POLYNOMIAL EXTENSIONS OF THE LUCAS AND GINSBURG IDENTITIES

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Abstract. We extend the well-known Lucas identity $F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n}$ and the Ginsburg identity $F_{n+2}^3 - 3F_n^3 + F_{n-2}^3 = 3F_{3n}$ to Fibonacci and Lucas polynomials. This yields interesting dividends to Pell and Pell-Lucas polynomials and numbers.

1. Introduction

Fibonacci polynomials $f_n(x)$ and Lucas polynomials $l_n(x)$ were originally studied by Catalan in 1883 and Bicknell in 1970; see [1, 5]. They belong to a larger integer family of *gibonacci* (*g*eneralized F*ibonacci*) polynomials $g_n(x)$, defined recursively as follows:

$$g_1(x) = a, \ g_2(x) = b$$
$$g_n(x) = xg_{n-1}(x) + g_{n-2}(x),$$

where $a = a(x), b = b(x)$, and $n \geq 3$. When $a = 1$ and $b = x$, $g_n(x) = f_n(x)$; and when $a = x$ and $b = x^2 + 2$, $g_n(x) = l_n(x)$. Clearly, $F_n = f_n(1)$ and $L_n = l_n(1)$.

Table 1 shows the first 10 Fibonacci and Lucas Polynomials.

n	$f_n(x)$	$l_n(x)$
1	1	x
2	x	x^2+2
3	x^2+1	x^3+3x
4	x^3+2x	x^4+4x^2+2
5	x^4+3x^2+1	x^5+5x^3+5x
6	x^5+4x^3+3x	$x^6+6x^4+9x^2+2$
7	$x^6+5x^4+6x^2+1$	$x^7+7x^5+14x^3+7x$
8	$x^7+6x^5+10x^3+4x$	$x^8+8x^6+20x^4+16x^2+2$
9	$x^8+7x^6+15x^4+10x^2+1$	$x^9+9x^7+27x^5+30x^3+9x$
10	$x^9+8x^7+21x^5+20x^3+5x$	$x^{10}+10x^8+35x^6+50x^4+25x^2+2$

TABLE 1. First 10 Fibonacci and Lucas Polynomials.

2. Pell and Pell-Lucas Polynomials

The polynomials $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$ are the *Pell* and *Pell-Lucas polynomials*, respectively; see [4, 6]. Table 2 shows the first 10 Pell and Pell-Lucas Polynomials. The numbers $P_n = p_n(1) = f_n(2)$ and $Q_n = \frac{1}{2}q_n(1) = \frac{1}{2}l_n(2)$ are the nth *Pell* and *Pell-Lucas numbers*, respectively.

n	$p_n(x)$	$q_n(x)$
1	1	$2x$
2	$2x$	$4x^2+2$
3	$4x^2+1$	$8x^3+6x$
4	$8x^3+4x$	$16x^4+16x^2+2$
5	$16x^4+12x^2+1$	$32x^5+40x^3+10x$
6	$32x^5+32x^3+6x$	$64x^6+96x^4+36x^2+2$
7	$64x^6+80x^4+24x^2+1$	$128x^7+224x^5+112x^3+14x$
8	$128x^7+192x^5+80x^3+8x$	$256x^8+512x^6+320x^4+64x^2+2$
9	$256x^8+448x^6+240x^4+40x^2+1$	$512x^9+1152x^7+864x^5+240x^3+18x$
10	$512x^9+1024x^7+672x^5+160x^3+10x$	$1024x^{10}+2560x^8+2240x^6+800x^4+100x^2+2$

TABLE 2. First 10 Pell and Pell-Lucas Polynomials in x.

3. Binet's Formula

The gibonacci polynomials $g_n(x)$ can also be defined explicitly by *Binet's Formula*:

$$g_n(x) = \frac{c\alpha^n - d\beta^n}{\alpha - \beta},$$

where $c = c(x) = a + (a-b)\beta, d = d(x) = a + (a-b)\alpha$, and $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are the solutions of the equation $t^2 - xt - 1 = 0$.

In the interest of brevity, clarity, and convenience, we let $\Delta = \Delta(x) = \alpha - \beta = \sqrt{x^2+4}$, and will denote $g_n(x), f_n(x), l_n(x), p_n(x)$, and $q_n(x)$ by g_n, f_n, l_n, p_n, and q_n, respectively.

It is easy to confirm that

$$g_{a+b} = g_{a+1}f_b + g_a f_{b-1} \tag{3.1}$$

$$g_{n+a}g_{n+b} - g_n g_{n+a+b} = (-1)^n \mu f_a f_b, \tag{3.2}$$

where $\mu = \mu(x) = a^2 + abx - b^2 = cd; \mu = 1$ when $g_n = f_n$; and $\mu = -\Delta^2$ when $g_n = l_n$. Identity (3.1) is the *addition formula* for the gibonacci family, and (3.2) is a generalized Cassini's identity for the family.

4. Lucas and Ginsburg Identities

In 1876 Lucas established a charming identity involving the cubes of three consecutive Fibonacci numbers: $F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n}$; see [5, 7, 10]. In 1986 Long discovered its Lucas counterpart: $L_{n+1}^3 + L_n^3 - L_{n-1}^3 = 5L_{3n}$ [7].

Interestingly, in 1953 Ginsburg noted that Lucas' identity is the only identity involving the cubes of Fibonacci numbers mentioned in Dickson's classic work *History of the Theory of Numbers* [2, 10]. He then developed an equally delightful identity involving the cubes of three Fibonacci numbers, separated by two spaces: $F_{n+2}^3 - 3F_n^3 + F_{n-2}^3 = 3F_{3n}$ [3, 10].

5. Polynomial Extensions

We will now extend the two identities to Fibonacci and Lucas polynomials. Obviously, they have Pell and Pell-Lucas implications. In both cases, their proofs involve some messy, but carefully crafted algebra. So we will omit a lot of details for the sake of brevity. We will capitalize on a powerful technique touched upon by Melham in 1999 [8].

We begin our pursuits with a lemma. Its proof is elementary; so we will omit it.

Lemma 5.1. *Let* $r_n = g_n^3 - g_{n+1}g_ng_{n-1}$. *Then* r_n *satisfies the recurrence* $r_n = -xr_{n-1} + r_{n-2}$.

We are now ready to establish a generalization of Lucas' identity. To this end, we will need the following identities:

$$\begin{array}{rclcrcl} f_{-n} &=& (-1)^{n-1}f_n & & f_{2n} &=& f_nl_n \\ f_{n+1}+f_{n-1} &=& l_n & & l_{n+1}+l_{n-1} &=& \Delta^2 f_n \\ f_{n+1}^2+f_n^2 &=& f_{2n+1} & & l_{n+1}^2+l_n^2 &=& \Delta^2 f_{2n+1}. \end{array}$$

Theorem 5.2.

$$g_{n+1}^3 + xg_n^3 - g_{n-1}^3 = \begin{cases} xf_{3n} & \text{if } g_k = f_k \\ x(x^2+4)l_{3n} & \text{if } g_k = l_k. \end{cases}$$

Proof. By Lemma 5.1, we have

$$\begin{aligned} g_{n+1}^3 - g_{n+2}g_{n+1}g_n &= -x(g_n^3 - g_{n+1}g_ng_{n-1}) + (g_{n-1}^3 - g_ng_{n-1}g_{n-2}) \\ g_{n+1}^3 + xg_n^3 - g_{n-1}^3 &= g_{n+2}g_{n+1}g_n + xg_{n+1}g_ng_{n-1} - g_ng_{n-1}g_{n-2} \\ &= (xg_{n+1}+g_n)g_{n+1}g_n + xg_{n+1}g_ng_{n-1} - g_ng_{n-1}(g_n - xg_{n-1}) \\ &= xg_{n+1}^2g_n + g_n^2(g_{n+1}-g_{n-1}) + xg_ng_{n-1}(g_{n+1}+g_{n-1}) \\ &= xg_{n+1}^2g_n + xg_n^3 + xg_ng_{n-1}(g_{n+1}+g_{n-1}). \end{aligned} \tag{5.1}$$

Suppose $g_k = l_k$. Then (5.1) yields

$$\begin{aligned} g_{n+1}^3 + xg_n^3 - g_{n-1}^3 &= xl_{n+1}^2l_n + xl_n^3 + xl_nl_{n-1}\Delta^2 f_n \\ &= xl_n(l_{n+1}^2 + l_n^2) + x\Delta^2 l_{n-1}f_{2n} \\ &= xl_n\Delta^2 f_{2n+1} + x\Delta^2 l_{n-1}f_{2n} \\ &= x\Delta^2(l_nf_{2n+1} + l_{n-1}f_{2n}) \\ &= x(x^2+4)l_{3n}, \end{aligned}$$

as desired. The other case can be handled similarly. □

For example,

$$\begin{aligned} f_5^3 + xf_4^3 - f_3^3 &= x^{12} + 10x^{10} + 36x^8 + 56x^6 + 35x^4 + 6x^2 \\ &= xf_{12}; \\ l_4^3 + xl_3^3 - l_2^3 &= x^{12} + 13x^{10} + 63x^8 + 138x^6 + 129x^4 + 36x^2 \\ &= x(x^2+4)l_9. \end{aligned}$$

We note that this theorem can also be established using the addition formula for g_n.

Clearly, both Lucas' and Long's identities follow from this theorem. It also follows from the theorem that

$$\begin{aligned} p_{n+1}^3 + 2xp_n^3 - p_{n-1}^3 &= 2xp_{3n} \\ q_{n+1}^3 + 2xq_n^3 - q_{n-1}^3 &= 8x(x^2+1)q_{3n} \\ P_{n+1}^3 + 2P_n^3 - P_{n-1}^3 &= 2P_{3n} \\ Q_{n+1}^3 + 2Q_n^3 - Q_{n-1}^3 &= 4Q_{3n}. \end{aligned}$$

Next we will generalize the Ginsburg identity in Theorem 2. Although it follows by Theorem 1, we will provide an independent proof. In addition, this approach will exemplify the power of Melham's technique. To this end, we will need the next two lemmas. The proof of Lemma 5.3 is also straightforward; so we omit that too.

Lemma 5.3. *Let* $s_n = g_{n+2}^3 - g_{n+4}g_{n+2}g_n$. *Then* $s_n = (x^2+2)s_{n-2} - s_{n-4}$.

Lemma 5.4.

$$g_{n+3}^2 - xg_{n+3}g_{n+2} - g_{n-1}g_{n-3} = \begin{cases} x(x^2+2)f_{2n} & \text{if } g_k = f_k \\ x(x^2+2)(x^2+4)f_{2n} & \text{if } g_k = l_k. \end{cases}$$

Proof. We have

$$\begin{aligned} g_{n+3}^2 - xg_{n+3}g_{n+2} - g_{n-1}g_{n-3} &= g_{n+3}(g_{n+3} - xg_{n+2}) - g_{n-1}g_{n-3} \\ &= g_{n+3}g_{n+1} - g_{n-1}g_{n-3} \\ &= g_{n+1}[(x^2+1)g_{n+1} + xg_n] - \\ &\quad (g_{n+1} - xg_n)[(x^2+1)g_{n+1} - (x^3+2x)g_n] \\ &= 2(x^3+2x)g_{n+1}g_n - x(x^3+2x)g_n^2 \\ &= (x^3+2x)g_n[g_{n+1} + (g_{n+1} - xg_n)] \\ &= (x^3+2x)g_n(g_{n+1} + g_{n-1}). \end{aligned} \tag{5.2}$$

Since $f_{n+1} + f_{n-1} = l_n$ and $l_{n+1} + l_{n-1} = (x^2+4)f_n$, the desired results follow from (5.2), as claimed. □

It follows by Lemma 5.4 that

$$x(x^2+2)\Delta^2 f_{2n-2} - l_{n+2}^2 + xl_{n+2}l_{n+1} + l_{n-2}l_{n-4} = 0.$$

We will employ this result in the proof of Theorem 5.8.

We are now ready to present the next generalization. In addition to Lemmas 5.3 and 5.4, we will need the following three identities:

$$\begin{aligned} f_{n+2} - f_{n-2} &= xl_n \\ l_{n+2} - l_{n-2} &= x\Delta^2 f_n \\ l_{n+1}f_{2n} + l_nf_{2n-1} &= l_{3n}. \end{aligned}$$

Theorem 5.5.

$$g_{n+2}^3 - (x^2+2)g_n^3 + g_{n-2}^3 = \begin{cases} x^2(x^2+2)f_{3n} & \text{if } g_k = f_k \\ x^2(x^2+2)(x^2+4)l_{3n} & \text{if } g_k = l_k. \end{cases}$$

Proof. Using Lemma 5.3, we have

$$\begin{aligned} g_{n+2}^3 - g_{n+4}g_{n+2}g_n &= (x^2+2)(g_n^3 - g_{n+2}g_ng_{n-2}) - (g_{n-2}^3 - g_ng_{n-2}g_{n-4}) \\ g_{n+2}^3 - (x^2+2)g_n^3 + g_{n-2}^3 &= g_{n+4}g_{n+2}g_n - (x^2+2)g_{n+2}g_ng_{n-2} + g_ng_{n-2}g_{n-4} \\ &= g_{n+2}g_n[(x^2+1)g_{n+2} + xg_{n+1}] - (x^2+2)g_{n+2}g_ng_{n-2} + g_ng_{n-2}g_{n-4} \\ &= (x^2+2)g_{n+2}g_n(g_{n+2} - g_{n-2}) - g_{n+2}^2g_n + \\ &\quad xg_{n+2}g_{n+1}g_n + g_ng_{n-2}g_{n-4}. \end{aligned} \tag{5.3}$$

Suppose $g_k = l_k$. By Lemma 5.4, we then have

$$\begin{aligned}
\text{LHS} &= (x^2+2)l_{n+2}l_n \cdot x\Delta^2 f_n - l_{n+2}^2 l_n + x l_{n+2}l_{n+1}l_n + l_n l_{n-2}l_{n-4}\\
&= x(x^2+2)\Delta^2 l_{n+2}f_{2n} - l_{n+2}^2 l_n + x l_{n+2}l_{n+1}l_n + l_n l_{n-2}l_{n-4}\\
&= x(x^2+2)\Delta^2 f_{2n}(x l_{n+1} + l_n) - l_{n+2}^2 l_n + x l_{n+2}l_{n+1}l_n + l_n l_{n-2}l_{n-4}\\
&= x^2(x^2+2)\Delta^2 l_{n+1}f_{2n} + x(x^2+2)\Delta^2 f_{2n}l_n -\\
&\quad l_{n+2}^2 l_n + x l_{n+2}l_{n+1}l_n + l_n l_{n-2}l_{n-4}\\
&= x^2(x^2+2)\Delta^2 l_{n+1}f_{2n} + x(x^2+2)\Delta^2 l_n(x f_{2n-1} + f_{2n-2}) -\\
&\quad l_{n+2}^2 l_n + x l_{n+2}l_{n+1}l_n + l_n l_{n-2}l_{n-4}\\
&= x^2(x^2+2)\Delta^2(l_{n+1}f_{2n} + l_n f_{2n-1}) +\\
&\quad l_n[x(x^2+2)\Delta^2 f_{2n-2} - l_{n+2}^2 + x l_{n+2}l_{n+1} + l_{n-2}l_{n-4}]\\
&= x^2(x^2+2)\Delta^2 l_{3n} + l_n \cdot 0\\
&= x^2(x^2+2)\Delta^2 l_{3n},
\end{aligned}$$

as claimed. The other case follows similarly. □

For example

$$\begin{aligned}
f_5^3 - (x^2+2)f_3^3 + f_1^3 &= x^{12} + 9x^{10} + 29x^8 + 40x^6 + 21x^4 + 2x^2\\
&= x^2(x^2+2)f_9;\\
l_5^3 - (x^2+2)l_3^3 + l_1^3 &= x^{15} + 15x^{13} + 89x^{11} + 264x^9 + 405x^7 + 294x^5 + 72x^3\\
&= x^2(x^2+2)(x^2+4)l_9.
\end{aligned}$$

Obviously, Theorem 5.5 also has Fibonacci, Lucas, Pell, and Pell-Lucas implications:

$$\begin{aligned}
F_{n+2}^3 - 3F_n^3 + F_{n-2}^3 &= 3F_{3n}\\
L_{n+2}^3 - 3L_n^3 + L_{n-2}^3 &= 15L_{3n}\\
p_{n+2}^3 - 2(2x^2+1)p_n^3 + p_{n-2}^3 &= 8x^2(2x^2+1)p_{3n}\\
q_{n+2}^3 - 2(2x^2+1)q_n^3 + q_{n-2}^3 &= 32x^2(x^2+1)(2x^2+1)q_{3n}\\
P_{n+2}^3 - 6P_n^3 + P_{n-2}^3 &= 24P_{3n}\\
Q_{n+2}^3 - 6Q_n^3 + Q_{n-2}^3 &= 48Q_{3n}.
\end{aligned}$$

Theorems 5.2 and 5.5 yield the following result.

Corollary 5.6.

$$g_{n+2}^3 - (x^3+2x)g_{n+1}^3 - (x^4+3x^2+2)g_n^3 + (x^3+2x)g_{n-1}^3 + g_{n-2}^3 = 0.$$

Interestingly, we can generalize Theorems 5.2 and 5.5 into a single relationship linking $g_{n+k}^3, g_n^3, g_{n-k}^3$, and g_{3n}, where $g_r = f_r$ or l_r. To this end, we need the next lemma.

Lemma 5.7.

$$(x^2+1)g_n^3 + 3g_{n-1}g_n g_{n+1} = \begin{cases} f_{3n} & \text{if } g_k = f_k\\ (x^2+4)l_{3n} & \text{if } g_k = l_k.\end{cases}$$

Proof. Suppose $g_k = l_k$. Using the identities $l_{a+b} = f_{a+1}l_b + f_a l_{b-1}, f_n l_n = f_{2n}, l_{n+1} + l_{n-1} = \Delta^2 f_n$, and $l_n^2 + l_{n+1}^2 = \Delta^2 f_{2n+1}$, we have

$$\begin{aligned}\Delta^2 l_{3n} &= \Delta^2(f_{2n+1}l_n + f_{2n}l_{n-1})\\ &= l_n(l_n^2 + l_{n+1}^2) + \Delta^2 f_n l_n l_{n-1}\\ &= l_n^3 + l_n(xl_n + l_{n-1})^2 + l_n l_{n-1}(l_{n+1} + l_{n-1})\\ &= (x^2+1)l_n^3 + 2l_{n-1}l_n(xl_n + l_{n-1}) + l_{n-1}l_n l_{n+1}\\ &= (x^2+1)l_n^3 + 3l_{n-1}l_n l_{n+1}.\end{aligned}$$

The other half follows similarly. □

We are now ready for the generalization.

Theorem 5.8.

$$g_{n+k}^3 - (-1)^k l_k g_n^3 + (-1)^k g_{n-k}^3 = \begin{cases} f_k^2 l_k f_{3n} & \text{if } g_r = f_r\\ (x^2+4) f_k^2 l_k l_{3n} & \text{if } g_r = l_r.\end{cases}$$

Proof. Suppose $g_r = l_r$. The corresponding proof requires Lemma 5.7, and the identities $f_{k+1} + f_{k-1} = l_k, f_{k+1}f_{k-1} - f_k^2 = (-1)^k$, and $l_{a-b} = (-1)^b(f_{b-1}l_a - f_b l_{a-1})$. We then have

$$\begin{aligned}l_{n+k}^3 + (-1)^k l_{n-k}^3 &= (f_{k+1}l_n + f_k l_{n-1})^3 + (f_{k-1}l_n - f_k l_{n-1})^3\\ &= l_n^3(f_{k+1}^3 + f_{k-1}^3) + 3f_k l_{n-1} l_n^2(f_{k+1}^2 - f_{k-1}^2) + 3f_k^2 l_{n-1}^2 l_n(f_{k+1} + f_{k-1})\\ &= l_n^3(f_{k+1} + f_{k-1})(f_{k+1}^2 - f_{k+1}f_{k-1} + f_{k-1}^2) + 3xf_k^2 l_k l_{n-1} l_n^2 + 3f_k^2 l_k l_{n-1}^2 l_n\\ &= l_n^3 l_k \left[(f_{k+1} - f_{k+1})^2 + f_{k+1}f_{k-1})\right] + 3f_k^2 l_k l_{n-1} l_n(xl_n + l_{n-1})\\ &= l_k l_n^3 \left\{x^2 f_k^2 + \left[f_k^2 + (-1)^k\right] l_k l_n^3\right\} + 3f_k^2 l_k l_{n-1} l_n l_{n+1}\\ &= f_k^2 l_k \left[(x^2+1)l_n^3 + 3l_{n-1}l_n l_{n+1}\right] + (-1)^k l_k l_n^3\\ &= \Delta^2 l_{3n} f_k^2 l_k + (-1)^k l_k l_n^3.\end{aligned}$$

This yields the desired cubic identity for Lucas polynomials.

The corresponding identity for Fibonacci polynomials follows similarly. □

Clearly, Theorems 5.2 and 5.5 follow from this. So do the following identities:

$$\begin{aligned}p_{n+k}^3 - (-1)^k q_k p_n^3 + (-1)^k p_{n-k}^3 &= p_k^2 q_k p_{3n}\\ q_{n+k}^3 - (-1)^k q_k q_n^3 + (-1)^k q_{n-k}^3 &= 4(x^2+1)p_k^2 q_k q_{3n}.\end{aligned}$$

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