AN INTEGRAL REPRESENTATION FOR THE FIBONACCI NUMBERS AND THEIR GENERALIZATION

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Abstract. We report on an integral representation for the Fibonacci sequence

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^n - \frac{2}{\pi}\int_0^\infty \frac{\sin(x/2)}{x}\frac{\cos(nx)-2\sin(nx)\sin x}{5\sin^2 x+\cos^2 x}\,dx$$

and give two different proofs, with or without invoking complex analysis. These proofs allow us to present some generalizations of this integral representation along two different directions.

1. Introduction

Years ago, when one of us was working on the electron gas in a magnetic field, whose quantum levels are expressible in terms of associated Laguerre functions, a uniform asymptotic expansion of the latter was needed beyond the leading term available [2]. The first author developed a procedure, based on obtaining a Fourier integral representation, for producing this uniform asymptotic expansion. Essentially, if one has a generating series $\mathcal{F}(z)=\sum_{n=0}^\infty A_n z^n$ for the sequence $\{A_n|n\in\mathbb{Z}_{\geq 0}\}$, then

$$A_{\lfloor u\rfloor} = \frac{1}{\pi}\int_0^\infty \frac{\sin(x/2)}{x}\left[e^{i(u-1/2)x}\mathcal{F}(e^{-ix})+e^{-i(u-1/2)x}\mathcal{F}(e^{ix})\right]dx + R,\quad u\in(0,+\infty)\smallsetminus\mathbb{Z}, \tag{1.1}$$

where the "remainder term" R comes in, if $\mathcal{F}(z)$ has singularities in the right-half complex plane. Specializing (1.1) to the generating function of the Fibonacci sequence $\mathcal{F}(z)=\sum_{n=0}^\infty F_n z^n = z/(1-z-z^2)$, one could deduce, after some algebra, the integral representation mentioned in the abstract (reproduced as (1.2) below).

In this brief note, we present two different proofs for the following integral representation of the Fibonacci sequence

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^n - \frac{2}{\pi}\int_0^\infty \frac{\sin(x/2)}{x}\frac{\cos(nx)-2\sin(nx)\sin x}{5\sin^2 x+\cos^2 x}\,dx,\quad n\in\mathbb{Z}_{\geq 0}, \tag{1.2}$$

drawing on the methods developed independently by the authors. In Section 2, we outline a proof of (1.1), thereby placing (1.2) in a complex-analytic context. In Section 3, we use real-analytic methods to establish an equivalent formulation of (1.2):

$$\frac{2}{\pi}\int_0^\infty \frac{\sin(x/2)}{x}\frac{\cos(nx)-2\sin(nx)\sin x}{5\sin^2 x+\cos^2 x}\,dx = \frac{(-1)^n}{\sqrt{5}}\left(\frac{\sqrt{5}-1}{2}\right)^n,\quad n\in\mathbb{Z}_{\geq 0}, \tag{1.2$'$}$$

and extend the result to an evaluation of the integral

$$I(m,n) := \frac{2}{\pi}\int_0^\infty \frac{\sin(x/2)}{x}\frac{\cos(nx)-2\sin(nx)\sin x}{m\sin^2 x+\cos^2 x}\,dx \tag{1.3}$$

for arbitrary $m>0, n\in\mathbb{R}$.

2. A Complex-Analytic Proof

For $n < u < n+1$, the nth Fibonacci number can be written $F_{\lfloor u \rfloor}$. Consider this as a function of u and let us take its Laplace Transform:

$$\int_0^\infty e^{-uz} F_{\lfloor u \rfloor}\, du = \sum_{k=0}^\infty F_k \int_k^{k+1} e^{-uz}\, du = \sum_{k=0}^\infty F_k e^{-kz} \int_0^1 e^{-tz}\, dt = \frac{e^{-z}}{z} \frac{1-e^{-z}}{1-e^{-z}-e^{-2z}}, \quad (2.1)$$

where we have noted the generating function

$$\mathcal{F}(z) = \sum_{n=0}^\infty F_n z^n = \frac{z}{1-z-z^2}. \quad (2.2)$$

One might also note that (2.1) is equal to $\frac{1-e^{-z}}{z}\mathcal{F}(e^{-z})$, a relation that remains valid when the aforementioned $\mathcal{F}$ is replaced by the generating function of other well-behaved sequences [2, Equation 4].

Now take the inverse Laplace transform to obtain

$$F_{\lfloor u \rfloor} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dz}{z} e^{(u-1/2)z} \frac{\sinh(z/2)}{\sinh z - 1/2}, \quad u \in (0,+\infty) \smallsetminus \mathbb{Z} \quad (2.3)$$

where $c > \sinh^{-1}(1/2) = z_0$, the only real-valued singularity of the integrand. All the singularities of the integrand that lie in the right half-plane can be enumerated as $z_k = z_0 + 2k\pi i, k \in \mathbb{Z}$.

By displacing the contour to the imaginary axis $z = iy, y \in \mathbb{R}$, we have

$$F_{\lfloor u \rfloor} = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{\sin(y/2)}{y} \frac{e^{i(u-1/2)y}}{i \sin y - 1/2}\, dy + \sum_{k=-\infty}^\infty I_k, \quad (2.4)$$

$$I_k = \frac{1}{2\pi i} \oint_{C_k} \frac{dz}{z} e^{(u-1/2)z} \frac{\sinh(z/2)}{\sinh z - 1/2}, \quad (2.5)$$

where the contour C_k is a small circle centered at z_k. The infinite sum $\sum_{k=-\infty}^\infty I_k$ in (2.4) is understood as $\lim_{N\to+\infty} \sum_{k=-N}^N I_k$. Such an inversion formula as (2.4) can be generalized into (1.1). However, we point out that it is generally hard to compute the residue contribution, namely, the "remainder term" R in (1.1). For the case of the Fibonacci sequence, the sum over the residues I_k can be evaluated in closed form, as we explain in the next paragraph.

By residue calculus,

$$\sum_{k=-\infty}^\infty I_k = \frac{\varphi^{u-2}}{\sqrt{5}} \left[\frac{1}{\ln \varphi} + 2 \sum_{k=1}^\infty \frac{\cos(2k\pi u) \ln \varphi + 2k\pi \sin(2k\pi u)}{\ln^2 \varphi + 4\pi^2 k^2} \right], \quad (2.6)$$

where $\varphi = (\sqrt{5}+1)/2$ is the Golden Ratio, and $\ln \varphi = z_0 = \sinh^{-1}(1/2)$. To evaluate the infinite sum in (2.6), we require the series (cf. [2, Equation 12] and [3, Equation 5.4.5(2)])

$$\sum_{k=1}^\infty \frac{\cos(2\pi k x)}{a^2 + k^2} = \frac{\pi}{2a} \frac{\cosh[2a\pi(x - \lfloor x \rfloor - \frac{1}{2})]}{\sinh(a\pi)} - \frac{1}{2a^2}, \text{ for all } x \in \mathbb{R}, ia \in \mathbb{C} \smallsetminus \mathbb{Z}, \quad (2.7)$$

and its derivative with respect to $x \in \mathbb{R} \smallsetminus \mathbb{Z}$. After some algebra, one can deduce

$$\sum_{k=-\infty}^\infty I_k = \left(\frac{\sqrt{5}+1}{2} \right)^{\lfloor u \rfloor} \frac{1}{\sqrt{5}}, \quad u \in (0,+\infty) \smallsetminus \mathbb{Z}. \quad (2.8)$$

The remaining integral in (2.4) is equal to

$$\frac{1}{\pi}\int_0^\infty \frac{dx}{x}\sin(x/2)\mathrm{Re}\left[\frac{e^{i(u-1/2)x}}{i\sin x-1/2}\right]. \tag{2.9}$$

Consequently, with $u=n+1/2$ for $n\in\mathbb{Z}_{\geq 0}$, one finds

$$\frac{2}{\pi}\int_0^\infty \sin(x/2)\frac{2\sin(nx)\sin x-\cos(nx)}{5\sin^2 x+\cos^2 x}\frac{dx}{x}=F_n-\frac{\varphi^n}{\sqrt{5}}. \tag{2.10}$$

By Wells' formula (see [1] and [4, p. 62]),

$$F_n=\left\lfloor\frac{\varphi^n}{\sqrt{5}}\right\rfloor \tag{2.11}$$

holds for non-negative even integers. So, for n even the integral in (2.10) is precisely the negative of the fractional part of $\varphi^n/\sqrt{5}$.

3. A Real-Analytic Proof

In this section, we base the integral formula in (1.2′) on the following theorem.

Theorem 3.1. *When* $n\in(2k-\frac{1}{2},2k+\frac{1}{2})\cap[0,+\infty)$ *for a given integer* $k\in\mathbb{Z}_{\geq 0}$*, we have*

$$I(m,n):=\frac{2}{\pi}\int_0^\infty \frac{\sin(x/2)}{x}\frac{\cos(nx)-2\sin(nx)\sin x}{m\sin^2 x+\cos^2 x}\,dx=\left(\frac{\sqrt{m}-1}{\sqrt{m}+1}\right)^k\frac{1}{\sqrt{m}},\quad \text{for all } m>0; \tag{3.1}$$

when $n\in(2k+1-\frac{1}{2},2k+1+\frac{1}{2})$ *for a given integer* $k\in\mathbb{Z}_{\geq 0}$*, we have*

$$I(m,n)=-\left(\frac{\sqrt{m}-1}{\sqrt{m}+1}\right)^k\frac{2}{\sqrt{m}(1+\sqrt{m})},\quad \text{for all } m>0; \tag{3.2}$$

when $n-\frac{1}{2}\in\mathbb{Z}_{\geq 0}$*, we can compute* $I(m,n)=\frac{I(m,n+0^+)+I(m,n-0^+)}{2}$.

Proof. The entire proof hinges on the following Poisson kernel expansion

$$\frac{\sqrt{m}}{m\sin^2 x+\cos^2 x}=1+2\sum_{k=1}^\infty\left(\frac{\sqrt{m}-1}{\sqrt{m}+1}\right)^k\cos(2kx),\quad \text{for all } m,x>0. \tag{3.3}$$

By elementary trigonometry, we have

$$\begin{aligned}&\frac{2\sin(x/2)}{x}[\cos(nx)-2\sin(nx)\sin x]\cos(2kx)\\&=\frac{1}{2x}\left[2\sin\left(2kx-nx+\frac{x}{2}\right)+2\sin\left(-2kx-nx+\frac{x}{2}\right)-\sin\left(-2kx-nx+\frac{3x}{2}\right)\right.\\&\left.+\sin\left(2kx+nx+\frac{3x}{2}\right)+\sin\left(-2kx+nx+\frac{3x}{2}\right)-\sin\left(2kx-nx+\frac{3x}{2}\right)\right].\end{aligned} \tag{3.4}$$

Bearing in mind that the Dirichlet integral evaluates to

$$\frac{2}{\pi}\int_0^\infty\frac{\sin\alpha x}{x}\,dx=\operatorname{sgn}\alpha\equiv\begin{cases}1, & \alpha>0\\0, & \alpha=0\\-1, & \alpha<0\end{cases} \tag{3.5}$$

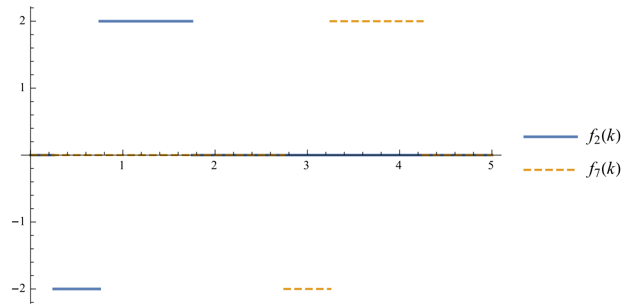

FIGURE 1. Two *typical* plots of $f_n(k)$ as a function of $k \in [0, +\infty)$. We note that for varying values of $n \in (3/2, +\infty)$, the shapes of the k-$f_n(k)$ plots are just horizontal translates of each other.

we can compute

$$\int_0^\infty \frac{2\sin(x/2)}{\pi x}[\cos(nx) - 2\sin(nx)\sin x]\cos(2kx)\,dx = \frac{f_n(k)}{4}, \tag{3.6}$$

where the function

$$\begin{aligned} f_n(k) = 2\operatorname{sgn}\left(2k - n + \frac{1}{2}\right) + 2\operatorname{sgn}\left(-2k - n + \frac{1}{2}\right) - \operatorname{sgn}\left(-2k - n + \frac{3}{2}\right) \\ + \operatorname{sgn}\left(2k + n + \frac{3}{2}\right) + \operatorname{sgn}\left(-2k + n + \frac{3}{2}\right) - \operatorname{sgn}\left(2k - n + \frac{3}{2}\right), \\ k \in [0, +\infty) \end{aligned} \tag{3.7}$$

is supported on a bounded interval $k \in [\frac{n}{2} - \frac{3}{4}, \frac{n}{2} + \frac{3}{4}] \cap [0, +\infty)$ (see Figures 1 and 2).

Judging from Figure 1, it is clear that whenever $n - \frac{1}{2} \in (1, +\infty) \smallsetminus \mathbb{Z}$, there are at most two terms in the series expansion for the Poisson kernel (see (3.3)) that can have a net contribution to the integral $I(m, n)$. Specifically, when $n \in (2k - \frac{1}{2}, 2k + \frac{1}{2}) \cap (3/2, +\infty)$ for a given integer $k \in \mathbb{Z}_{>0}$, only the term $\cos(2kx)$ matters, which leads to

$$I(m, n) = \frac{2f_n(k)}{4}\left(\frac{\sqrt{m} - 1}{\sqrt{m} + 1}\right)^k \frac{1}{\sqrt{m}} = \left(\frac{\sqrt{m} - 1}{\sqrt{m} + 1}\right)^k \frac{1}{\sqrt{m}}; \tag{3.8}$$

when $n \in (2k + 1 - \frac{1}{2}, 2k + 1 + \frac{1}{2}) \cap (3/2, +\infty)$ for a given integer $k \in \mathbb{Z}_{>0}$, the terms $\cos(2kx)$ and $\cos[2(k+1)x]$ both come into play, which results in

$$\begin{aligned} I(m, n) &= \left[\frac{2f_n(k+1)}{4}\left(\frac{\sqrt{m} - 1}{\sqrt{m} + 1}\right)^{k+1} + \frac{2f_n(k)}{4}\left(\frac{\sqrt{m} - 1}{\sqrt{m} + 1}\right)^k\right]\frac{1}{\sqrt{m}} \\ &= \left[\left(\frac{\sqrt{m} - 1}{\sqrt{m} + 1}\right)^{k+1} - \left(\frac{\sqrt{m} - 1}{\sqrt{m} + 1}\right)^k\right]\frac{1}{\sqrt{m}} = -\left(\frac{\sqrt{m} - 1}{\sqrt{m} + 1}\right)^k \frac{2}{\sqrt{m}(1 + \sqrt{m})}. \end{aligned} \tag{3.9}$$

So far, we have confirmed (3.10) and (3.11) under the additional constraint that $n > 3/2$.

When $0 \le n < 3/2$, we will need to cope with the $k = 0$ term (*i.e.* the leading constant "1") in the Poisson kernel expansion. The leading constant "1" is exactly half of the value "2" that precedes each $\cos(2kx), k \in \mathbb{Z}_{>0}$ term in the Fourier series expansion; in the meantime, the actual value of $f_n(0), 0 \le n < 3/2$ also doubles what would come from a direct extrapolation of the $f_n(0), n > 3/2$ scenario (see Figure 2). These two rescaling effects cancel each other, so the validity of (3.10) and (3.11) remains unshaken for $0 \le n < 3/2$.

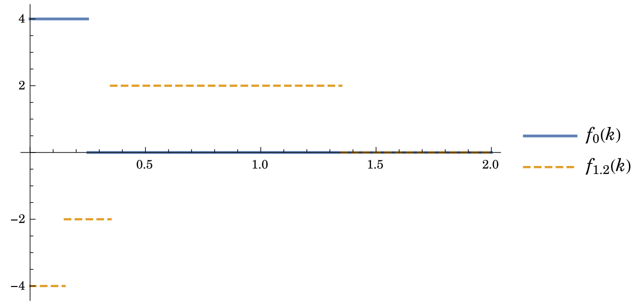

FIGURE 2. Two *atypical* plots of $f_n(k)$ as a function of $k \in [0, +\infty)$. We note that for varying values of $n \in [0, 3/2)$, the actual value of $f_n(0)$ doubles what is anticipated from a naïve horizontal translation of the plot in Figure 1.

Finally, the identity $f_n(k) = \lim_{\varepsilon \to 0^+} \frac{f_{n+\varepsilon}(k)+f_{n-\varepsilon}(k)}{2}$ brings us $I(m,n) = \frac{I(m,n+0^+)+I(m,n-0^+)}{2}$, as claimed. □

We note that a similar discussion can be carried out for $n < 0$. We record the results in the theorem below, and leave the proof to interested readers.

Theorem 3.2. *When $-n \in (2k - \frac{1}{2}, 2k + \frac{1}{2}) \cap (0, +\infty)$ for a given integer $k \in \mathbb{Z}_{\geq 0}$, we have*

$$I(m,n) := \frac{2}{\pi} \int_0^\infty \frac{\sin(x/2)}{x} \frac{\cos(nx) - 2\sin(nx)\sin x}{m \sin^2 x + \cos^2 x} \, dx = \left(\frac{\sqrt{m}-1}{\sqrt{m}+1} \right)^k \frac{1}{\sqrt{m}}, \quad \text{for all } m > 0; \tag{3.10}$$

when $-n \in (2k + 1 - \frac{1}{2}, 2k + 1 + \frac{1}{2})$ for a given integer $k \in \mathbb{Z}_{\geq 0}$, we have

$$I(m,n) = + \left(\frac{\sqrt{m}-1}{\sqrt{m}+1} \right)^k \frac{2}{\sqrt{m}(1+\sqrt{m})}, \quad \text{for all } m > 0; \tag{3.11}$$

when $n - \frac{1}{2} \in \mathbb{Z}_{<0}$, we can compute $I(m,n) = \frac{I(m,n+0^+)+I(m,n-0^+)}{2}$. □

Specializing to the case $m = 5$, and combining the results for $I(5,n)$ and $I(5,-n)$, we obtain the following integral representations for the even and odd terms in the Fibonacci sequence:

$$F_{2n} = \frac{1}{\sqrt{5}} \left(\frac{\sqrt{5}+1}{2} \right)^{2n} - \frac{2}{\pi} \int_0^\infty \frac{\sin(x/2)}{x} \frac{\cos(2nx)}{5\sin^2 x + \cos^2 x} \, dx, \tag{3.12}$$

$$F_{2n+1} = \frac{1}{\sqrt{5}} \left(\frac{\sqrt{5}+1}{2} \right)^{2n+1} + \frac{4}{\pi} \int_0^\infty \frac{\sin(x/2)}{x} \frac{\sin[(2n+1)x]\sin x}{5\sin^2 x + \cos^2 x} \, dx, \tag{3.13}$$

where $n \in \mathbb{Z}_{\geq 0}$.

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MSC2010: 11B33, 44A20,

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