GIBONOMIAL COEFFICIENTS WITH INTERESTING BYPRODUCTS

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Abstract. We investigate a new class of polynomial functions, called *gibonomial coefficients*, and extract some of their properties. We then deduce the corresponding properties for Fibonacci, Lucas, Pell, and Pell-Lucas polynomials and numbers.

1. INTRODUCTION

Gibonacci (*g*eneralized F*ibonacci*) *polynomials* $g_n(x)$ satisfy the recurrence $g_n(x) = xg_{n-1}(x) + g_{n-2}(x)$, where $a = a(x) = g_1(x)$ and $b = b(x) = g_2(x)$ are arbitrary polynomials, and $n \geq 3$. Obviously, the definition can be extended to negative subscripts. When $g_1(x) = 1$ and $g_2(x) = x$, $g_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $g_1(x) = x$ and $g_2(x) = x^2 + 2$, $g_n(x) = l_n(x)$, the nth *Lucas polynomial*. In particular, $g_n(1) = G_n$, the nth *gibonacci number*; $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number.

The Binet-like formula

$$g_n(x) = \frac{c\alpha^n - d\beta^n}{\alpha - \beta}$$

can be employed to extract a number of properties of gibonacci polynomials, where $\alpha = \alpha(x) = \dfrac{x + \Delta}{2}$, $\beta = \beta(x) = \dfrac{x - \Delta}{2}$, and $\Delta = \Delta(x) = \sqrt{x^2 + 4}$, $c = c(x) = a + (a - b)\beta$, and $d = d(x) = a + (a - b)\alpha$. For instance, we can establish the gibonacci *addition formula* $g_{m+k} = g_{m+1}f_k + g_m f_{k-1}$, where $m, k \in \mathbb{N}$.

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1)$ and $2Q_n = q_n(1)$, respectively.

Vieta polynomials $V_n(x)$ and *Vieta-Lucas polynomials* $v_n(x)$ are also related to $f_n(x)$ and $l_n(x)$, respectively: $V_n(ix) = i^{n-1}f_n(x)$ and $v_n(x) = i^n l_n(x)$, where $i = \sqrt{-1}$. Likewise, the *Jacobsthal polynomial* $J_n(x)$ is related to $f_n(x)$ and the *Chebyshev polynomial of the second kind* $U_n(x)$ to $V_{n+1}(2x)$: $J_{n+1}(x) = x^{n/2}f_{n+1}(1/\sqrt{x})$ and $U_n(x) = V_{n+1}(2x)$ [9, 14].

In the interest of brevity and convenience, we will omit the argument in the functional notation; so $g_n = g_n(x)$.

2. FIBONOMIAL COEFFICIENTS

Generalized binomial coefficients were originally studied by G. Fontené in 1915, and then independently by M. Ward in 1936 [5, 13], where the upper and lower numbers are arbitrary. In 1949, D. Jarden investigated the special case when the upper and lower numbers are Fibonacci numbers [13].

Fibonomial coefficients (the equivalent of binomial coefficients for Fibonacci numbers) are defined by

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{F_n^*}{F_r^* F_{n-r}^*},$$

where $F_k^* = F_k F_{k-1} \dots F_2 F_1, F_0^* = 1$, and $0 \le r \le n$ [6, 10, 12, 15]. The bracketed bi-level notation for Fibonomial coefficients was introduced by Torretto and Fuchs in 1964 [15]. In 1970, D. Lind established that every Fibonomial coefficient is an integer [12]. Since $\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n \\ n-r \end{bmatrix}$, it follows that $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1 = \begin{bmatrix} n \\ n \end{bmatrix}$ and $\begin{bmatrix} n \\ 1 \end{bmatrix} = F_n = \begin{bmatrix} n \\ n-1 \end{bmatrix}$.

2.1. **Brennan's Equation.** In 1964, T. A. Brennan established that

$$\sum_{r=0}^{n+1} (-1)^{r(r+1)/2} \begin{bmatrix} n+1 \\ r \end{bmatrix} x^{n-r+1} = 0$$

is the characteristic equation of the product of n Fibonacci recurrences $y_{n+2} = y_{n+1} + y_n$ [2]. When $n = 2$, it yields $x^3 - 2x^2 - 2x + 1 = 0$. Correspondingly, $G_{n+3}^2 = 2G_{n+2}^2 + 2G_{n+1}^2 - G_n^2$. Likewise, $x^4 - 3x^3 - 6x^2 + 3x + 1 = 0$. This implies $G_{n+4}^3 = 3G_{n+3}^3 + 6G_{n+2}^3 - 3G_{n+1}^3 - G_n^3$. In particular, $F_{n+4}^3 = 3F_{n+3}^3 + 6F_{n+2}^3 - 3F_{n+1}^3 - F_n^3$; D. Zeitlin discovered this identity in 1963 [16].

More generally,

$$g_{n+4}^3 = (x^3 + 2x) g_{n+3}^3 + (x^4 + 3x^2 + 2) g_{n+2}^3 - (x^3 + 2x) g_{n+1}^3 - g_n^3. \tag{2.1}$$

Its proof involves some messy algebra; so we omit it. But we will revisit it shortly.

It follows from recurrence (2.1) that

$$\begin{aligned} f_{n+4}^3 &= (x^3 + 2x) f_{n+3}^3 + (x^4 + 3x^2 + 2) f_{n+2}^3 - (x^3 + 2x) f_{n+1}^3 - f_n^3 \\ l_{n+4}^3 &= (x^3 + 2x) l_{n+3}^3 + (x^4 + 3x^2 + 2) l_{n+2}^3 - (x^3 + 2x) l_{n+1}^3 - l_n^3 \\ p_{n+4}^3 &= 4(2x^3 + x) p_{n+3}^3 + 2(8x^4 + 6x^2 + 1) p_{n+2}^3 - 4(2x^3 + x) p_{n+1}^3 - p_n^3 \\ q_{n+4}^3 &= 4(2x^3 + x) q_{n+3}^3 + 2(8x^4 + 6x^2 + 1) q_{n+2}^3 - 4(2x^3 + x) q_{n+1}^3 - q_n^3 \\ P_{n+4}^3 &= 12 P_{n+3}^3 + 30 P_{n+2}^3 - 12 P_{n+1}^3 - P_n^3 \\ Q_{n+4}^3 &= 12 Q_{n+3}^3 + 30 Q_{n+2}^3 - 12 Q_{n+1}^3 - Q_n^3. \end{aligned}$$

3. Gibonomial Coefficients

The nth *gibonomial coefficient* $\left[\begin{bmatrix} n \\ r \end{bmatrix}\right]$ is defined by

$$\left[\begin{bmatrix} n \\ r \end{bmatrix}\right] = \frac{f_n^*}{f_r^* f_{n-r}^*}, \tag{3.1}$$

where $f_k^* = f_k f_{k-1} \dots f_2 f_1, f_0^* = 1$, and $0 \le r \le n$. Clearly, $\left[\begin{bmatrix} n \\ r \end{bmatrix}\right] = \left[\begin{bmatrix} n \\ n-r \end{bmatrix}\right]$, $\left[\begin{bmatrix} n \\ 0 \end{bmatrix}\right] = \left[\begin{bmatrix} n \\ n \end{bmatrix}\right]$ and $\left[\begin{bmatrix} n \\ 1 \end{bmatrix}\right] = \left[\begin{bmatrix} n \\ n-1 \end{bmatrix}\right]$. Also when $x = 1$, $\left[\begin{bmatrix} n \\ r \end{bmatrix}\right] = \begin{bmatrix} n \\ r \end{bmatrix}$.

3.1. **Gibonomial Recurrences.** Gibonomial coefficients satisfy two Pascal-like recurrences:

$$\left[\begin{bmatrix} n \\ r \end{bmatrix}\right] = \left[\begin{bmatrix} n-1 \\ r \end{bmatrix}\right] f_{r+1} + \left[\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}\right] f_{n-r-1} \tag{3.2}$$

$$= \left[\begin{bmatrix} n-1 \\ r \end{bmatrix}\right] f_{r-1} + \left[\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}\right] f_{n-r+1}. \tag{3.3}$$

These recurrences can be established using the addition formula and definition (3.1).

For example,

$$\begin{aligned}
\left[\!\!\left[\begin{matrix}n-1\\ r\end{matrix}\right]\!\!\right] f_{r+1} + \left[\!\!\left[\begin{matrix}n-1\\ r-1\end{matrix}\right]\!\!\right] f_{n-r-1} &= \frac{f_{n-1}^*}{f_r^* f_{n-r-1}^*} f_{r+1} + \frac{f_{n-1}^*}{f_{r-1}^* f_{n-r}^*} f_{n-r-1}\\
&= \frac{f_{n-1}^*}{f_r^* f_{n-r}^*}(f_{r+1}f_{n-r} + f_{n-r-1}f_r)\\
&= \frac{f_{n-1}^* f_n}{f_r^* f_{n-r}^*}\\
&= \left[\!\!\left[\begin{matrix}n\\ r\end{matrix}\right]\!\!\right].
\end{aligned}$$

It follows from recurrence (3.3) that $\left[\!\!\left[\begin{matrix}n\\ 1\end{matrix}\right]\!\!\right] = f_n = \left[\!\!\left[\begin{matrix}n\\ n-1\end{matrix}\right]\!\!\right]$.

Recurrence (3.2) or (3.3), coupled with the initial conditions $\left[\!\!\left[\begin{matrix}0\\ 0\end{matrix}\right]\!\!\right] = 1 = \left[\!\!\left[\begin{matrix}1\\ 0\end{matrix}\right]\!\!\right]$, implies that every gibonomial coefficient is an integer-valued polynomial.

The recurrences can be used to construct the *gibonomial triangle* in Figure 1.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & x & & 1 & & \\
 & 1 & & x^2+1 & & x^2+1 & & 1 & \\
1 & & x^3+2x & & x^4+3x^2+2 & & x^3+2x & & 1
\end{array}$$

Figure 1. Gibonomial Triangle

Since $f_{k+1} + f_{k-1} = l_k$, it follows by recurrences (3.2) and (3.3) that

$$2\left[\!\!\left[\begin{matrix}n\\ r\end{matrix}\right]\!\!\right] = \left[\!\!\left[\begin{matrix}n-1\\ r\end{matrix}\right]\!\!\right] l_r + \left[\!\!\left[\begin{matrix}n-1\\ r-1\end{matrix}\right]\!\!\right] l_{n-r}. \tag{3.4}$$

Consequently, $2f_n = f_{n-r}l_r + f_r l_{n-r}$ and hence, $2V_n = V_{n-r}v_r + V_r v_{n-r}$.

It also follows from equation (3.4) that

$$2\begin{bmatrix}n\\ r\end{bmatrix} = \begin{bmatrix}n-1\\ r\end{bmatrix} L_r + \begin{bmatrix}n-1\\ r-1\end{bmatrix} L_{n-r}.$$

Brennan discovered this formula in 1963 [1].

3.2. **Central Gibonomial Coefficients.**

The *central gibonomial coefficients* $\left[\!\!\left[\begin{matrix}2n\\ n\end{matrix}\right]\!\!\right]$ satisfy the following property:

$$\begin{aligned}
\left[\!\!\left[\begin{matrix}2n\\ n\end{matrix}\right]\!\!\right] &= \left[\!\!\left[\begin{matrix}2n-1\\ n\end{matrix}\right]\!\!\right] f_{n+1} + \left[\!\!\left[\begin{matrix}2n-1\\ n-1\end{matrix}\right]\!\!\right] f_{n-1}\\
&= \left[\!\!\left[\begin{matrix}2n-1\\ n\end{matrix}\right]\!\!\right] (f_{n+1} + f_{n-1})\\
&= \left[\!\!\left[\begin{matrix}2n-1\\ n\end{matrix}\right]\!\!\right] l_n. \qquad (3.5)
\end{aligned}$$

It follows from identity (3.5) that $f_{2n} = f_n l_n$, and hence, $V_{2n} = V_n v_n$.

3.3. **Star of David Property.** Gibonomial coefficients satisfy the *Star of David property*

$$\left[\left[\begin{matrix}n-1\\r-1\end{matrix}\right]\right]\left[\left[\begin{matrix}n\\r+1\end{matrix}\right]\right]\left[\left[\begin{matrix}n+1\\r\end{matrix}\right]\right]=\left[\left[\begin{matrix}n-1\\r\end{matrix}\right]\right]\left[\left[\begin{matrix}n+1\\r+1\end{matrix}\right]\right]\left[\left[\begin{matrix}n\\r-1\end{matrix}\right]\right];$$

see Figure 2.

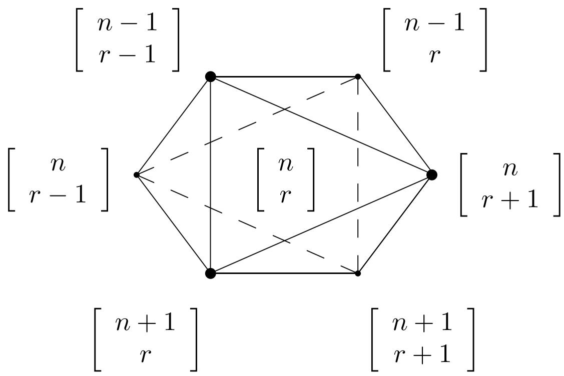

Figure 2.

This property also can be established algebraically:

$$\begin{aligned}\text{LHS}&=\frac{f^*_{n-1}}{f^*_{r-1}f^*_{n-r}}\cdot\frac{f^*_n}{f^*_{r+1}f^*_{n-r-1}}\cdot\frac{f^*_{n+1}}{f^*_rf^*_{n-r+1}}\\&=\frac{f^*_{n-1}}{f^*_rf^*_{n-r-1}}\cdot\frac{f^*_{n+1}}{f^*_{r+1}f^*_{n-r}}\cdot\frac{f^*_n}{f^*_{r-1}f^*_{n-r+1}}\\&=\left[\left[\begin{matrix}n-1\\r\end{matrix}\right]\right]\left[\left[\begin{matrix}n+1\\r+1\end{matrix}\right]\right]\left[\left[\begin{matrix}n\\r-1\end{matrix}\right]\right]\\&=\text{RHS}.\end{aligned}$$

Hoggatt and Hansel discovered the binomial version of the Star of David property in 1971 [8, 10].

3.4. **Applications of Gibonomial Coefficients.** Following the spirit of Brennan's equation, the characteristic equation of the product of n polynomial recurrences $y_{n+2}=xy_{n+1}+y_n$ is given by

$$\sum_{r=0}^{n+1}(-1)^{r(r+1)/2}\left[\left[\begin{matrix}n+1\\r\end{matrix}\right]\right]z^{n-r+1}=0.$$

When $n=2,3,$ and 4, this yields

$$\begin{aligned}z^3-(x^2+1)z^2-(x^2+1)z+1&=0\\z^4-(x^3+2x)z^3-(x^4+3x^2+2)z^2+(x^3+2x)z+1&=0\\z^5-(x^4+3x^2+1)z^4-(x^6+5x^4+7x^2+2)z^3+&\\(x^6+5x^4+7x^2+2)z^2+(x^4+3x^2+1)z-1&=0,\end{aligned}$$

respectively. These equations imply that

$$g_{n+3}^2 = (x^2+1)g_{n+2}^2 + (x^2+1)g_{n+1}^2 - g_n^2 \tag{3.6}$$

$$g_{n+4}^3 = (x^3+2x)g_{n+3}^3 + (x^4+3x^2+2)g_{n+2}^3 - (x^3+2x)g_{n+1}^3 - g_n^3 \tag{3.7}$$

$$\begin{aligned} g_{n+5}^4 = {} & (x^4+3x^2+1)g_{n+4}^4 + (x^6+5x^4+7x^2+2)g_{n+3}^4 - \\ & (x^6+5x^4+7x^2+2)g_{n+2}^4 - (x^4+3x^2+1)g_{n+1}^4 + g_n^4. \end{aligned} \tag{3.8}$$

It follows from equation (3.8) that

$$\begin{aligned} f_{n+5}^4 &= (x^4+3x^2+1)f_{n+4}^4 + (x^6+5x^4+7x^2+2)f_{n+3}^4 - \\ &\quad (x^6+5x^4+7x^2+2)f_{n+2}^4 - (x^4+3x^2+1)f_{n+1}^4 + f_n^4 \\ l_{n+5}^4 &= (x^4+3x^2+1)l_{n+4}^4 + (x^6+5x^4+7x^2+2)l_{n+3}^4 - \\ &\quad (x^6+5x^4+7x^2+2)l_{n+2}^4 - (x^4+3x^2+1)l_{n+1}^4 + l_n^4 \\ p_{n+5}^4 &= (16x^4+12x^2+1)p_{n+4}^4 + (64x^6+80x^4+28x^2+2)p_{n+3}^4 - \\ &\quad (64x^6+80x^4+28x^2+2)p_{n+2}^4 - (16x^4+12x^2+1)p_{n+1}^4 + p_n^4 \\ q_{n+5}^4 &= (16x^4+12x^2+1)q_{n+4}^4 + (64x^6+80x^4+28x^2+2)q_{n+3}^4 - \\ &\quad (64x^6+80x^4+28x^2+2)q_{n+2}^4 - (16x^4+12x^2+1)q_{n+1}^4 + q_n^4. \end{aligned}$$

Consequently,

$$\begin{aligned} F_{n+5}^4 &= 5F_{n+4}^4 + 15F_{n+3}^4 - 15F_{n+2}^4 - 5F_{n+1}^4 + F_n^4 \\ L_{n+5}^4 &= 5L_{n+4}^4 + 15L_{n+3}^4 - 15L_{n+2}^4 - 5L_{n+1}^4 + L_n^4 \\ P_{n+5}^4 &= 29P_{n+4}^4 + 174P_{n+3}^4 - 174P_{n+2}^4 - 29P_{n+1}^4 + P_n^4 \\ Q_{n+5}^4 &= 29Q_{n+4}^4 + 174Q_{n+3}^4 - 174Q_{n+2}^4 - 29Q_{n+1}^4 + Q_n^4. \end{aligned}$$

Similar results follow from equations (3.6) and (3.7).

3.5. **Generating Function for Gibonomial Coefficients.** Using the *Gaussian binomial coefficients*

$$\left\{ \begin{matrix} m \\ r \end{matrix} \right\} = \frac{(1-q^m)(1-q^{m-1})\ldots(1-q^{m-r+1})}{(1-q)(1-q^2)\ldots(1-q^r)},$$

we have

$$\prod_{r=0}^{m-1}(1-q^r z) = \sum_{r=0}^{m}(-1)^r \left\{ \begin{matrix} m \\ r \end{matrix} \right\} q^{r(r-1)/2} z^r \tag{3.9}$$

$$\prod_{r=0}^{m-1} \frac{1}{1-q^r z} = \sum_{r=0}^{\infty} \left\{ \begin{matrix} m+r-1 \\ r \end{matrix} \right\} z^r, \tag{3.10}$$

where q is a dummy variable [3, 4, 10]. Letting $q = \beta/\alpha$,

$$\begin{aligned}\left\{ m \atop r \right\} &= \frac{(\alpha^m - \beta^m)(\alpha^{m-1} - \beta^{m-1}) \dots (\alpha^{m-r+1} - \beta^{m-r+1})}{(\alpha - \beta)(\alpha^2 - \beta^2) \dots (\alpha^r - \beta^r)} \cdot \alpha^{-r(m-r)} \\ &= \frac{f_m f_{m-1} \dots f_{m-r+1} \cdot \Delta^r}{f_1 f_2 \dots f_r \cdot \Delta^r} \cdot \alpha^{-r(m-r)} \\ &= \frac{f_m^*}{f_r^* f_{m-r}^*} \alpha^{-r(m-r)} \\ &= \left[\left[m \atop r \right]\right] \alpha^{-r(m-r)}. \end{aligned} \tag{3.11}$$

Likewise,

$$\left\{ m+r-1 \atop r \right\} = \left[\left[m+r-1 \atop r \right]\right] \alpha^{-r(m-1)}. \tag{3.12}$$

Since

$$\begin{aligned}(-1)^r \left(\frac{\beta}{\alpha}\right)^{r(r-1)/2} &= (-1)^r (-\alpha^{-2})^{r(r-1)/2} \\ &= (-1)^{r(r+1)/2} \alpha^{-r(r-1)},\end{aligned}$$

replacing z with $\alpha^{m-1}z$ and letting $q = \beta/\alpha$, identities (3.9) and (3.11) then yield

$$\begin{aligned}\prod_{r=0}^{m-1} (1 - \beta^r \alpha^{m-r-1} z) &= \sum_{r=0}^{m} (-1)^{r(r+1)/2} \alpha^{-r(r-1)} \left[\left[m \atop r \right]\right] \alpha^{-r(m-r)} \cdot (\alpha^{m-1} z)^r \\ &= \sum_{r=0}^{m} (-1)^{r(r+1)/2} \left[\left[m \atop r \right]\right] z^r.\end{aligned}$$

Identities (3.10) and (3.12) then imply that

$$\begin{aligned}\frac{1}{\sum\limits_{r=0}^{m} (-1)^{r(r+1)/2} \left[\left[m \atop r \right]\right] z^r} &= \sum_{r=0}^{\infty} \left[\left[m+r-1 \atop r \right]\right] \alpha^{-r(m-1)} \cdot \left[\alpha^{m-1} z\right]^r \\ &= \sum_{r=0}^{\infty} \left[\left[m+r-1 \atop r \right]\right] z^r \\ &= \sum_{r=0}^{\infty} \left[\left[m+r-1 \atop m-1 \right]\right] z^r \\ \frac{z^{m-1}}{\sum\limits_{r=0}^{m} (-1)^{r(r+1)/2} \left[\left[m \atop r \right]\right] z^r} &= \sum_{n=0}^{\infty} \left[\left[n \atop m-1 \right]\right] z^n, \end{aligned} \tag{3.13}$$

where $m \geq 1$. This is the desired generating function.

When $m = 2$ and $m = 3$, equation (3.13) gives

$$\frac{z}{1 - xz - z^2} = \sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty} \left[\begin{bmatrix} n \\ 1 \end{bmatrix}\right] z^n;$$

$$\frac{z^2}{1 - (x^2+1)z - (x^2+1)z^2 + z^3} = z^2 + (x^2+1)z^3 + (x^4 + 3x^2 + 2)z^4 + (x^6 + 5x^4 + 7x^2 + 2)z^5 + \cdots = \sum_{n=0}^{\infty} \left[\begin{bmatrix} n \\ 2 \end{bmatrix}\right] z^n,$$

respectively. In particular, equation (3.13) gives a generating function for Fibonomial coefficients [4, 6, 7]:

$$\frac{z^{m-1}}{\sum_{r=0}^{m} (-1)^{r(r+1)/2} \begin{bmatrix} m \\ r \end{bmatrix} z^r} = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ m-1 \end{bmatrix} z^n,$$

where $m \geq 1$.

3.6. **Addition Formula.** Torretto and Fuchs developed an addition formula involving the sum of products of $m+1$ terms of sequences satisfying the same general second-order recurrence [15]. The identity

$$\sum_{r=0}^{m} (-1)^{r(r+3)/2} \begin{bmatrix} m \\ r \end{bmatrix} F_{n+m-r}^{m+1} = F_m^* F_{(m+1)(n+m/2)}$$

is a special case of their formula (5).

This identity has an analogous result for f_k:

$$\sum_{r=0}^{m} (-1)^{r(r+3)/2} \left[\begin{bmatrix} m \\ r \end{bmatrix}\right] f_{n+m-r}^{m+1} = f_m^* f_{(m+1)(n+m/2)}. \tag{3.14}$$

When $m = 1$, this yields the familiar Lucas-like identity $f_{n+1}^2 + f_n^2 = f_{2n+1}$; and when $m = 2$, it yields

$$\left[\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right] f_{n+2}^3 + \left[\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right] f_{n+1}^3 - \left[\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right] f_n^3 = f_1 f_2 f_{3(n+1)}$$

$$f_{n+2}^3 + x f_{n+1}^3 - f_n^3 = x f_{3(n+1)}.$$

This generalization of the Lucas identity $F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n}$ is established in [11].

Letting $m = 3$ and $m = 4$, we get

$$f_{n+3}^4 + (x^2+1) f_{n+2}^4 - (x^2+1) f_{n+1}^4 - f_n^4 = x(x^2+1) f_{4n+6} \tag{3.15}$$

$$f_{n+4}^5 + (x^3+2x) f_{n+3}^5 - (x^4+3x^2+2) f_{n+2}^5 - (x^3+2x) f_{n+1}^5 + f_n^5 = x(x^2+1)(x^3+2x) f_{5n+10}. \tag{3.16}$$

It follows from identities (3.15) and (3.16) that

$$F_{n+3}^4 + 2F_{n+2}^4 - 2F_{n+1}^4 - F_n^4 = 2F_{4n+6} \tag{3.17}$$

$$p_{n+3}^4 + (4x^2+1)p_{n+2}^4 - (4x^2+1)p_{n+1}^4 - p_n^4 = 2x(4x^2+1)p_{4n+6}$$

$$P_{n+3}^4 + 5P_{n+2}^4 - 5P_{n+1}^4 - P_n^4 = 10P_{4n+6}$$

$$F_{n+4}^5 + 3F_{n+3}^5 - 6F_{n+2}^5 - 3F_{n+1}^5 + F_n^5 = 6F_{5n+10} \tag{3.18}$$

$$\begin{aligned} p_{n+4}^5 &+ 4x(2x^2+1)p_{n+3}^5 - 2(8x^4+6x^2+1)p_{n+2}^5 \\ &- 4x(2x^2+1)p_{n+1}^5 + p_n^5 = 8x^2(2x^2+1)(4x^2+1)p_{5n+10} \end{aligned}$$

$$P_{n+4}^5 + 12P_{n+3}^5 - 30P_{n+2}^5 - 12P_{n+1}^5 + P_n^5 = 120P_{5n+10}.$$

Identities (3.17) and (3.18) appear in [13].

Letting $m = 5$, identity (3.14) yields

$$\sum_{r=0}^{5} (-1)^{r(r+3)/2} \left[\begin{bmatrix}5\\r\end{bmatrix}\right] f_{n+5-r}^6 = f_1 f_2 f_3 f_4 f_5 f_{6n+15}$$

$$\begin{aligned} &\left[\begin{bmatrix}5\\0\end{bmatrix}\right] f_{n+5}^6 + \left[\begin{bmatrix}5\\1\end{bmatrix}\right] f_{n+4}^6 - \left[\begin{bmatrix}5\\2\end{bmatrix}\right] f_{n+3}^6 - \left[\begin{bmatrix}5\\3\end{bmatrix}\right] f_{n+2}^6 + \left[\begin{bmatrix}5\\4\end{bmatrix}\right] f_{n+1}^6 - \left[\begin{bmatrix}5\\5\end{bmatrix}\right] f_n^6 \\ &\quad = f_1 f_2 f_3 f_4 f_5 f_{6n+15}; \end{aligned}$$

that is,

$$\begin{aligned} f_{n+5}^6 &+ (x^4+3x^2+1)f_{n+4}^6 - (x^6+5x^4+7x^2+2)f_{n+3}^6 - (x^6+5x^4+7x^2+2)f_{n+2}^6 \\ &+ (x^4+3x^2+1)f_{n+1}^6 - f_n^6 = x(x^2+1)(x^3+2x)(x^4+3x^2+1)f_{6n+15}. \end{aligned}$$

In particular, we have

$$F_{n+5}^6 + 5F_{n+4}^6 - 15F_{n+3}^6 - 15F_{n+2}^6 + 5F_{n+1}^6 - F_n^6 = 30F_{6n+15}$$

$$\begin{aligned} &p_{n+5}^6 + (16x^4+12x^2+1)p_{n+4}^6 \\ &- 2(32x^6+40x^4+14x^2+1)p_{n+3}^6 \\ &- 2(32x^6+40x^4+14x^2+1)p_{n+2}^6 \\ &+ (16x^4+12x^2+1)p_{n+1}^6 - p_n^6 = x(x^2+1)(x^3+2x)(x^4+3x^2+1)p_{6n+15} \end{aligned}$$

$$\begin{aligned} P_{n+5}^6 &+ 29P_{n+4}^6 - 174P_{n+3}^6 - 174P_{n+2}^6 \\ &+ 29P_{n+1}^6 - P_n^6 = 30P_{6n+15}. \end{aligned}$$

Finally, we add that the above Fibonacci identities have Vieta, Chebyshev, and Jacobsthal counterparts. For example, it follows from the identity $f_{n+1}^3 + xf_n^3 - f_{n-1}^3 = xf_{3n}$ that $V_{n+1}^3 - xV_n^3 + V_{n-1}^3 = xV_{3n}$.

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