DIFFERENCES OF GIBONACCI PRODUCTS WITH THE SAME ORDER

THOMAS KOSHY

Abstract. We investigate differences of the form $\prod_{i\geq 1} g_{n+r_i}^{a_i} - \prod_{i\geq 1} g_{n+s_i}^{b_i}$, where $g_j = g_j(x)$ denotes the jth gibonacci (Fibonacci, Lucas, Pell, or Pell-Lucas) polynomial; n, r_i, and s_i are integers; $a_i, b_i \geq 0$; $\sum a_i = \sum b_i$ denotes the order m of each product, and $m = 2$ or 3. This investigation yields interesting byproducts.

1. Introduction

Gibonacci polynomials $g_n(x)$ are defined by the recurrence $g_n(x) = xg_{n-1}(x) + g_{n-2}(x)$, where $g_1(x) = a = a(x)$ and $g_2(x) = b = b(x)$ are arbitrary polynomials, and $n \geq 3$. Clearly, $g_0(x) = b - ax$. When $a = 1$ and $b = x$, $g_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $a = x$ and $b = x^2 + 2$, $g_n(x) = l_n(x)$, the nth *Lucas polynomial*. In particular, $g_n(1) = G_n$, the nth gibonacci number; $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number.

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1)$ and $2Q_n = q_n(1)$, respectively.

In the interest of brevity and convenience, we will omit the argument in the functional notation; so g_n will mean $g_n(x)$, although it is technically incorrect. Also we will confine our discussion to Fibonacci, Lucas, Pell, and Pell-Lucas polynomials.

It can be confirmed by induction that $g_n = af_{n-2} + bf_{n-1}$, where $n \geq 0$. Consequently, $g_{-n} = (-1)^{n+1}(af_{n+2} - bf_{n+1})$; so g_n is well-defined for all integers n.

1.1. Binet-like formula.

Gibonacci polynomials g_n can also be defined by the Binet-like formula

$$g_n = \frac{c\alpha^n - d\beta^n}{\alpha - \beta},$$

where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are solutions of the characteristic equation $t^2 - xt - 1 = 0, c = c(x) = a + (ax - b)\beta, d = d(x) = a + (ax - b)\alpha$, and $n \geq 0$. Then $cd = a^2 + abx - b^2$; we will denote this by $\mu = \mu(x)$. When $g_n = f_n, \mu = 1$; and when $g_n = l_n, \mu = -(x^2 + 4)$.

It is well-known that

$$f_{n+k}f_{n-k} - f_n^2 = (-1)^{n-k+1}f_k^2 \tag{1.1}$$

$$l_{n+k}l_{n-k} - l_n^2 = (-1)^{n-k}(x^2 + 4)f_k^2 \tag{1.2}$$

$$f_{m+k}f_{n-k} - f_mf_n = (-1)^{n-k+1}f_kf_{m-n+k} \tag{1.3}$$

$$l_{m+k}l_{n-k} - l_ml_n = (-1)^{n-k}(x^2 + 4)f_kf_{m-n+k}. \tag{1.4}$$

Identity (1.1) generalizes the Catalan identity $F_{n+k}F_{n-k} - F_n^2 = (-1)^{n-k+1}F_k^2$, discovered by Eugene C. Catalan. This, in turn, is a generalization of the Cassini formula $F_{n+1}F_{n-1} - F_n^2 =$

$(-1)^n$, named after Giovanni D. Cassini. Identity (1.2) is the Lucas counterpart of identity (1.1).

Identity (1.3) generalizes the d'Ocagne identity $F_{m+k}F_{n-k} - F_m F_n = (-1)^{n-k+1} F_k F_{m-n+k}$, found by Philbert Maurice d'Ocagne. Identity (1.4) is its Lucas counterpart. d'Ocagne's identity is a slight variation of the identity $F_{n+h}F_{n+k} - F_n F_{n+h+k} = (-1)^n F_h F_k$, discovered by A. Tagiuri in 1901 [1, 2].

As can be predicted, identities (1.1) and (1.2) have a gibonacci version

$$g_{n+k}g_{n-k} - g_n^2 = (-1)^{n-k+1}\mu\, f_k^2; \tag{1.5}$$

so do identities (1.3) and (1.4), and Tagiuri's identity:

$$g_{m+k}g_{n-k} - g_m g_n = (-1)^{n-k+1}\mu\, f_k f_{m-n+k} \tag{1.6}$$

$$g_{n+h}g_{n+k} - g_n g_{n+h+k} = (-1)^n \mu\, f_h f_k. \tag{1.7}$$

These gibonacci identities can be established using the Binet-like formula.

An interesting observation: The left-hand side of each identity in (1.5), (1.6), and (1.7) is the difference of two gibonacci products of order two.

2. Differences of Cubic Gibonacci Products

Recently, R. S. Melham discovered a charming formula for the difference of two Fibonacci products of order three [3]:

$$F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n. \tag{2.1}$$

Two years later, using extensive computer research, S. Fairgrieve and H. W. Gould found an equally beautiful formula [2]:

$$F_n F_{n+4} F_{n+5} - F_{n+3}^3 = (-1)^{n+1} F_{n+6}. \tag{2.2}$$

They also found two additional cubic identities:

$$F_n F_{n+3}^2 - F_{n+2}^3 = (-1)^{n+1} F_{n+1} \tag{2.3}$$

$$F_n^2 F_{n+3} - F_{n+1}^3 = (-1)^{n+1} F_{n+2}. \tag{2.4}$$

The left-hand sides of identities (2.2)–(2.4) are also differences of Fibonacci products of order three.

We will now extend the cubic identities (2.1)–(2.4) to the gibonacci family.

We will begin our pursuit with the gibonacci version of Melham's identity.

Theorem 2.1. *Let $n \geq 0$. Then*

$$g_{n+1}g_{n+2}g_{n+6} - g_{n+3}^3 = (-1)^n \mu (x^3 g_{n+2} - g_{n+1}). \tag{2.5}$$

Proof. By the gibonacci recurrence, we have

$$g_{n+6} = (x^4 + 3x^2 + 1)g_{n+2} + (x^3 + 2x)g_{n+1}$$

$$g_{n+1}g_{n+2}g_{n+6} = (x^4 + 3x^2 + 1)g_{n+2}^2 g_{n+1} + (x^3 + 2x)g_{n+2}g_{n+1}^2$$

$$g_{n+3}^3 = x^3 g_{n+2}^3 + 3x^2 g_{n+2}^2 g_{n+1} + 3x g_{n+2} g_{n+1}^2 + g_{n+1}^3.$$

Then, by identity (1.5) and some basic algebra, we have

$$\begin{aligned}
g_{n+1}g_{n+2}g_{n+6} - g_{n+3}^3 &= (x^4+1)g_{n+2}^2 g_{n+1} + (x^3 - x)g_{n+2}g_{n+1}^2 - x^3 g_{n+2}^3 - g_{n+1}^3 \\
&= x^3 g_{n+2}^2(xg_{n+1} - g_{n+2}) + g_{n+2}g_{n+1}(g_{n+2} - xg_{n+1}) + x^3 g_{n+2}g_{n+1}^2 - g_{n+1}^3 \\
&= -x^3 g_{n+2}^2 g_n + g_{n+2}g_{n+1}g_n + x^3 g_{n+1}^2 g_{n+2} - g_{n+1}^3 \\
&= (g_{n+1}^2 - g_{n+2}g_n)(x^3 g_{n+2} - g_{n+1}) \\
&= (-1)^n \mu (x^3 g_{n+2} - g_{n+1}),
\end{aligned}$$

as desired. □

It follows by Theorem 2.1 that

$$\begin{aligned}
f_{n+1}f_{n+2}f_{n+6} - f_{n+3}^3 &= (-1)^n (x^3 f_{n+2} - f_{n+1}) \qquad (2.6)\\
l_{n+1}l_{n+2}l_{n+6} - l_{n+3}^3 &= (-1)^{n+1}(x^2+4)(x^3 l_{n+2} - l_{n+1}) \\
p_{n+1}p_{n+2}p_{n+6} - p_{n+3}^3 &= (-1)^n (8x^3 p_{n+2} - p_{n+1}) \\
q_{n+1}q_{n+2}q_{n+6} - q_{n+3}^3 &= (-1)^{n+1} 4(x^2+1)(8x^3 q_{n+2} - q_{n+1}).
\end{aligned}$$

Clearly, identity (2.6) yields Melham's identity.

Similarly, we have

$$\begin{aligned}
L_{n+1}L_{n+2}L_{n+6} - L_{n+3}^3 &= (-1)^{n+1} 5L_n \\
P_{n+1}P_{n+2}P_{n+6} - P_{n+3}^3 &= (-1)^n (8P_{n+2} - P_{n+1}) \\
Q_{n+1}Q_{n+2}Q_{n+6} - Q_{n+3}^3 &= (-1)^{n+1} 2(8Q_{n+2} - Q_{n+1}).
\end{aligned}$$

Theorem 2.1 has an additional byproduct. It follows from identity (2.5) that $G_{n+1}G_{n+2}G_{n+6} - G_{n+3}^3 = (-1)^n \mu(1) G_n$, so $(G_{n+1}G_{n+2}G_{n+6} - G_{n+3}^3)^2 = \mu^2(1) G_n^2$. This implies

$$4G_{n+1}G_{n+2}G_{n+3}^3 G_{n+6} + \mu^2(1) G_n^2 = (G_{n+1}G_{n+2}G_{n+6} + G_{n+3}^3)^2.$$

In particular,

$$\begin{aligned}
4F_{n+1}F_{n+2}F_{n+3}^3 F_{n+6} + F_n^2 &= (F_{n+1}F_{n+2}F_{n+6} + F_{n+3}^3)^2 \\
4L_{n+1}L_{n+2}L_{n+3}^3 L_{n+6} + 25L_n^2 &= (L_{n+1}L_{n+2}L_{n+6} + L_{n+3}^3)^2.
\end{aligned}$$

The next theorem gives a companion formula for the difference of three gibonacci products.

Theorem 2.2. *Let $n \geq 0$. Then*

$$g_n g_{n+4} g_{n+5} - g_{n+3}^3 = (-1)^{n+1} \mu (x^3 g_{n+4} + g_{n+5}). \qquad (2.7)$$

Proof. By the gibonacci recurrence, $g_n = (x^2+1)g_{n+4} - (x^3+2x)g_{n+3}$. Then

$$g_n g_{n+4} g_{n+5} = (x^2+1)g_{n+4}^2 g_{n+5} - (x^3+2x)g_{n+3}g_{n+4}g_{n+5}.$$

We also have

$$\begin{aligned}
g_{n+3}^3 &= (g_{n+5} - xg_{n+4})^3 \\
&= g_{n+5}^3 - 3xg_{n+4}g_{n+5}^2 + 3x^2 g_{n+4}^2 g_{n+5} - x^3 g_{n+4}^3 \\
&= (g_{n+5} - xg_{n+4})(g_{n+5} - 2xg_{n+4})g_{n+5} + x^2 g_{n+4}^2 g_{n+5} - x^3 g_{n+4}^3 \\
&= g_{n+3}(g_{n+5} - 2xg_{n+4})g_{n+5} + x^2 g_{n+4}^2 g_{n+5} - x^3 g_{n+4}^3.
\end{aligned}$$

Therefore,

$$\begin{aligned} g_n g_{n+4} g_{n+5} - g_{n+3}^3 &= g_{n+4}^2 g_{n+5} - x^3 g_{n+3} g_{n+4} g_{n+5} - g_{n+3} g_{n+5}^2 + x^3 g_{n+4}^3 \\ &= (g_{n+4}^2 - g_{n+3} g_{n+5})(x^3 g_{n+4} + g_{n+5}) \\ &= (-1)^{n+1} \mu (x^3 g_{n+4} + g_{n+5}), \end{aligned}$$

as claimed. □

It follows by Theorem 2.2 that

$$\begin{aligned} f_n f_{n+4} f_{n+5} - f_{n+3}^3 &= (-1)^{n+1}(x^3 f_{n+4} + f_{n+5}) \\ l_n f_{n+4} l_{n+5} - l_{n+3}^3 &= (-1)^n (x^2+4)(x^3 l_{n+4} + l_{n+5}) \\ p_n p_{n+4} p_{n+5} - p_{n+3}^3 &= (-1)^{n+1}(8x^3 p_{n+4} + p_{n+5}) \\ q_n q_{n+4} q_{n+5} - q_{n+3}^3 &= (-1)^n 4(x^2+1)(8x^3 q_{n+4} + q_{n+5}). \end{aligned}$$

The above identities imply that

$$\begin{aligned} F_n F_{n+4} F_{n+5} - F_{n+3}^3 &= (-1)^{n+1} F_{n+6} \\ L_n L_{n+4} L_{n+5} - L_{n+3}^3 &= (-1)^n 5 L_{n+6} \\ P_n P_{n+4} P_{n+5} - P_{n+3}^3 &= (-1)^{n+1}(8P_{n+4} + P_{n+5}) \\ Q_n Q_{n+4} Q_{n+5} - Q_{n+3}^3 &= (-1)^n 2(8Q_{n+4} + Q_{n+5}). \end{aligned}$$

Theorem 2.2 also has an additional consequence. It follows from identity (2.7) that $G_n G_{n+4} G_{n+5} - G_{n+3}^3 = (-1)^{n+1}\mu(1) G_{n+6}$; so $(G_n G_{n+4} G_{n+5} - G_{n+3}^3)^2 = \mu^2(1) G_{n+6}^2$. Consequently,

$$4G_n G_{n+3}^3 G_{n+4} G_{n+5} + \mu^2(1) G_{n+6}^2 = (G_n G_{n+4} G_{n+5} + G_{n+3}^3)^2.$$

In particular, this implies

$$\begin{aligned} 4F_n F_{n+3}^3 F_{n+4} F_{n+5} + F_{n+6}^2 &= (F_n F_{n+4} F_{n+5} + F_{n+3}^3)^2 \\ 4L_n L_{n+3}^3 L_{n+4} L_{n+5} + 25 L_{n+6}^2 &= (L_n L_{n+4} L_{n+5} + L_{n+3}^3)^2. \end{aligned}$$

The next theorem generalizes identity (2.3).

Theorem 2.3. *Let $n \geq 0$. Then*

$$g_n g_{n+3}^2 - g_{n+2}^3 = (-1)^{n+1}\mu(x^2 g_{n+2} - g_n). \tag{2.8}$$

Proof. By the gibonacci recurrence, we have

$$\begin{aligned} g_n g_{n+3}^2 &= g_n (x g_{n+2} + g_{n+1})^2 \\ &= x^2 g_n g_{n+2}^2 + 2x g_n g_{n+1} g_{n+2} + g_n g_{n+1}^2. \end{aligned}$$

But

$$\begin{aligned} 2x g_n g_{n+1} g_{n+2} &= (g_{n+2} - x g_{n+1})(g_{n+2} - g_n) g_{n+2} + g_n (g_{n+2} - g_n) g_{n+2} \\ &= g_{n+2}^3 - x g_{n+1} g_{n+2} (g_{n+2} - g_n) - g_n^2 g_{n+2} \\ &= g_{n+2}^3 - x^2 g_{n+1}^2 g_{n+2} - g_n^2 g_{n+2}. \end{aligned}$$

Therefore,

$$\begin{aligned} g_n g_{n+3}^2 - g_{n+2}^3 &= x^2 g_n g_{n+2}^2 - x^2 g_{n+1}^2 g_{n+2} - g_n^2 g_{n+2} + g_n g_{n+1}^2 \\ &= (g_n g_{n+2} - g_{n+1}^2)(x^2 g_{n+2} - g_n) \\ &= (-1)^{n+1}\mu(x^2 g_{n+2} - g_n), \end{aligned}$$

as desired. □

As can be predicted, this theorem also has interesting ramifications:

$$\begin{aligned} f_n f_{n+3}^2 - f_{n+2}^3 &= (-1)^{n+1}(x^2 f_{n+2} - f_n) \qquad (2.9)\\ l_n l_{n+3}^2 - l_{n+2}^3 &= (-1)^n (x^2+4)(x^2 l_{n+2} - l_n)\\ p_n p_{n+3}^2 - p_{n+2}^3 &= (-1)^{n+1}(4x^2 p_{n+2} - p_n)\\ q_n q_{n+3}^2 - q_{n+2}^3 &= (-1)^n 4(x^2+1)(4x^2 q_{n+2} - q_n). \end{aligned}$$

The above polynomial identities have additional Fibonacci, Lucas, Pell, and Pell-Lucas consequences. For example, identity (2.3) follows from (2.9).

It also follows from identity from (2.8) that $G_n G_{n+3}^2 - G_{n+2}^3 = (-1)^{n+1}\mu(1)G_{n+1}$. As before, this yields

$$4G_n G_{n+2}^3 G_{n+3}^2 + \mu^2(1)G_{n+1}^2 = (G_n G_{n+3}^2 + G_{n+2}^3)^2.$$

This implies

$$\begin{aligned} 4F_n F_{n+2}^3 F_{n+3}^2 + F_{n+1}^2 &= (F_n F_{n+3}^2 + F_{n+2}^3)^2\\ 4L_n L_{n+2}^3 L_{n+3}^2 + 25L_{n+1}^2 &= (L_n L_{n+3}^2 + L_{n+2}^3)^2. \end{aligned}$$

The following theorem generalizes identity (2.4). Its proof is also short and neat.

Theorem 2.4. *Let $n \geq 0$. Then*

$$g_n^2 g_{n+3} - g_{n+1}^3 = (-1)^{n+1}\mu(g_{n+3} - x^2 g_{n+1}). \qquad (2.10)$$

Proof. By the gibonacci recurrence, we have

$$\begin{aligned} g_n^2 g_{n+3} - g_{n+1}^3 &= (g_{n+2} - x g_{n+1})^2 g_{n+3} - g_{n+1}(g_{n+3} - x g_{n+2})^2\\ &= g_{n+2}^2 g_{n+3} + x^2 g_{n+1}^2 g_{n+3} - g_{n+1} g_{n+3}^2 - x^2 g_{n+1} g_{n+2}^2\\ &= (g_{n+1} g_{n+3} - g_{n+2}^2)(x^2 g_{n+1} - g_{n+3})\\ &= (-1)^{n+1}\mu(g_{n+3} - x^2 g_{n+1}). \end{aligned}$$ □

It follows from identity (2.10) that

$$\begin{aligned} f_n^2 f_{n+3} - f_{n+1}^3 &= (-1)^{n+1}(f_{n+3} - x^2 f_{n+1})\\ l_n^2 l_{n+3} - l_{n+1}^3 &= (-1)^n (x^2+4)(l_{n+3} - x^2 l_{n+1})\\ p_n^2 p_{n+3} - p_{n+1}^3 &= (-1)^{n+1}(p_{n+3} - 4x^2 p_{n+1})\\ q_n^2 q_{n+3} - q_{n+1}^3 &= (-1)^n 4(x^2+1)(q_{n+3} - 4x^2 q_{n+1}). \end{aligned}$$

Theorem 2.4 has another interesting consequence. It also follows from identity (2.10) that $G_n^2 G_{n+3} - G_{n+1}^3 = (-1)^{n+1}\mu(1)G_{n+2}$. Again, as before, this yields

$$4G_n^2 G_{n+1}^3 G_{n+3} + \mu^2(1)G_{n+2}^2 = (G_n^2 G_{n+3} + G_{n+1}^3)^2.$$

Consequently,

$$\begin{aligned} 4F_n^2 F_{n+1}^3 F_{n+3} + F_{n+2}^2 &= (F_n^2 F_{n+3} + F_{n+1}^3)^2\\ 4L_n^2 L_{n+1}^3 L_{n+3} + 25L_{n+2}^2 &= (L_n^2 L_{n+3} + L_{n+1}^3)^2. \end{aligned}$$

3. Acknowledgement

The author would like to thank the referee for his/her thoughtful and constructive comments and suggestions for improving the quality of the exposition of the original version.

References

[1] L. E. Dickson, *History of the Theory of Numbers*, Vol. 1, Chelsea, New York, 1966.

[2] S. Fairgrieve and H. W. Gould, *Product difference Fibonacci identities of Simson, Gelin-Cesàro, Tagiuri, and generalizations*, The Fibonacci Quarterly, **43.2** (2005), 137–141.

[3] R. S. Melham, *A Fibonacci identity in the spirit of Simson and Gelin-Cesàro*, The Fibonacci Quarterly, **41.2** (2003), 142–143.

MSC2010: 11B39

Department of Mathematics, Framingham State University, Framingham, Massachusetts 01701
E-mail address: tkoshy@emeriti.framingham.edu