ON CERTAIN FAMILIES OF FINITE RECIPROCAL SUMS THAT INVOLVE GENERALIZED FIBONACCI NUMBERS

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Abstract. In this paper, we find closed forms, in terms of rational numbers, for certain finite sums. Our most general results are for finite sums where the denominator of the summand is a product of terms from a sequence that generalizes both the Fibonacci and Lucas numbers.

1. Introduction

The Fibonacci and Lucas numbers are defined, respectively, for all integers n, by

$$F_n = F_{n-1} + F_{n-2},\ F_0 = 0,\ F_1 = 1,$$

$$L_n = L_{n-1} + L_{n-2},\ L_0 = 2,\ L_1 = 1.$$

To appreciate the many topics in the study of Fibonacci numbers, one need only glance at the chapter headings in the lovely book of Hoggatt, Jr. [2]. Chapter 10 deals with the topic of Fibonacci identities. Two well-known Fibonacci identities that occur in chapter 10 are

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \tag{1.1}$$

and

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} - F_n^4 = -1. \tag{1.2}$$

Identities (1.1) and (1.2) are known as Simson's identity, and the Gelin-Cesàro identity, respectively. Although these identities are quite old, they continue to provoke research. In this regard, we refer the interested reader to [1] and [3].

The present paper deals with *reciprocal* summation involving Fibonacci numbers, a topic not covered in [2]. Specifically, we continue the line of research in [4, 5, 6], where we give closed forms for certain finite sums in which the denominator of the summand consists of a *product* of Fibonacci (or generalized Fibonacci) numbers. Two instances of such finite sums are

$$\sum_{i=1}^{n-1} \frac{1}{F_iF_{i+1}F_{i+2}F_{i+3}} = \frac{7}{4} - \frac{1}{2}\left(\frac{F_{n-1}}{F_n} + \frac{3F_n}{F_{n+1}} + \frac{F_{n+1}}{F_{n+2}}\right), \tag{1.3}$$

and

$$8\sum_{i=1}^{n-1} \frac{L_{2(i+1)}L_{2(i+2)}}{F_{2i}F_{2(i+1)}F_{2(i+2)}F_{2(i+3)}} = 2 + F_{2(n-2)}\left(\frac{7}{3F_{2n}} - \frac{9}{8F_{2(n+1)}} + \frac{1}{3F_{2(n+2)}}\right), \tag{1.4}$$

both valid for $n \geq 2$. The sums (1.3) and (1.4) occur in [4] and [5], respectively.

In order to indicate the flavor of the results that we present in this paper, we begin by writing down the closed forms for two finite sums that involve the Fibonacci and Lucas numbers, respectively. For $n \geq 2$, we have

$$6\sum_{i=1}^{n-1} \frac{(-1)^iF_{2i+3}}{F_iF_{i+1}F_{i+2}F_{i+3}} + 5 = \frac{2F_{n-2}F_{n+4}}{F_nF_{n+2}}. \tag{1.5}$$

Again, for $n \geq 2$, we have

$$5922 \sum_{i=1}^{n-1} \frac{L_{4i+6}}{L_{2i}L_{2i+2}L_{2i+4}L_{2i+6}} - 41 = \frac{F_{2(n-2)}}{20}\left(\frac{2538}{L_{2n}} - \frac{329}{L_{2n+2}} + \frac{378}{L_{2n+4}}\right). \tag{1.6}$$

The finite sum (1.5) is an instance of a six-parameter family of identities that we present in Section 2, while (1.6) is an instance of a seven-parameter family of similar identities that we also present in Section 2.

We now introduce the two other pairs of integer sequences that are featured in this paper. Let $a \geq 0$ and $b \geq 0$ be integers with $(a,b) \neq (0,0)$. For p a positive integer, we define, for all integers n, the sequences $\{W_n\}$ and $\{\overline{W}_n\}$ by

$$W_n = pW_{n-1} + W_{n-2},\ W_0 = a,\ W_1 = b,$$

and

$$\overline{W}_n = W_{n-1} + W_{n+1}.$$

For $(a,b,p) = (0,1,1)$, we have $\{W_n\}$=$\{F_n\}$, and $\{\overline{W}_n\}$=$\{L_n\}$. Retaining the parameter p, and taking $(a,b) = (0,1)$, we write $\{W_n\}$=$\{U_n\}$, and $\{\overline{W}_n\}$=$\{V_n\}$, which are integer sequences that generalize the Fibonacci and Lucas numbers, respectively. Set $\Delta = p^2+4$. Then $\overline{U}_n = V_n$, and $\overline{V}_n = \Delta U_n$, so that $\overline{F}_n = L_n$, and $\overline{L}_n = 5F_n$.

Let $\alpha = \left(p+\sqrt{\Delta}\right)/2$, and $\beta = \left(p-\sqrt{\Delta}\right)/2$, denote the two distinct real roots of $x^2 - px - 1 = 0$. Set $A = b - a\beta$, and $B = b - a\alpha$. Then the closed forms (the Binet forms) for $\{W_n\}$ and $\{\overline{W}_n\}$ are, respectively,

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

and

$$\overline{W}_n = A\alpha^n + B\beta^n.$$

We require also the constant $e_W = AB = b^2 - pab - a^2$.

In Section 5, we prove one of our theorems with the use of a method that can be used to prove all the results that we present in this paper. Indeed, this method of proof is used in [5] and [6]. To this end, we choose to write

$$\begin{aligned}
U_n &= \left(\alpha^n + (-1)^{n+1}\alpha^{-n}\right)/\sqrt{\Delta},\\
V_n &= \alpha^n + (-1)^n\alpha^{-n},\\
W_n &= \left(\left(b + a\alpha^{-1}\right)\alpha^n + (-1)^{n+1}\left(b - a\alpha\right)\alpha^{-n}\right)/\sqrt{\Delta},\\
\overline{W}_n &= \left(b + a\alpha^{-1}\right)\alpha^n + (-1)^n\left(b - a\alpha\right)\alpha^{-n},
\end{aligned} \tag{1.7}$$

where these closed forms are valid for all integers n. We note also that $p = \alpha - \alpha^{-1}$.

There is a finite sum that we feature throughout. For integers $0 \leq l_1 < l_2$, $k \geq 1$, $m \geq 0$, and $n \geq 2$, this finite sum is

$$\Omega_W(k,m,n,l_1,l_2) = \sum_{i=l_1}^{l_2-1} \frac{(-1)^{ki}}{W_{k(i+2)+m}W_{k(i+n)+m}}. \tag{1.8}$$

If, for instance, on the right side of (1.8) we replace each occurrence of W by U, we denote the resulting sum by $\Omega_U(k,m,n,l_1,l_2)$.

We now give an identity involving Ω_W that is required for the proofs of all the theorems in this paper. We give this identity, whose proof can be found in [5], as a lemma.

Lemma 1.1. *With the constraints on l_1, l_2, k, m, and n, given above,*

$$U_{k(n-1)}\Omega_W(n+1) - U_{k(n-2)}\Omega_W(n) = \frac{(-1)^{k(n+l_1)}U_{k(l_2-l_1)}}{W_{k(n+l_1)+m}W_{k(n+l_2)+m}}.$$

The reader will notice that, in the statement of Lemma 1.1, we take $\Omega_W(n)$ to mean $\Omega_W(k,m,n,l_1,l_2)$. Likewise, in the sequel, we suppress certain arguments from quantities when there is no danger of confusion. We do this to prevent the formulas in question from becoming too unwieldy. Again, in Theorem 2.1, $\Omega_U(m_1,m_2)$ means $\Omega_U(k,m,n,m_1,m_2)$. A similar interpretation applies to the statement of all the theorems in this paper.

As we have already stated, the sums (1.5) and (1.6) are particular instances of two of our results. We begin in Section 2 with finite sums in which the denominator of the summand consists of a product of four factors. For finite sums of the type considered in this paper, where the denominator of the summand consists of fewer than four factors, see [5]. In Sections 3 and 4, we consider finite sums in which the denominator of the summand consists of a product of five and six factors, respectively. In Section 5, we present a sample proof that sets forth a method by which all of our results can be proved.

2. The Summand has Four Factors in the Denominator

Throughout this paper, $k \geq 1$, $m \geq 0$, and $n \geq 2$ always represent integers, and henceforth we do not restate this. In this section, $0 < m_1 < m_2 < m_3$ represent integers. We now present the six-parameter family of finite reciprocal sums that yields (1.5) as a special case. Define the finite sum

$$S_1(k,m,n,m_1,m_2,m_3) = \sum_{i=1}^{n-1} \frac{(-1)^{ki}U_{k(2i+m_1+m_2)+2m}}{U_{ki+m}U_{k(i+m_1)+m}U_{k(i+m_2)+m}U_{k(i+m_3)+m}}.$$

For $0 \leq i \leq 3$, define $a_i = a_i(k,m_1,m_2,m_3)$ by

$$\begin{aligned}
a_0 &= U_{m_1k}U_{m_2k}U_{m_3k}U_{(m_3-m_2)k}U_{(m_3-m_1)k},\\
a_1 &= U_{(m_1+m_2)k}U_{(m_3-m_2)k}U_{(m_3-m_1)k},\\
a_2 &= (-1)^{km_2}U_{m_1k}U_{(m_3-m_2)k}U_{(m_3-m_2-m_1)k},\\
a_3 &= -U_{m_1k}U_{m_2k}U_{(2m_3-m_2-m_1)k}.
\end{aligned}$$

Then we have the following theorem.

Theorem 2.1. *With S_1 and the a_i as defined above,*

$$\begin{aligned}
a_0\left(S_1(n) - S_1(2)\right) = U_{k(n-2)}\left(a_1\Omega_U(0,m_1) + a_2\Omega_U(m_1,m_2)\right.\\
\left. + a_3\Omega_U(m_2,m_3)\right).
\end{aligned}$$

In Theorem 2.1, taking $(p,k,m,m_1,m_2,m_3) = (1,1,0,1,2,3)$, we obtain (1.5). In the numerator of the summand of S_1, replace m_1+m_2 by m_1+m_3, or simply by m_3. Then we have also obtained closed forms for the corresponding finite sums, together with several other similarly defined sums. We do not present these closed forms here.

In the summand of S_1, we replaced each occurrence of U by V, but were unable to find the closed form of the corresponding finite sum. Likewise, in the numerator of the summand of S_1, we replaced U by V, but were unable to find the closed form of the corresponding finite sum. However, by taking $m_3 = m_1+m_2$, we were able to find the closed forms of the aforementioned sums. The condition $m_3 = m_1+m_2$ encompasses all instances where 0, m_1, m_2, and m_3 are in arithmetic progression. Assuming this condition, we succeeded in finding closed forms for sums analogous to S_1 that involve the more general sequences $\{W_n\}$ and $\{\overline{W}_n\}$. We now give two of these results (we have discovered several others).

Define

$$S_2(k,m,n,m_1,m_2,m_3) = \sum_{i=1}^{n-1} \frac{(-1)^{ki} W_{k(2i+m_1+m_2)+2m}}{W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}W_{k(i+m_3)+m}}.$$

For $0 \le i \le 3$, define $b_i = b_i(k,m_1,m_2)$ by

$$\begin{aligned} b_0 &= e_W U_{m_1k}U_{m_2k}U_{(m_1+m_2)k}U_{(m_2-m_1)k}, \\ b_1 &= (-1)^{k(m_1+m_2)+1}U_{(m_2-m_1)k}W_{-(m_1+m_2)k}, \\ b_2 &= (-1)^{k(m_1+m_2)}W_0U_{2m_1k}, \\ b_3 &= -U_{(m_2-m_1)k}W_{(m_1+m_2)k}. \end{aligned}$$

We then have the following theorem.

Theorem 2.2. *Let $0 < m_1 < m_2$ be integers, and let $m_3 = m_1 + m_2$. Then*

$$\begin{aligned} b_0\left(S_2(n) - S_2(2)\right) = U_{k(n-2)}\,&(b_1\Omega_W(0,m_1) + b_2\Omega_W(m_1,m_2) \\ &+ b_3\Omega_W(m_2,m_3)). \end{aligned}$$

In Theorem 2.2, taking $(p,a,b,k,m,m_1,m_2) = (1,2,1,2,0,1,2)$, we obtain (1.6).

Next, define

$$S_3(k,m,n,m_1,m_2,m_3) = \sum_{i=1}^{n-1} \frac{(-1)^{ki} \overline{W}_{k(2i+m_1+m_2)+2m}}{W_{ki+m}W_{k(i+m_1)+m}W_{k(i+m_2)+m}W_{k(i+m_3)+m}}.$$

For $0 \le i \le 3$, define $c_i = c_i(k,m_1,m_2)$ as

$$\begin{aligned} c_0 &= e_W U_{m_1k}U_{m_2k}U_{(m_1+m_2)k}U_{(m_2-m_1)k}, \\ c_1 &= (-1)^{k(m_1+m_2)}U_{(m_2-m_1)k}\overline{W}_{-(m_1+m_2)k}, \\ c_2 &= (-1)^{k(m_1+m_2)+1}\overline{W}_0U_{2m_1k}, \\ c_3 &= U_{(m_2-m_1)k}\overline{W}_{(m_1+m_2)k}. \end{aligned}$$

We then have the following theorem.

Theorem 2.3. *Let $0 < m_1 < m_2$ be integers, and let $m_3 = m_1 + m_2$. Then*

$$\begin{aligned} c_0\left(S_3(n) - S_3(2)\right) = U_{k(n-2)}\,&(c_1\Omega_W(0,m_1) + c_2\Omega_W(m_1,m_2) \\ &+ c_3\Omega_W(m_2,m_3)). \end{aligned}$$

It is easy to see that, with $W_n = F_n$, or $W_n = L_n$, Theorem 2.3 yields a four-parameter family of sums that are analogous to (1.5) and (1.6), and where the summand of each member of this family involves both F_n and L_n.

To conclude this section, we give closed forms for two finite sums where, in each case, the numerator of the summand contains a square. For integers $0 < m_1 < m_2$, define

$$S_4(k,m,n,m_1,m_2) = \sum_{i=1}^{n-1} \frac{(-1)^{ki} W^2_{k(i+m_2)+m}}{W_{ki+m}W_{k(i+m_1)+m}W_{k(i+2m_2-m_1)+m}W_{k(i+2m_2)+m}}.$$

For $0 \le i \le 2$, define $d_i = d_i(k, m_1, m_2)$ as

$$\begin{aligned}
d_0 &= U_{m_1 k} V_{m_2 k} V_{(m_2-m_1)k} U_{(2m_2-m_1)k},\\
d_1 &= U_{m_2 k} V_{(m_2-m_1)k},\\
d_2 &= 2(-1)^{k(m_1+m_2)} U_{m_1 k}.
\end{aligned}$$

Then we have the following theorem.

Theorem 2.4. *We have*

$$\begin{aligned}
d_0\left(S_4(n) - S_4(2)\right) = U_{k(n-2)}\,&(d_1 \Omega_W(0, m_1) + d_2 \Omega_W(m_1, 2m_2 - m_1)\\
&+ d_1 \Omega_W(2m_2 - m_1, 2m_2)).
\end{aligned}$$

Next, for integers $0 < m_1 < m_2$, define

$$S_5(k, m, n, m_1, m_2) = \sum_{i=1}^{n-1} \frac{(-1)^{ki}\overline{W}^2_{k(i+m_2)+m}}{W_{ki+m} W_{k(i+m_1)+m} W_{k(i+2m_2-m_1)+m} W_{k(i+2m_2)+m}}.$$

For $0 \le i \le 2$, define $f_i = f_i(k, m_1, m_2)$ by

$$\begin{aligned}
f_0 &= U_{m_1 k} U_{m_2 k} U_{(m_2-m_1)k} U_{(2m_2-m_1)k},\\
f_1 &= V_{m_2 k} U_{(m_2-m_1)k},\\
f_2 &= 2(-1)^{k(m_1+m_2+1)} U_{m_1 k}.
\end{aligned}$$

Then we have the following theorem.

Theorem 2.5. *We have*

$$\begin{aligned}
f_0\left(S_5(n) - S_5(2)\right) = U_{k(n-2)}\,&(f_1 \Omega_W(0, m_1) + f_2 \Omega_W(m_1, 2m_2 - m_1)\\
&+ f_1 \Omega_W(2m_2 - m_1, 2m_2)).
\end{aligned}$$

Taking $(p, a, b, k, m, m_1, m_2) = (1, 0, 1, 2, 0, 1, 2)$ in Theorem 2.5, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n-1} \frac{L^2_{2(i+2)}}{F_{2i} F_{2(i+1)} F_{2(i+3)} F_{2(i+4)}}\\
&= \frac{36}{385} + \frac{F_{2(n-2)}}{12}\left(\frac{7}{6F_{2n}} - \frac{1}{8F_{2(n+1)}} - \frac{1}{21F_{2(n+2)}} + \frac{7}{110F_{2(n+3)}}\right).
\end{aligned}$$

3. The Summand has Five Factors in the Denominator

In this section, and in the sections that follow, we require quantities analogous to a_i, b_i, c_i, d_i and f_i, defined in the previous section. For simplicity, we use the same pro-numerals here (and in subsequent sections) to describe these quantities. In this section, we take $0 < m_1 < m_2 < m_3 < m_4$ to be integers. Define the sum

$$T_1(k, m, n, m_1, \dots, m_4) = \sum_{i=1}^{n-1} \frac{(-1)^{ki} V_{k(3i+m_1+m_2+m_3)+3m}}{U_{ki+m} U_{k(i+m_1)+m} \cdots U_{k(i+m_4)+m}}.$$

We were unable to find the closed form for the sums that we present in this section without certain simplifying assumptions on the m_i. We chose to take $m_3 = 2m_2 - m_1$, and $m_4 = 2m_2$. These assumptions encompass all instances where 0, m_1, m_2, m_3, and m_4 are in arithmetic progression, and also bring a nice symmetry to the subscripts in the denominator of the summand. Indeed, under these assumptions, the successive differences between the subscripts in the denominator of the summand are km_1, $k(m_2 - m_1)$, $k(m_2 - m_1)$, and km_1. For convenience, we record these assumptions as follows: For integers $0 < m_1 < m_2$, define integers m_4 and m_5 as

$$m_3 = 2m_2 - m_1, \text{ and } m_4 = 2m_2. \tag{3.1}$$

We now define quantities $a_i = a_i(k, m_1, m_2)$, for $0 \le i \le 2$, that help us to succinctly give the closed form for T_1. Define

$$\begin{aligned} a_0 &= U_{m_1k} U_{(2m_2-m_1)k} U_{m_2k}^2 U_{(m_2-m_1)k}^2, \\ a_1 &= \left(V_{2m_2k} + (-1)^{km_2+1}\right) U_{(m_2-m_1)k}^2, \\ a_2 &= (-1)^{k(m_1+m_2)+1} U_{m_1k} U_{(2m_2-m_1)k}. \end{aligned}$$

We then have the following theorem.

Theorem 3.1. *Let $0 < m_1 < m_2$ be integers. Let m_3 and m_4 be as defined in (3.1). Then*

$$\begin{aligned} a_0\left(T_1(n) - T_1(2)\right) = U_{k(n-2)} &\left(a_1\Omega_U(0, m_1) + a_2\Omega_U(m_1, m_2)\right. \\ &\left. - a_2\Omega_U(m_2, m_3) - a_1\Omega_U(m_3, m_4)\right). \end{aligned} \tag{3.2}$$

We now state a *dual* result for Theorem 3.1 that we obtained by interchanging the roles of U and V in T_1. In the summand of T_1, replace V by U, and replace each occurrence of U by V. Then (3.2) remains valid provided we multiply the left side by Δ^2, and replace each occurrence of Ω_U by Ω_V. Where we have discovered dual results for other results in this paper, we present these as well.

Define the sum

$$T_2(k, m, n, m_1, \ldots, m_4) = \sum_{i=1}^{n-1} \frac{(-1)^{ki} U_{k(3i+m_1+m_2+m_3)+3m}}{U_{ki+m} U_{k(i+m_1)+m} \cdots U_{k(i+m_4)+m}}.$$

For $0 \le i \le 2$, define $b_i = b_i(k, m_1, m_2)$ as

$$\begin{aligned} b_0 &= U_{m_1k} U_{2m_2k} U_{2(m_2-m_1)k} U_{(2m_2-m_1)k}, \\ b_1 &= \left(V_{2m_2k} + (-1)^{km_2}\right) U_{2(m_2-m_1)k}, \\ b_2 &= \left((-1)^{k(m_1+m_2)+1} V_{(2m_2-m_1)k} - 2V_{m_1k}\right) U_{m_1k}. \end{aligned}$$

We then have the following theorem.

Theorem 3.2. *Let $0 < m_1 < m_2$ be integers. Let m_3 and m_4 be as defined in (3.1). Then*

$$\begin{aligned} b_0\left(T_2(n) - T_2(2)\right) = U_{k(n-2)} &\left(b_1\Omega_U(0, m_1) + b_2\Omega_U(m_1, m_2)\right. \\ &\left. + b_2\Omega_U(m_2, m_3) + b_1\Omega_U(m_3, m_4)\right). \end{aligned} \tag{3.3}$$

In the summand of T_2, replace each occurrence of U by V. Then (3.3) remains valid provided we multiply the left side by Δ, and replace each occurrence of Ω_U by Ω_V.

Next, define the sum

$$T_3(k, m, n, m_1, \ldots, m_4) = \sum_{i=1}^{n-1} \frac{(-1)^{ki} V_{2k(i+m_2)+2m} V_{k(i+m_2)+m}}{U_{ki+m} U_{k(i+m_1)+m} \cdots U_{k(i+m_4)+m}}.$$

For $0 \le i \le 2$, define $c_i = c_i(k, m_1, m_2)$ as

$$\begin{aligned}c_0 &= U_{m_1k}U_{(2m_2-m_1)k}U^2_{m_2k}U^2_{(m_2-m_1)k},\\ c_1 &= V_{2m_2k}U^2_{(m_2-m_1)k},\\ c_2 &= 2(-1)^{k(m_1+m_2)+1}U_{m_1k}U_{(2m_2-m_1)k}.\end{aligned}$$

Then we have the following theorem.

Theorem 3.3. *Let $0 < m_1 < m_2$ be integers. Let m_3 and m_4 be as defined in (3.1). Then*

$$\begin{aligned}c_0\left(T_3(n) - T_3(2)\right) = U_{k(n-2)}\,&(c_1\Omega_U(0, m_1) + c_2\Omega_U(m_1, m_2)\\ &- c_2\Omega_U(m_2, m_3) - c_1\Omega_U(m_3, m_4)). \qquad (3.4)\end{aligned}$$

In the summand of T_3, replace $V_{k(i+m_2)+m}$ by $U_{k(i+m_2)+m}$, and leave the quantity $V_{2k(i+m_2)+2m}$ unchanged. Also replace each occurrence of U by V. Then (3.4) remains valid provided we multiply the left side by Δ^2, and replace each occurrence of Ω_U by Ω_V.

To conclude this section, we give the closed form for a finite sum in which the numerator of the summand contains a cube. Define the sum

$$T_4(k, m, n, m_1, \dots, m_4) = \sum_{i=1}^{n-1} \frac{(-1)^{ki}\overline{W}^3_{k(i+m_2)+m}}{W_{ki+m}W_{k(i+m_1)+m}\cdots W_{k(i+m_4)+m}}.$$

For $0 \le i \le 2$, define $d_i = d_i(k, m_1, m_2)$ by

$$\begin{aligned}d_0 &= U_{m_1k}U_{(2m_2-m_1)k}U^2_{m_2k}U^2_{(m_2-m_1)k},\\ d_1 &= U^2_{(m_2-m_1)k}V^2_{m_2k},\\ d_2 &= 4(-1)^{k(m_1+m_2)+1}U_{m_1k}U_{(2m_2-m_1)k}.\end{aligned}$$

We then have the following theorem.

Theorem 3.4. *Let $0 < m_1 < m_2$ be integers. Let m_3 and m_4 be as defined in (3.1). Then*

$$\begin{aligned}d_0\left(T_4(n) - T_4(2)\right) = U_{k(n-2)}\,&(d_1\Omega_W(0, m_1) + d_2\Omega_W(m_1, m_2)\\ &- d_2\Omega_W(m_2, m_3) - d_1\Omega_W(m_3, m_4)). \qquad (3.5)\end{aligned}$$

In Theorem 3.4, take $(p, a, b, k, m, m_1, m_2) = (1, 2, 1, 1, 0, 1, 2)$. Then

$$\begin{aligned}&\sum_{i=1}^{n-1} \frac{(-1)^i F^3_{i+2}}{L_iL_{i+1}L_{i+2}L_{i+3}L_{i+4}} + \frac{2}{231}\\ &= \frac{F_{n-2}}{125}\left(\frac{3}{2L_n} - \frac{1}{L_{n+1}} - \frac{4}{7L_{n+2}} + \frac{9}{22L_{n+3}}\right).\end{aligned}$$

Before leaving this section, we remark that in the definitions of T_1, T_2, and T_3, we replaced each occurrence of U and V by W and $\overline{W}$, respectively. Then, with the same assumptions on the m_i, we attempted to find closed forms for the corresponding finite sums. Our motivation was to achieve greater generality. However, we were unsuccessful. The same can be said for all other results in this paper that involve the sequences $\{U_n\}$ and $\{V_n\}$.

4. The Summand has Six Factors in the Denominator

Let $0 < m_1 < m_2 < m_3$ be integers. The first three theorems in this section require that

$$m_4 = m_2 + m_3 - m_1, \text{ and } m_5 = m_2 + m_3. \tag{4.1}$$

Define the sum

$$X_1(k, m, n, m_1, \ldots, m_5) = \sum_{i=1}^{n-1} \frac{U_{k(2i+m_2+m_3)+2m}}{U_{ki+m} U_{k(i+m_1)+m} \cdots U_{k(i+m_5)+m}}.$$

For $0 \le i \le 2$, define $a_i = a_i(k, m, m_1, m_2, m_3)$ as

$$\begin{aligned} a_0 &= (-1)^{m+1} U_{m_1 k} U_{m_2 k} U_{m_3 k} U_{(m_2-m_1)k} U_{(m_3-m_1)k} U_{(m_2+m_3-m_1)k}, \\ a_1 &= -U_{(m_2-m_1)k} U_{(m_3-m_1)k}, \\ a_2 &= (-1)^{km_1} U_{m_1 k} U_{(m_2+m_3-m_1)k}. \end{aligned}$$

We then have the following theorem.

Theorem 4.1. *Let $0 < m_1 < m_2 < m_3$ be integers. Let m_4 and m_5 be as defined in (4.1). Then*

$$\begin{aligned} a_0 \left(X_1(n) - X_1(2) \right) = U_{k(n-2)} \big(& a_1 \Omega_U(0, m_1) + a_2 \Omega_U(m_1, m_2) \\ & - a_2 \Omega_U(m_3, m_4) - a_1 \Omega_U(m_4, m_5) \big). \end{aligned} \tag{4.2}$$

In the denominator of the summand of X_1, replace each occurrence of U by V. Then (4.2) remains valid provided we multiply the left side by $-\Delta^2$, and replace each occurrence of Ω_U by Ω_V.

Define the sum

$$X_2(k, m, n, m_1, \ldots, m_5) = \sum_{i=1}^{n-1} \frac{(-1)^{ki} U_{k(4i+2m_2+2m_3)+4m}}{U_{ki+m} U_{k(i+m_1)+m} \cdots U_{k(i+m_5)+m}}.$$

Interestingly, the b_i that we now define have a connection to the a_i in Theorem 4.1. For $0 \le i \le 2$, define $b_i = b_i(k, m, m_1, m_2, m_3)$ as

$$\begin{aligned} b_0 &= (-1)^{m+1} a_0, \\ b_1 &= -V_{(m_2+m_3)k} a_1, \\ b_2 &= (-1)^{km_2+1} V_{(m_3-m_2)k} a_2. \end{aligned}$$

Then we have the following theorem.

Theorem 4.2. *Let $0 < m_1 < m_2 < m_3$ be integers. Let m_4 and m_5 be as defined in (4.1). Then*

$$\begin{aligned} b_0 \left(X_2(n) - X_2(2) \right) = U_{k(n-2)} \big(& b_1 \Omega_U(0, m_1) + b_2 \Omega_U(m_1, m_2) \\ & - b_2 \Omega_U(m_3, m_4) - b_1 \Omega_U(m_4, m_5) \big). \end{aligned} \tag{4.3}$$

In the denominator of the summand of X_2, replace each occurrence of U by V. Then (4.3) remains valid provided we multiply the left side by Δ^2, and replace each occurrence of Ω_U by Ω_V.

Define the sum

$$X_3(k, m, n, m_1, \ldots, m_5) = \sum_{i=1}^{n-1} \frac{(-1)^{ki} U^2_{k(2i+m_2+m_3)+2m}}{U_{ki+m} U_{k(i+m_1)+m} \cdots U_{k(i+m_5)+m}}.$$

Some of the c_i that we now define have a connection to the a_i in Theorem 4.1. For $0 \le i \le 3$, define $c_i = c_i(k, m, m_1, m_2, m_3)$ by

$$\begin{aligned} c_0 &= (-1)^{m+1} a_0, \\ c_1 &= -U_{(m_2+m_3)k} a_1, \\ c_2 &= (-1)^{km_1+1} U_{m_1 k} \left((-1)^{km_3} U_{m_2 k} U_{(m_2-m_1)k} + (-1)^{km_2} U_{m_3 k} U_{(m_3-m_1)k} \right), \\ c_3 &= 2(-1)^{k(m_1+m_3)+1} U_{m_1 k} U_{m_2 k} U_{(m_2-m_1)k}. \end{aligned}$$

We then have the following theorem.

Theorem 4.3. *Let $0 < m_1 < m_2 < m_3$ be integers. Let m_4 and m_5 be as defined in (4.1). Then*

$$\begin{aligned} c_0 \left(X_3(n) - X_3(2) \right) = U_{k(n-2)} &\left(c_1 \Omega_U(0, m_1) + c_2 \Omega_U(m_1, m_2) \right. \\ &+ c_3 \Omega_U(m_2, m_3) + c_2 \Omega_U(m_3, m_4) \\ &\left. + c_1 \Omega_U(m_4, m_5) \right). \end{aligned} \tag{4.4}$$

In the denominator of the summand of X_3, replace each occurrence of U by V. Then (4.4) remains valid provided we multiply the left side by Δ^2, and replace each occurrence of Ω_U by Ω_V.

For our next result, we require m_1 and m_2 to be positive integers with $0 < 2m_1 < m_2$. Then, with $m_3 = 2m_1 + m_2$, $m_4 = 2m_2$, and $m_5 = 2m_1 + 2m_2$, we define $X_4(n) = X_4(k, m, n, m_1, m_2)$ to be the sum

$$\sum_{i=1}^{n-1} \frac{(-1)^{ki} W^4_{k(i+m_1+m_2)+m}}{W_{ki+m} W_{k(i+2m_1)+m} W_{k(i+m_2)+m} W_{k(i+m_3)+m} W_{k(i+m_4)+m} W_{k(i+m_5)+m}}.$$

For $0 \le i \le 3$, define $d_i = d_i(k, m_1, m_2)$ by

$$\begin{aligned} d_0 &= U_{2m_1 k} U_{m_2 k} U_{2m_2 k} U_{(m_2-2m_1)k} U_{(2m_1+m_2)k} V_{(m_2-m_1)k} V_{(m_1+m_2)k}, \\ d_1 &= U_{(m_2-2m_1)k} U^3_{(m_1+m_2)k} V_{(m_2-m_1)k}, \\ d_2 &= (-1)^{k(m_1+m_2)+1} U_{2m_1 k} U_{m_2 k} \Big((-1)^{km_1} U_{2m_1 k} U_{(m_2-m_1)k} V_{(m_1+m_2)k} \\ &\qquad - 4 U_{m_1 k} U_{(m_2-m_1)k} V_{m_2 k} + 2(-1)^{km_2} U_{(m_1+m_2)k} U_{(3m_1-m_2)k} \Big), \\ d_3 &= 2(-1)^{km_2} U_{m_1 k} U_{(m_2-2m_1)k} \left(U_{m_1 k} U_{(m_2-m_1)k} V_{(m_1+m_2)k} \right. \\ &\qquad \left. + 2 U_{m_2 k} U_{(m_1+m_2)k} \right). \end{aligned}$$

We have the following theorem.

Theorem 4.4. *Let m_1 and m_2 be integers with $0 < 2m_1 < m_2$. Let m_3, m_4, and m_5 be as given in the definition of X_4. Then*

$$\begin{aligned} d_0 \left(X_4(n) - X_4(2) \right) = U_{k(n-2)} &\left(d_1 \Omega_W(0, 2m_1) + d_2 \Omega_W(2m_1, m_2) \right. \\ &+ d_3 \Omega_W(m_2, m_3) + d_2 \Omega_W(m_3, m_4) \\ &\left. + d_1 \Omega_W(m_4, m_5) \right). \end{aligned}$$

For the final theorem in this paper, we require m_1, m_2, and m_3 to be positive integers with $0 < m_1 < 2m_2 < 2m_3$. Then, with $m_4 = 2m_2 + 2m_3 - m_1$, and $m_5 = 2m_2 + 2m_3$, we define

$X_5(n) = X_5(k, m, n, m_1, m_2, m_3)$ to be the sum

$$\sum_{i=1}^{n-1} \frac{W^2_{k(i+m_2+m_3)+m}}{W_{ki+m}W_{k(i+m_1)+m}W_{k(i+2m_2)+m}W_{k(i+2m_3)+m}W_{k(i+m_4)+m}W_{k(i+m_5)+m}}.$$

For $0 \le i \le 3$, define $f_i = f_i(k, m, m_1, m_2, m_3)$ as

$$\begin{aligned}
f_0 &= e_W U_{m_1k}U_{2m_2k}U_{2m_3k}U_{(2m_2-m_1)k}U_{(2m_3-m_1)k}U_{(2m_2+2m_3-m_1)k} \\
&\quad \times V_{(m_3-m_2)k}V_{(m_2+m_3-m_1)k}V_{(m_2+m_3)k}, \\
f_1 &= (-1)^m U_{(m_2+m_3)k}U_{(2m_2-m_1)k}U_{(2m_3-m_1)k}V_{(m_3-m_2)k}V_{(m_2+m_3-m_1)k}, \\
f_2 &= (-1)^{k(m_1+m_2+m_3)+m+1}U_{m_1k}V_{(m_3-m_2)k}\Big((-1)^{k(m_2+m_3)}U_{(m_2+m_3-m_1)k} \\
&\quad \times U_{(3m_2+3m_3-m_1)k} - U_{(2m_2-m_1)k}U_{(2m_3-m_1)k} + (-1)^{km_1}U^2_{(m_3-m_2)k}\Big), \\
f_3 &= 2(-1)^{k(m_1+m_2+m_3)+m+1}U_{m_1k}U_{2m_2k}U_{(2m_2-m_1)k}V_{(3m_3+m_2-m_1)k}.
\end{aligned}$$

We have the following theorem.

Theorem 4.5. *Let m_1, m_2, and m_3 be integers with $0 < m_1 < 2m_2 < 2m_3$. Let m_4 and m_5 be as given in the definition of X_5. Then*

$$\begin{aligned}
f_0\left(X_5(n) - X_5(2)\right) = U_{k(n-2)}\big(&f_1\Omega_W(0, m_1) + f_2\Omega_W(m_1, 2m_2) \\
&+ f_3\Omega_W(2m_2, 2m_3) + f_2\Omega_W(2m_3, m_4) \\
&+ f_1\Omega_W(m_4, m_5)\big).
\end{aligned}$$

In Theorems 4.1 to 4.5, we have presented closed forms for eight finite sums. This includes the dual results for Theorems 4.1 to 4.3. We have discovered closed forms for a further eight finite sums of a similar nature that we do not present here. In the paragraph that follows, we briefly indicate the form of these results.

In the definitions of X_1, X_2, and X_3, replace each occurrence of U in the numerator of the summand by V, and denote these finite sums by X_1^V, X_2^V, and X_3^V, respectively. Then we have discovered closed forms for X_1^V, X_2^V, and X_3^V, and we have found dual results in each case. For $1 \le i \le 3$, the dual result for X_i^V is the closed form for the following finite sum: In X_i^V, replace each occurrence of U in the denominator of the summand by V. Finally, in X_4 and X_5, replace each occurrence of W in the numerator of the summand by $\overline{W}$. Then we have discovered closed forms for the two corresponding finite sums.

As an example, in Theorem 4.2, take $(p, k, m, m_1, m_2, m_3) = (1, 1, 0, 1, 2, 3)$. Then

$$\begin{aligned}
&240\sum_{i=1}^{n-1} \frac{(-1)^i F_{4i+10}}{F_iF_{i+1}F_{i+2}F_{i+3}F_{i+4}F_{i+5}} + 377 \\
&= F_{n-2}\left(\frac{440}{F_n} - \frac{60}{F_{n+1}} + \frac{24}{F_{n+3}} - \frac{55}{F_{n+4}}\right).
\end{aligned}$$

In Theorem 4.3, also take $(p, k, m, m_1, m_2, m_3) = (1, 1, 0, 1, 2, 3)$. Then

$$\begin{aligned}
&720\sum_{i=1}^{n-1} \frac{(-1)^i F^2_{2i+5}}{F_iF_{i+1}F_{i+2}F_{i+3}F_{i+4}F_{i+5}} + 507 \\
&= F_{n-2}\left(\frac{600}{F_n} - \frac{60}{F_{n+1}} - \frac{80}{F_{n+2}} - \frac{24}{F_{n+3}} + \frac{75}{F_{n+4}}\right).
\end{aligned}$$

5. A Sample Proof

In [5], we give a sample proof with the use of two methods: with the use of generalized Fibonacci identities, and also with the use of the closed forms of the sequences involved. In [6], we give a sample proof that uses the closed forms. Once again, in order for this paper to be self contained, we demonstrate the method of proof that employs the closed forms. The advantages of this method are that it is transparent, mechanical, and it applies to all the results in this paper. To assist with the lengthy algebra, we make use of the computer algebra system *Mathematica 8*. What follows is a proof of Theorem 3.4.

Proof. All the finite sums in this paper are defined for $n \geq 2$, and so it is for these values of n that the following argument holds. In the statement of Theorem 3.4, denote the quantities on the left and right sides of (3.5) by $L(n)$ and $R(n)$, respectively. Since $m_3 = 2m_2 - m_1$, and $m_4 = 2m_2$, the expression for $L(n+1) - L(n)$ is

$$\frac{(-1)^{kn} U_{m_1 k} U_{(2m_2-m_1)k} U_{m_2 k}^2 U_{(m_2-m_1)k}^2 \overline{W}_{k(n+m_2)+m}^3}{W_{kn+m} W_{k(n+m_1)+m} W_{k(n+m_2)+m} W_{k(n+2m_2-m_1)+m} W_{k(n+2m_2)+m}}. \tag{5.1}$$

Next, we require the difference $R(n+1) - R(n)$, which we obtain with the use of Lemma 1.1. Since this difference is quite lengthy, it is convenient to express it in terms of five quantities $g_i = g_i(k, m, n, m_1, m_2)$ that we define as follows:

$$\begin{aligned}
g_1 &= -U_{m_1 k} U_{(m_2-m_1)k}, \\
g_2 &= (-1)^{km_1} U_{(m_2-m_1)k} V_{m_2 k}^2 W_{kn+m} W_{k(n+m_1)+m} W_{k(n+m_2)+m}, \\
g_3 &= 4(-1)^{km_1} U_{(2m_2-m_1)k} W_{kn+m} W_{k(n+m_1)+m} W_{k(n+2m_2)+m}, \\
g_4 &= 4(-1)^{km_2} U_{(2m_2-m_1)k} W_{kn+m} W_{k(n+2m_2-m_1)+m} W_{k(n+2m_2)+m}, \\
g_5 &= U_{(m_2-m_1)k} V_{m_2 k}^2 W_{k(n+m_2)+m} W_{k(n+2m_2-m_1)+m} W_{k(n+2m_2)+m}.
\end{aligned}$$

The expression for $R(n+1) - R(n)$ is then

$$\frac{(-1)^{kn} g_1 \left(g_2 - g_3 + g_4 - g_5\right)}{W_{kn+m} W_{k(n+m_1)+m} W_{k(n+m_2)+m} W_{k(n+2m_2-m_1)+m} W_{k(n+2m_2)+m}}. \tag{5.2}$$

Our aim is to prove that $L(n+1) - L(n) = R(n+1) - R(n)$, and we achieve this by proving that the numerators in (5.1) and (5.2) are equal. Quite simply, we express each of these numerators in terms of the closed forms given in (1.7), and then consider their difference. Upon expansion, we see that this difference reduces to zero. This, together with the fact that $L(2) = R(2) = 0$, proves Theorem 3.4. □

For the sake of anyone who may wish to duplicate our proof, we remark that the algebra involved becomes much easier if one intervenes, and evaluates any power of -1 based on the parities of k, m, n, m_1, and m_2. Accordingly, to prove that the numerators in (5.1) and (5.2) are equal, one is required to consider thirty-two cases. This was incorporated into our programming, and was easily accomplished.

6. Concluding Comments

Numerical evidence suggests that results, analogous to those presented here, exist for finite sums in which the denominator of the summand consists of seven or more factors. However, such results become more unwieldy as the number of factors in the denominator of the summand increases. We hope that the presentation here has given the reader an appreciation of the types of results that are possible.

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References

[1] S. Fairgrieve and H. W. Gould, *Product difference Fibonacci identities of Simson, Gelin-Cesàro, Tagiuri and generalizations*, The Fibonacci Quarterly, **43.2** (2005), 137–141.
[2] V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers*, Boston:Houghton-Mifflin, 1969. Reprinted, The Fibonacci Association, 1972.
[3] R. S. Melham,*On product difference Fibonacci identities*, Integers, **11** (2011), #A10.
[4] R. S. Melham, *Finite sums that involve reciprocals of products of generalized Fibonacci numbers*, Integers, **13** (2013), #A40.
[5] R. S. Melham, *More on finite sums that involve reciprocals of products of generalized Fibonacci numbers*, Integers, **14** (2014), #A4.
[6] R. S. Melham, *Finite reciprocal sums involving summands that are balanced products of generalized Fibonacci numbers*, Journal of Integer Sequences, **17** (2014), Article 14.6.5.

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