined extensions and variations of these ideas [1, 2, 3, 4, 5, 6, 9, 12, 10, 11, 14, 15, 16]. Coverings are used quite extensively in these investigations, but the concept of periodicity also plays a major role.

It is well-known that the Fibonacci sequence is periodic modulo any integer $m \geq 2$ [8, 21, 26]. Because of this periodicity, there must be a least positive integer r such that $F_r \equiv 0 \pmod{m}$. We call this value of r the *rank of apparition* of m in $\{F_n\}$ (also known as the *restricted period* of the Fibonacci sequence modulo m). We let $\mathcal{P}(m)$ denote the period of $\{F_n\}$ modulo m, and $\mathcal{R}(m)$ denote the rank of apparition of m. The following two theorems, due to Vinson [25], are useful in the calculation of $\mathcal{P}(m)$.

Theorem 2.3. *Let m be a positive integer, and let $m = \prod_{i=1}^{t} p_i^{a_i}$ be the canonical factorization of m into distinct prime powers. Then*

$$\mathcal{P}(m) = \operatorname{lcm}\left(\mathcal{P}(p^{a_i})\right).$$

In this paper, we only need to calculate $\mathcal{P}(m)$ for square-free values of m. By Theorem 2.3, this reduces our calculations down to the calculation of $\mathcal{P}(p)$, where p is a prime. Unfortunately, there is no known formula for $\mathcal{P}(p)$, simply in terms of the prime p. The best tool we have for calculating $\mathcal{P}(p)$ is the following.

Theorem 2.4. *Let p be a prime. If $p = 2$, then $\mathcal{P}(2) = \mathcal{R}(2) = 3$. If p is odd, then*

$$\mathcal{P}(p) = \begin{cases} \mathcal{R}(p) & \text{if } \mathcal{R}(p) \equiv 2 \pmod 4 \\ 2\mathcal{R}(p) & \text{if } \mathcal{R}(p) \equiv 0 \pmod 4 \\ 4\mathcal{R}(p) & \text{if } \mathcal{R}(p) \equiv 1 \pmod 2. \end{cases}$$

Remark 2.5. Any prime p such that $F_{\mathcal{R}(p)} \equiv 0 \pmod p$ is known as a *primitive (prime) divisor* of $F_{\mathcal{R}(p)}$.

Computations in this paper were performed using `Maple, MAGMA, Pari/GP` and `Primo`.

3. The Proof of Theorem 1.1

To establish Theorem 1.1, we use a procedure similar to one used in [12]. Suppose that $\mathbb{C} = \{(z_i, m_i, p_i)\}$ is a covering, where p_i is odd for all i. Recall that $10^{m_i} \equiv 1 \pmod{p_i}$, and that no prime p_i is repeated. Then $10^n \equiv 10^{z_i} \pmod{p_i}$ when $n \equiv z_i \pmod{m_i}$. Define $L_{\mathbb{C}} := \operatorname{lcm}_i \{p_i - 1\}$. Note that $L_{\mathbb{C}}$ is independent of the list of residues in $\mathbb{C}$. Using (2.1) with $\kappa = F_\nu^r$, we wish to determine the values of r for which there exist infinitely many Fibonacci numbers F_ν such that

$$10^n F_\nu^r + \frac{10^n - 1}{9}$$

is composite for all integers $n \geq 1$. We begin with an analysis of the following somewhat simpler problem, and then "layer" on the Fibonacci restriction. We first determine the values of r for which there exist infinitely many positive integers k such that

$$s_n := 10^n k^r + \frac{10^n - 1}{9}$$

is composite for all integers $n \geq 1$. The general strategy is to use $\mathbb{C}$ in our search for values of k such that $s_n \equiv 0 \pmod{p_i}$ for each i, and then piece together the results using the Chinese Remainder Theorem. Because of the periodicity, it is clear that we only need to check values of r with $0 \leq r \leq L_{\mathbb{C}} - 1$. We use (2.1) and proceed as follows. Let r be a fixed integer with $0 \leq r \leq L_{\mathbb{C}} - 1$. Then, for each i with $p_i \neq 3$, we calculate the values k^r, with $0 \leq k \leq p_i - 1$, to determine whether there is a k such that

$$k^r \equiv -\frac{1 - 10^{-z_i}}{9} \pmod{p_i}. \tag{3.1}$$

If $p_i = 3$, then we cannot use (3.1) since 9 is not invertible modulo 3. To avoid this complication, when $p_i = 3$ we require that m_i be chosen so that $m_i \equiv 0 \pmod 3$. Then $n \equiv z_i \pmod{m_i}$ implies that $n \equiv z_i \pmod 3$. Then we can use the first formula in (2.1) to deduce that $k \equiv -z_i \pmod 3$. If, for each i, we are able to find a solution β_i for k, then we can use the Chinese Remainder Theorem to solve the resulting system of congruences $k \equiv \beta_i \pmod{p_i}$.

Now we add the Fibonacci layer, which can be done in one of two ways. The first way is to find the set of Fibonacci numbers $F_{\nu_i} \equiv k \pmod{p_i}$ for each i, if any exist. Then, if such numbers exist for all i, determine if the intersection of all these sets is nonempty to find a solution to the entire problem. This method was employed in [19]. We prefer a second approach. Let $\rho = \prod_{p_i \in \mathbb{C}} p_i$. We search for a Fibonacci number F_ν such that $F_\nu \equiv k \pmod{\rho}$. If such a Fibonacci number F_ν exists, it must be that $0 \le \nu < \mathrm{lcm}_{p_i \in \mathbb{C}} (\mathcal{P}(p_i))$ by Theorem 2.3. Using a computer to carry out this search is relatively painless as long as $\mathbb{C}$ is not too large. We illustrate this technique with an example.

Example 3.1. Let $r = 2$ and suppose we want to find a Fibonacci number F_ν such that F_ν^2 is 1-composite. That is, we want

$$10^n F_\nu^2 + \frac{10^n - 1}{9}$$

to be composite for all integers $n \ge 1$. Let

$$\mathbb{C} = \{(z_i, m_i, p_i)\} = \{(1, 2, 11), (2, 3, 3), (4, 6, 7), (0, 6, 13)\}.$$

It is easy to check that $\mathbb{C}$ is a covering. Solving each congruence

$$k^2 \equiv \begin{cases} -\dfrac{1 - 10^{-z_i}}{9} \pmod{p_i} & \text{if } p_i \ne 3 \\ -z_i \pmod{p_i} & \text{if } p_i = 3 \end{cases}$$

for k gives the following sets of solutions:

- $\{1, 10\}$ modulo 11
- $\{1, 2\}$ modulo 3
- $\{2, 5\}$ modulo 7
- $\{0\}$ modulo 13.

Hence, there are 8 possible sets of residues for k. They are

$$\{[1, 2, 5, 0], [1, 1, 5, 0], [1, 2, 2, 0], [10, 1, 2, 0], [1, 1, 2, 0], [10, 1, 5, 0], [10, 2, 2, 0], [10, 2, 5, 0]\}.$$

Consider first the possibility $[1, 2, 5, 0]$. Here $\rho = 11 \cdot 3 \cdot 7 \cdot 13 = 3003$, and so using the Chinese Remainder Theorem to solve for k gives $k = 1937 \pmod{3003}$. Since the periods of the primes 11, 3, 7 and 13 are, respectively, 10, 8, 16 and 28, we have that

$$\mathcal{P}(\rho) = \mathrm{lcm}(10, 8, 16, 28) = 560.$$

We now search for a Fibonacci number F_ν such that

$$F_\nu \equiv 1937 \pmod{3003}. \tag{3.2}$$

If such a Fibonacci number F_ν exists, it must be that $0 \le \nu < 560$ by Theorem 2.3. Using a computer to conduct this search yields the solutions F_{21}, F_{91}, F_{469} and F_{539}. Thus, F_{21}^2, F_{91}^2, F_{469}^2 and F_{539}^2 are 1-composite.

Remark 3.2. Although it is known that there can be at most 4 solutions to (3.2) [20, 24], there is no known way to predict exactly how many solutions there will be. In other words, it is unknown how to determine, *a priori*, the frequency of a particular residue modulo a prime p in a period of the Fibonacci sequence modulo p.

For a given value of r, when we find such a Fibonacci number, we say that we have *captured* this value of r. This process can be repeated for every list of residues for which a covering exists, either keeping the same list of primes, or rearranging the list of primes if the new arrangement still satisfies the conditions that $10^{m_i} \equiv 1 \pmod{p_i}$, and that no prime p_i is repeated. In addition, this process can be repeated for coverings with different lists of moduli. To combine all of these results in a sensible manner, one must take care since the values of r captured using one list of moduli must be "meshed" with the values of r captured using a different list of moduli. This can be done by examining values of $r \pmod{\mathcal{L}}$, where $\mathcal{L} = \mathrm{lcm}_{\mathbb{C}} L_{\mathbb{C}}$, for all coverings $\mathbb{C}$ under consideration. Then the density of the set of captured values of r will be the cardinality of the union of these various sets divided by $\mathcal{L}$. We call this density a *1-composite Fibonacci r-density*, and we denote it as $r_F(1)$. Ideally, we would like to achieve $r_F(1) = 1$. However, we have only been able to achieve $r_F(1) = 8/9$.

Proof of Theorem 1.1. Consider the lists

$$M = [2, 3, 6, 6], \quad P_1 = [11, 3, 13, 37], \quad \text{and} \quad P_2 = [11, 37, 3, 7],$$

where M is a list of moduli to be used to construct a covering, and P_1 and P_2 are lists of corresponding primes. Note that here $\mathcal{L} = 180$. There are only 12 coverings having M as the moduli. Then, when we apply the procedure outlined above to the 12 coverings using P_1, we capture 145 r-values out of the total $\mathcal{L}$. Using P_2, we capture 155 r-values out of $\mathcal{L}$. When we take the union of these two sets of r-values, we get 160 total r-values out of a possible $\mathcal{L} = 180$. Thus, $r_F(1) = 8/9$. Further inspection reveals that the missing r-values are exactly the values of r such that $r \equiv 0, 12, 18, 24 \pmod{36}$. □

Remark 3.3. Since so few coverings are used in the proof of Theorem 1.1, and the coverings used are quite simple, one might speculate that the missing r-values might be achieved using more complicated coverings. Our analysis indicates that this is not the case–even before introducing the Fibonacci restriction. For example, we have not been able to show that there exist infinitely many positive integers k such that k^{12} is 1-composite. However, we also cannot prove that these missing r-values can never be achieved with our methods.

4. The Proof of Theorem 1.2

Some of the items in Theorem 1.2 follow immediately from a careful analysis of the specific details of the proof of Theorem 1.1. In any case, only minor tweaking and variations of the general methods described in Section 3 (for example, changing the moduli) are required for the proofs. We give below the basic details of the proof of each item.

For example, to establish item (1), we use the single covering

$$\mathbb{C} = \{(1, 2, 11), (2, 3, 3), (0, 6, 13), (4, 6, 7)\}.$$

We let $\kappa = k^2$ in (2.1). Then, for each p_i in $\mathbb{C}$, we solve the congruence $s_n \equiv 0 \pmod{p_i}$ for k^2 to get:

$$\begin{aligned} k^2 &\equiv 1 \pmod{11} \\ k^2 &\equiv 1 \pmod{3} \\ k^2 &\equiv 0 \pmod{13} \\ k^2 &\equiv 4 \pmod{7}. \end{aligned} \tag{4.1}$$

Note that each residue in (4.1) is a square modulo the corresponding prime p_i. So, we take a square root in each case:

$$\begin{aligned} k &\equiv 10 \pmod{11} \\ k &\equiv 2 \pmod{3} \\ k &\equiv 0 \pmod{13} \\ k &\equiv 2 \pmod{7}. \end{aligned} \tag{4.2}$$

Then we apply the Chinese Remainder Theorem to (4.2) to get infinitely many solutions $k \equiv 65 \pmod{3003}$. Hence, $k^2 = 65^2$ is 1-composite. To verify that it is the smallest 1-composite square, we use `Pari/GP` or `Primo` to certify that a prime is reached in each sequence $\{s_n\}_{n=1}^{\infty}$ for each $\kappa = k^2 < 65^2$.

To establish item (2), we use the covering

$$\mathbb{C} = \{(0,2,11),(0,3,3),(5,6,13),(1,6,7)\}\,.$$

The list of square roots we use here is $[0,0,1,3]$. Using the Chinese Remainder Theorem, this list produces the arithmetic progression $k \equiv 66 \pmod{3003}$. Combining this with item (1) yields the result.

For item (3), we use the covering

$$\mathbb{C} = \{(1,3,3),(2,3,37),(3,6,7),(0,6,13)\}\,,$$

and proceed as in the proof of item (1).

Item (4) follows from using the covering

$$\mathbb{C} = \{(1,2,11),(1,3,3),(0,6,13),(2,6,7)\}\,. \tag{4.3}$$

The values of k modulo the corresponding primes that yield $k \equiv 44 \pmod{3003}$, using the Chinese Remainder Theorem, are $[0,2,5,2]$.

For item (5), we use the covering (4.3), which produces the arithmetic progression $k \equiv 1937 \pmod{3003}$ of 1-composite integers. Observe that F_{21} is in this progression. Then, from Example 3.1, we have that F_{21}^2 is also 1-composite using the same primes, and the result follows.

5. The Proof of Theorem 1.3

For each $d \in \{1,3,7,9\}$, we first construct a covering $\mathbb{C}_d = \{(z_i,m_i,p_i)\}$ to find d-composite squares. We require that this collection of coverings satisfies the following properties:

- $10^{m_i} \equiv 1 \pmod{p_i}$ for all i,
- no prime p_i is repeated in any individual covering,
- the system of congruences

$$\sigma \equiv \begin{cases} -d\,(1-10^{-z_i})\,9^{-1} \pmod{p_i} & \text{if } p_i \neq 3 \\ -dz_i \pmod{p_i} & \text{if } p_i = 3 \end{cases} \tag{5.1}$$

  is consistent, and
- each residue in each congruence in (5.1) is a square modulo p_i.

The following coverings have been constructed accordingly:

$$\begin{aligned}
\mathbb{C}_1 &= \{(0,2,11),(0,3,37),(1,6,13),(5,6,3)\},\\
\mathbb{C}_3 &= \{(0,2,11),(0,3,37),(1,4,101),(7,8,73),(11,24,99990001),(0,7,239),(5,7,4649),\\
&\qquad (11,14,909091),(1,21,43),(10,21,1933),(13,21,10838689),(23,28,281)\},\\
\mathbb{C}_7 &= \{(0,2,11),(3,6,37),(5,6,3),(1,30,9091),(7,30,241),\\
&\qquad (13,30,211),(19,30,271),(25,30,41)\},\\
\mathbb{C}_9 &= \{(0,2,11),(1,6,7),(3,6,37),(5,6,13)\}.
\end{aligned}$$

From these coverings we construct the set of congruences (5.1). We add the two additional congruences $\sigma \equiv 1 \pmod 2$ and $\sigma \equiv 1 \pmod 5$ to ensure that $\gcd(\sigma,10) = 1$. With redundancies removed, the resulting system of congruences in σ is given below:

$$\begin{array}{ll}
\sigma \equiv 1 \pmod{2} & \sigma \equiv 109 \pmod{211} \\
\sigma \equiv 1 \pmod{5} & \sigma \equiv 0 \pmod{239} \\
\sigma \equiv 1 \pmod{3} & \sigma \equiv 205 \pmod{241} \\
\sigma \equiv 4 \pmod{7} & \sigma \equiv 7 \pmod{271} \\
\sigma \equiv 0 \pmod{11} & \sigma \equiv 175 \pmod{281} \\
\sigma \equiv 9 \pmod{13} & \sigma \equiv 917 \pmod{1933} \\
\sigma \equiv 0 \pmod{37} & \sigma \equiv 33 \pmod{4649} \\
\sigma \equiv 0 \pmod{41} & \sigma \equiv 6363 \pmod{9091} \\
\sigma \equiv 4 \pmod{43} & \sigma \equiv 333 \pmod{909091} \\
\sigma \equiv 3 \pmod{73} & \sigma \equiv 817266 \pmod{10838689} \\
\sigma \equiv 30 \pmod{101} & \sigma \equiv 66659997 \pmod{99990001}.
\end{array} \tag{5.2}$$

Then, any value of σ satisfying all congruences in (5.2), is pandigital-composite. Note also that each residue in each congruence in (5.2) is a square modulo the respective prime.

We now incorporate the Sierpiński layer. We seek a covering $\mathbb{C}_R = \{(z_i, m_i, p_i)\}$ such that

- $2^{m_i} \equiv 1 \pmod{p_i}$ for all i,
- no prime p_i is repeated,
- the system of congruences

$$\sigma \equiv -2^{-z_i} \pmod{p_i} \tag{5.3}$$

  is consistent with the system (5.2),
- and -2^{-z_i} in each congruence in (5.3) is a square modulo p_i.

The following covering $\mathbb{C}_S := \{(z_i, m_i, p_i)\}$, which was used by Sierpiński [23], satisfies all criteria above:

$$\mathbb{C}_S = \{(1,2,3),(2,4,5),(4,8,17),(8,16,257),(16,32,65537),(32,64,641),(0,64,6700417)\}.$$

The resulting system of congruences in σ is

$$\begin{array}{ll}
\sigma \equiv 1 \pmod{3} & \sigma \equiv 1 \pmod{65537} \\
\sigma \equiv 1 \pmod{5} & \sigma \equiv 1 \pmod{641} \\
\sigma \equiv 1 \pmod{17} & \sigma \equiv -1 \pmod{6700417} \\
\sigma \equiv 1 \pmod{257}. &
\end{array} \tag{5.4}$$

We now incorporate the Riesel layer. We seek a covering $\mathbb{C}_R = \{(z_i, m_i, p_i)\}$ such that

- $2^{m_i} \equiv 1 \pmod{p_i}$ for all i,
- no prime p_i is repeated,
- the system of congruences

$$\sigma \equiv 2^{-z_i} \pmod{p_i} \tag{5.5}$$

  is consistent with the systems (5.2) and (5.4),
- and 2^{-z_i} in each congruence in (5.5) is a square modulo p_i.

We have constructed the following covering $\mathbb{C}_R$ in accordance with the conditions above. We list the elements in $\mathbb{C}_R$ with the moduli in increasing order:

$$
\begin{aligned}
\mathbb{C}_R = \{&(0,2,3),(1,3,7),(3,5,31),(4,7,127),(2,11,23),(0,11,89),(5,15,151),(9,21,337),\\
&(16,25,601),(11,25,1801),(24,27,262657),(3,33,599479),(14,35,71),(24,35,122921),\\
&(2,45,631),(32,45,23311),(41,49,4432676798593),(34,55,881),(49,55,3191),\\
&(29,55,201961),(12,63,649657),(36,63,92737),(15,66,20857),(71,75,10567201),\\
&(56,75,100801),(41,77,581283643249112959),(33,81,97685839),(6,81,71119),\\
&(69,81,2593),(17,90,18837001),(55,98,4363953127297),(39,99,199),\\
&(75,99,33057806959),(54,99,153649),(29,105,152041),(89,105,29191),(44,105,106681),\\
&(117,126,77158673929),(92,135,271),(15,135,49971617830801),(42,135,348031),\\
&(87,147,2741672362528725535068727),(101,150,1133836730401),(49,154,617),\\
&(133,154,78233),(141,162,272010961),(3,162,135433),\\
&(117,165,2048568835297380486760231),(139,175,5353476247914885528371 51),\\
&(69,175,60816001),(104,175,39551),(183,189,2076174855442583929707535 27),\\
&(120,189,1560007),(45,210,664441),(87,210,1564921),(126,225,13861369826299351),\\
&(171,225,1348206751),(81,225,617401),(201,225,115201),(63,231,463),\\
&(27,231,4982397651178256151338302204762057),(34,245,1471),\\
&(209,245,252359902034571016856214298851708529738525821631),(231,270,15121),\\
&(40,275,4074891477354886815033308087379995347151),(95,275,382027665134363932751),\\
&(219,297,17088661882314173808183095080729277164831359 9433),(21,297,8950393),\\
&(0,315,2972830715596370681022843537 8401),(90,315,870031),(27,315,983431),\\
&(315,330,415365721),(57,378,126127),(363,378,309583),\\
&(182,385,3105534168119044447812671975596513457115147392576553 2041),\\
&(42,385,1971764055031),(314,385,55441),\\
&(204,405,1764566555621340010737060208115573728140 6841),\\
&(339,405,11096527935003481),(201,405,537841),\\
&(440,441,708642357485397214797008608843 4689),\\
&(314,441,4487533753346305838985313),(146,441,5828257),\\
&(381,450,2819414729537101777586472 01),(21,450,4714696801),\\
&(111,462,7018079616527704034924570385 1057),\\
&(419,490,5064728203579612588500033 0641),(120,495,991),\\
&(417,495,3342029347647379514385947 46151),\\
&(329,495,6084777159537635796550536863741698483921),\\
&(209,525,4201),(524,525,7351),(384,525,181165951),\\
&(174,525,3259855088755275876696070972226675571162211390901315 14801)\}.
\end{aligned}
$$

We give below in (5.6) the list of the residues, in the same order as the elements in $\mathbb{C}_R$, for the resulting system of congruences in (5.5):

$$\begin{aligned}
&[1,4,4,8,6,1,118,52,512,175,8,74935,25,2048,158,8192,256,372,64,57812,\\
&140685,27289,8997,16,20283,68719476736,69640886,18891,1503,14119559,\\
&68186767614,182,16777216,101553,6021,7154,75371,512,7,46285662011189,\\
&36281,1152921504606846976,566935142416,266,63094,2097152,118504,\\
&281474976710656,68719476736,30982137,31058,64,144933,533857,1302777,\\
&12617341819044219,998927987,120298,73071,55,\\
&271392961703758036325237462853614 1,1332,68719476736,14062,\\
&150251442981525502332046286841271612164 4,1485827250033231705 88,\\
&302231454903657293676544,3015748,1,62776,661245,32768,123646,\\
&195076,21443776412950837228588868509503911186712795997631910894,\\
&355066555671,14277,100266206152789866661878603563759790627774 37,\\
&6162055000061295,492419,2,27119152716895818664450 14,2767532,\\
&590295810358705651712,3230057134,66464381596548351255816362526061,\\
&2361183241434822606848,451,302231454903657293676544,\\
&40174053106081768675393715413111776849 43,706,2,115894483,\\
&325889727904223469615959700533470662792244967894995039663].
\end{aligned} \tag{5.6}$$

Each of the residues in (5.6) is a square modulo the corresponding prime, and the only primes that are ever repeated in any of (5.2), (5.4) and $\mathbb{C}_R$, are 3, 5, 7 and 271. Note that, in each of these cases, the congruences for σ are consistent. Thus, the systems in σ are all consistent and, since all residues are squares modulo the corresponding prime, we can let $\sigma = k^2$ and take square roots. Choosing a particular set of these square roots, and applying the Chinese Remainder Theorem, we get infinitely many solutions for k. Hence, any such k^2 satisfies the conclusion of the theorem. For the purposes of illustration and validation, we give a particular solution for k^2 that contains 2394 digits, which is not necessarily the smallest possible solution:

$k^2 =$ 43021946782373329533140445608206926620070245513074401153451532921179010431900416163226050146484723979174551652351652102117979149819561066496619899317886761385753419318551938667628848956588369955129998362175881336037737674353511098899315297768203257120145102201071231900200412844150356407609725706928501431754788923063687882255606949662375255267649453321627445009689281153450615483220291827714969818141315505430317824599835334905109030569798413870327659098164712922151213673548098134922352839372410298012060699679102999826659558444654602754396368466961056825321349230220434365709386489922747993758098174851058640377876718771593365694464592316621278595224245210816098929506627324545669474898464571017394931902418399852590050412662109871584501347072712689774583046743851064184716229392099558277971689499167948283702502257210411495677010113276749186746441931599596856689685601511872416340349034277630620739860907844020876895536473275521427012707943957056688102014717850343363466607552366912185062275112784316880842852873709305046392396463839764129578953843753697896634666636768961947116002896209045964345805035925883583913734653799627137628391027801772338434301527300725225503218873568734341492399011490883263595961146690719767310500292250394043493638555094077690256055354372561744861589184090818518035390761273124942402876672723391015920833210308235210634463437256238627248448447030160046332583627649358691602277791370057259949854150880123998637912389174820387664201647625372577606139018456922969596372780971869215811283876715945310888936601735142584464755599366746384394298482835152105288710533547907705915719797157866879102030182556311317808176755138090550893421754408578008829931949633865446862182890342989561118156298845071720570659770626245224762545616320748294142636569809628311629751560235512789799997347052508259542187853300672534060157988491840597911575044277563308086439929063916707522619872394862583422425038280871448343198687750093298038699321383589908320092636642991417923369299789032821115680092762518752207728666353868414166075080830369204930612498357289855409789824460762119004492955119735551651745466716268407694783997000512140299387643765595416823330572055900813223792082843509566038683119446613600125972379656745938732765448454736789698743409072985264243349259127133068274648532712697489981353613757305122943584677048165201519912596482928996525570248793373376579857853398426961.

6. d-COMPOSITES WHEN $d \in \{3, 7, 9\}$

As mentioned in Section 1, analogous results can, in many cases, be established for d-composite numbers when $d \in \{3, 7, 9\}$. Since the proofs are similar to the proofs in the 1-composite situation, we present only one such theorem.

Theorem 6.1.

(1) *There exist infinitely many Fibonacci numbers that are 3-composite. The smallest known one is* F_{570}.
(2) *There exist infinitely many Fibonacci numbers that are 7-composite. The smallest known one is* F_{1250}.

Proof. To prove item (1), we use the covering

$$\mathbb{C} = \{(0, 2, 11), (0, 3, 37), (1, 6, 7), (5, 6, 13)\}.$$

For item (2), we use the covering

$$\mathbb{C} = \{(0, 2, 11), (1, 3, 37), (3, 6, 13), (5, 6, 3)\}.$$

In both cases, the techniques are similar to the techniques used in the proof of Theorem 1.2. □

Remark 6.2. *Curiously, we have been unable to find a Fibonacci number that is 9-composite.*

References

[1] B. Banks, C. Finch, F. Luca, C. Pomerance, and P. Stănică, *Sierpiński and Carmichael numbers*, Trans. Amer. Math. Soc., (to appear).
[2] Y. G. Chen, *On integers of the form* $2^n \pm p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, Proc. Amer. Math. Soc., **128** (2000), 1613–1616.
[3] Y. G. Chen, *On integers of the form* $k2^n + 1$, Proc. Amer. Math. Soc., **129** (2001), 355–361.
[4] Y. G. Chen, *On integers of the forms* $k - 2^n$ *and* $k2^n + 1$, J. Number Theory, **89** (2001), 121–125.
[5] Y. G. Chen, *On integers of the forms* $k^r - 2^n$ *and* $k^r 2^n + 1$, J. Number Theory, **98** (2003), 310–319.
[6] Y. G. Chen, *On integers of the forms* $k \pm 2^n$ *and* $k2^n \pm 1$, J. Number Theory, **125** (2007), 14–25.
[7] P. Erdős, *On integers of the form* $2^k + p$ *and some related problems*, Summa Brasil. Math., (1950), 113–123.
[8] G. Everest, A. van der Poorten, I. Shparlinski, and T. Ward, *Recurrence Sequences*, **104**, American Mathematical Society, (2003).
[9] M. Filaseta, *Coverings of the integers associated with an irreducibility theorem of A. Schinzel*, Number Theory for the Millennium, II, (2002) 1–24.
[10] M. Filaseta, C. Finch, and M. Kozek, *On powers associated with Sierpiński numbers, Riesel numbers and Polignac's conjecture*, J. Number Theory, **128** (2008), 1916–1940.
[11] M. Filaseta and J. Harrington, *A polynomial investigation inspired by work of Schinzel and Sierpiński*, Acta Arithmetica, (to appear).
[12] C. Finch, J. Harrington, and L. Jones, *Nonlinear Sierpiński and Riesel numbers*, J. Number Theory, **133.2** (2013), 534-544.
[13] J. Grantham, W. Jarnicki, J. Rickert, and S. Wagon, *Repeatedly appending any digit to generate composite numbers*, Amer. Math. Monthly, (to appear).
[14] A. S. Izotov, *A note on Sierpiński numbers*, The Fibonacci Quarterly, **33.3** (1995), 206–207.
[15] L. Jones, *Variations on a theme of Sierpiński*, J. Integer Seq., **10** (2007), Article 07.4.4, 15 pp.
[16] L. Jones, *Polynomial variations on a theme of Sierpiński*, Int. J. Number Theory, **5** (2009), 1–17.
[17] L. Jones, *Using Lucas sequences to generalize a theorem of Sierpiński*, Acta Arith., **152.3** (2012), 307-322.
[18] L. Jones, *When does appending the same digit repeatedly on the right of a positive integer generate a sequence of composite integers?*, Amer. Math. Monthly, **118** (2011), 153–160.
[19] F. Luca and V. J. Meja Huguet, *Fibonacci-Riesel and Fibonacci-Sierpiński numbers*, The Fibonacci Quarterly, **46/47.3** (2008/09), 216-219.
[20] H. Niederreiter, A. Schinzel, and L. Somer, *Maximal frequencies of elements in second-order linear recurring sequences over a finite field*, Elem. Math., **46.5** (1991), 139-143.
[21] Pisano Period, `http://en.wikipedia.org/wiki/Pisano_period.`
[22] H. Riesel, *Några stora primtal*, Elementa, **39** (1956), 258–260.
[23] W. Sierpiński, *Sur un problème concernant les nombres* $k.2^n + 1$, Elem. d. Math., **15** (1960), 73–74.
[24] L. Somer, *Distribution of Residues of Certain Second-Order Linear Recurrences Modulo p*, Applications of Fibonacci Numbers, Vol. 3. Ed. G. E. Bergum, A. N. Philippou, and A. F. Horadam.

[25] J. Vinson, *The relation of the period modulo m to the rank of apparition of m in the Fibonacci sequence*, The Fibonacci Quarterly, **1.1** (1963), 37–45.
[26] D. D. Wall, *Fibonacci series modulo m*, American Mathematical Monthly, 67 (1960), 525-532.

MSC2010: 11B39, 11B25, 11Y55, 11B75

Department of Mathematics, Shippensburg University, Shippensburg, PA 17257
E-mail address: `lkjone@ship.edu`

Department of Mathematics, Shippensburg University, Shippensburg, PA 17257
E-mail address: `mm0283@ship.edu`