ON IDENTITIES BY LARCOMBE–FENNESSEY AND CASSINI

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Abstract. A recent identity of Larcombe and Fennessey is derived via a weighted version of Cassini's identity for Fibonacci numbers.

1. The Identities

Let $M = \begin{pmatrix} V & U \\ W & 0 \end{pmatrix}$ and $\alpha_n = \begin{pmatrix} 1 & 0 \end{pmatrix} M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

In [2], the non-linear identity

$$(-1)^n U^n W^n = \alpha_n^2 - UW\alpha_{n-1}^2 - V\alpha_n\alpha_{n-1}$$

($n \geq 1$) was presented. Actually in [2], V and W were replaced by $-V$ and $-W$, respectively, and the quantities U, V, W could depend on a parameter x.

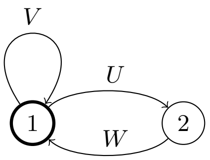
Here, we want to link this identity to the classical Cassini identity

$$F_{n+1}^2 - F_nF_{n+2} = (-1)^n$$

for Fibonacci numbers; we will deduce the Larcombe–Fennessey identity from the Cassini identity.

2. Interpretation as Walks in a Graph

Consider the following graph:



Then α_n may be interpreted as the sum over all walks of length n from state 1 to state 1, where each walk is coded by the letter attached to the directed edge. For example,

$$\alpha_4 = V^4 + UWV^2 + UWUW + VUWV + V^2UW.$$

Since a walk can start either with V or UW, we have the recursion formula

$$\alpha_n = V\alpha_{n-1} + UW\alpha_{n-2}.$$

This works for $n \geq 1$, provided we set $\alpha_{-1} = 0$.

Consequently, we have

$$\alpha_{n+1}\alpha_{n-1} = V\alpha_{n-1}\alpha_n + UW\alpha_{n-1}^2.$$

Therefore, the Larcombe–Fennessey identity follows from the simpler identity

$$\alpha_n^2 - \alpha_{n+1}\alpha_{n-1} = (-1)^n U^n W^n.$$

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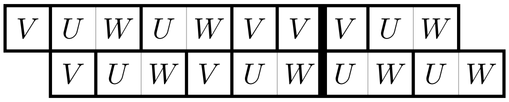
We will deduce this one from Cassini's identity.

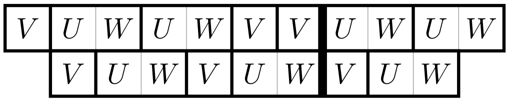
3. Interpretation as Tilings of an $n \times 1$ Rectangle

We want to tile an $n \times 1$ rectangle using 1×1 and 2×1 rectangles. Each such tiling is in obvious correspondence with a walk, where the edge V corresponds to a 1×1 rectangle, and the two consecutive edges UW correspond to a 2×1 rectangle. For example, the walk $VUWUWVVVUW$ can be interpreted as

It is plain to see, compare [1, p. 1], that the number of tilings of an $n \times 1$ rectangle is F_{n+1}, a Fibonacci number. We refer to the graphical proof of Cassini's identity in [1, p. 8] which we repeat here for the readers' convenience. Consider two such tilings, which we arrange in 2 rows, but the second one shifted one unit to the right. Their number is F_{n+1}^2; we call this a type 1 tiling. Here is an example:



The rightmost vertical line that is common to both tilings is especially indicated. Now the part to the right of this line will be flipped: top and bottom are exchanged; the result we will call a type 2 tiling:



The number of type 2 tilings is $F_{n+2}F_n$. Note that this operation is reversible, and this mapping is "almost" a bijection. There is a correction to be made, namely when a common vertical line does not exist. Let $n = 2m$ be even. Then there is a tiling of the first type, namely both rows are $(UW)^m$, which has no correspondence of the second type. On the other hand, if $n = 2m + 1$ is odd, there is a tiling of the second type, namely $(UW)^{m+1}$ in the first row and $(UW)^m$ in the second row, which has no corresponding element of the first type. In [1, p. 8], this is only used for the numbers of tilings, but the operation is *weight preserving.* Putting things together, we have shown that

$$\alpha_n^2 - \alpha_{n-1}\alpha_{n+1} = \begin{cases} (UW)^m (UW)^m & \text{for } n = 2m, \\ -(UW)^{m+1}(UW)^m & \text{for } n = 2m+1, \end{cases}$$

which is the identity that we needed to prove.

References

[1] A. T. Benjamin and J. J. Quinn, *Proofs That Really Count*, The Mathematical Association of America, Washington D. C., 2003.

[2] P. J. Larcombe and E. J. Fennessey, *A non-linear identity for a particular class of polynomial families*, The Fibonacci Quarterly, **52.1** (2014), 75–79.

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