GENERALIZED BERNOULLI NUMBERS AND A FORMULA OF LUCAS

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Abstract. An overlooked formula of E. Lucas for the generalized Bernoulli numbers is proved using generating functions. This is then used to provide a new proof and a new form of a sum involving classical Bernoulli numbers studied by K. Dilcher. The value of this sum is then given in terms of the Meixner-Pollaczek polynomials.

1. Introduction

The goal of this paper is to provide a unified approach to two topics that have appeared in the literature. The first one is an expression for the generalized Bernoulli numbers $B_n^{(p)}$ defined by the exponential generating function

$$\sum_{n=0}^{\infty} B_n^{(p)} \frac{z^n}{n!} = \left(\frac{z}{e^z - 1} \right)^p. \tag{1.1}$$

For $n \in \mathbb{N}$, the coefficients $B_n^{(p)}$ are polynomials in p named after Nörlund in [1]. The first few are

$$B_0^{(p)} = 1,\ B_1^{(p)} = -\tfrac{1}{2}p,\ B_2^{(p)} = -\tfrac{1}{12}p + \tfrac{1}{4}p^2,\ B_3^{(p)} = \tfrac{1}{8}p^2(1-p). \tag{1.2}$$

In his 1878 paper, E. Lucas [5] gave the formula

$$B_n^{(p)} = \frac{(-1)^{p-1}}{(p-1)!} \frac{n!}{(n-p)!} \beta^{n-p+1}(1+\beta)\cdots(p-1+\beta) \tag{1.3}$$

for $n \geq p$. This is a symbolic formula: to obtain the value of $B_n^{(p)}$, expand the expression (1.3) and replace β^j by the ratio B_j/j. Here B_j is the classical Bernoulli number $B_n = B_n^{(1)}$ in the notation from (1.1).

The second topic is an expression established by K. Dilcher [2] for the sums of products of Bernoulli numbers

$$S_N(n) := \sum \binom{2n}{2j_1,\, 2j_2,\, \ldots,\, 2j_N} B_{2j_1} B_{2j_2} \cdots B_{2j_N}, \tag{1.4}$$

where the sum is taken over all nonnegative integers $j_1, \ldots, j_N$ such that $j_1 + \cdots + j_N = n$, and where

$$\binom{2n}{2j_1,\, 2j_2,\, \ldots,\, 2j_N} = \frac{(2n)!}{(2j_1)! \cdots (2j_N)!} \tag{1.5}$$

is the multinomial coefficient and B_{2k} is the classical Bernoulli number. One of the main results of [2] is the evaluation, for $N \leq 2n$,

$$S_N(n) = \frac{(2n)!}{(2n-N)!} \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} b_k^{(N)} \frac{B_{2n-2k}}{2n-2k}, \tag{1.6}$$

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where the coefficients $b_k^{(N)}$ are defined by the recurrence

$$b_k^{(N+1)} = -\frac{1}{N} b_k^{(N)} + \frac{1}{4} b_{k-1}^{(N-1)}, \tag{1.7}$$

with $b_0^{(1)} = 1$ and $b_k^{(N)} = 0$ for $k < 0$ and for $k > \lfloor (N-1)/2 \rfloor$.

Lucas's original proof of (1.3) is recalled in Section 2. This section also contains an extension of Lucas's formula for $B_n^{(p)}$ to $0 \leq n \leq p-1$ in terms of the Stirling numbers of the first kind. A unified proof of the two formulas for $B_n^{(p)}$ based on generating functions is given in Section 3. Another proof of Lucas's formula, based on recurrences, is given in Section 4 while Section 6 contains a proof of

$$S_N(n) = \sum_{k=0}^{N} \frac{(2n)!}{(2n-k)!} 2^{-k} \binom{N}{k} B_{2n-k}^{(N-k)} \tag{1.8}$$

that expresses Dilcher's sum (1.4) explicitly in terms of the generalized Bernoulli numbers. Expressing this result in hypergeometric form leads to a formula for $S_N(n)$ in terms of the Meixner-Pollaczek polynomials

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1 \left(\begin{matrix} -n & \lambda + ix \\ & 2\lambda \end{matrix} \middle| 1 - e^{-2i\phi} \right). \tag{1.9}$$

It is then established that the recurrence (1.7), provided by Dilcher in [2], is equivalent to the classical three-term relation for this orthogonal family of polynomials.

2. Lucas's Theorem

In his paper [5], E. Lucas gave an expression for the generalized Bernoulli numbers $B_n^{(p)}$, for $n \geq p$. This section presents an outline of his proof and an extension of this expression for $B_n^{(p)}$ to the case $0 \leq n \leq p-1$. A proof based on generating functions is given in the next section. Lucas's formula uses the translation

$$\beta^n = \frac{B_n}{n} \tag{2.1}$$

coming from umbral calculus. Observe, for example, that

$$\begin{aligned} B_3^{(2)} &= \frac{(-1)^1}{1!} \frac{3!}{1!} \beta^2 (1+\beta) = -6(\beta^2 + \beta^3) \\ &= -6 \left(\frac{B_2}{2} + \frac{B_3}{3} \right) = -3B_2 = -\frac{1}{2}. \end{aligned}$$

Observe also that the symbolic substitution (2.1) should be performed only *after* all the terms have been expanded. For example,

$$\beta^2(1+\beta) = \beta^2 + \beta^3 = \frac{B_2}{2} + \frac{B_3}{3} = -\frac{1}{4} \tag{2.2}$$

but

$$\beta^2(1+\beta) \neq \frac{B_2}{2}\left(1 + \frac{B_1}{1}\right) = \frac{1}{24}. \tag{2.3}$$

Theorem 2.1 (Lucas)**.** *For $n \geq p$, the generalized Bernoulli numbers $B_n^{(p)}$ are given by*

$$B_n^{(p)} = \frac{(-1)^{p-1}}{(p-1)!} \frac{n!}{(n-p)!} \beta^{n-p+1} (1+\beta)(2+\beta) \cdots (p-1+\beta) \tag{2.4}$$

where, in symbolic notation,

$$\beta^n = \frac{B_n}{n}. \tag{2.5}$$

Proof. We exhibit here Lucas's proof as it can be found in [5]. A similar proof is provided by Vandiver in [9]. Lucas's argument begins with the identity

$$pB_n^{(p+1)} = (p-n)B_n^{(p)} - pnB_{n-1}^{(p)} \tag{2.6}$$

which follows directly from the identity for generating functions

$$x\frac{d}{dx}\left(\frac{x}{e^x-1}\right)^p = p(1-x)\left(\frac{x}{e^x-1}\right)^p - p\left(\frac{x}{e^x-1}\right)^{p+1}. \tag{2.7}$$

Shifting n to $n-1$ it follows that

$$pB_{n-1}^{(p+1)} = (p-n+1)B_{n-1}^{(p)} - p(n-1)B_{n-2}^{(p)}. \tag{2.8}$$

Now multiplying (2.6) by $n(p+1)$ and (2.8) by $(p-n+1)$ leads to

$$\begin{aligned} p(p+1)B_n^{(p+2)} &= (p-n+1)(p-n)B_n^{(p)} - (p-n+1)(p+p+1)nB_{n-1}^{(p)} \\ &\quad + p(p+1)n(n-1)B_{n-2}^{(p)} \end{aligned}$$

and then, by the same methods, he produces

$$\begin{aligned} (p+2)(p+1)pB_n^{(p+3)} &= (p-n+2)(p-n+1)(p-n)B_n^{(p)} \\ &\quad - (p-n+2)(p-n+1)(p+p+1+p+2)nB_{n-1}^{(p)} \\ &\quad + (p-n+2)(p(p+1)+p(p+2)+(p+1)(p+2))n(n-1)B_{n-2}^{(p)} \\ &\quad - p(p+1)(p+2)n(n-1)(n-2)B_{n-3}^{(p)} \end{aligned}$$

and then, stating '*and so on*', Lucas concludes the proof. □

The following alternate proof of Lucas's Theorem using generating functions requires an expression for $B_n^{(p)}$ in the range $0 \le n \le p-1$, of the kind given in (2.4). This cannot be obtained directly from (2.4) which holds only for $n \ge p$; however, using (2.4) with $n \le p$ and a limiting argument yields the desired result as follows: first, the Stirling numbers of the first kind $s_k^{(p)}$ are used to produce an equivalent formulation of $B_n^{(p)}$. These numbers are defined by the generating function

$$z(z-1)(z-2)\cdots(z-(p-1)) = \sum_{k=1}^{p} s_k^{(p)} z^k. \tag{2.9}$$

Then (2.4) is written as

$$\begin{aligned} B_n^{(p)} &= \frac{(-1)^{p-1}}{(p-1)!}n(n-1)\cdots(n-(p-1))\beta^{n-p}(-1)^p\sum_{k=1}^{p}s_k^{(p)}(-\beta)^k \\ &= -\frac{1}{(p-1)!}n(n-1)\cdots(n-(p-1))\sum_{k=1}^{p}s_k^{(p)}(-1)^k\frac{B_{n-p+k}}{n-p+k}. \end{aligned}$$

Observe that the index n varies in the range $0 \le n \le p-1$, therefore the prefactor $n(n-1)\cdots(n-(p-1))$ always vanishes. On the other hand, all the summands are finite, except for the value of the index $k = p-n$: the corresponding denominator is then equal to zero, but canceled by its null counterpart in the prefactor. Hence only this term is non zero, and equal to

$$-\frac{1}{(p-1)!}n(n-1)\cdots 1\times(-1)(-2)\cdots(-(p-1-n)))s_{p-n}^{(p)}(-1)^{p-n}=\frac{s_{p-n}^{(p)}}{\binom{p-1}{n}}.$$

This gives the following theorem.

Theorem 2.2. *The generalized Bernoulli numbers $B_n^{(p)}$, with $0\le n\le p-1$ are given by*

$$B_n^{(p)}=\frac{s_{p-n}^{(p)}}{\binom{p-1}{n}}. \tag{2.10}$$

In fact, this is a classical result. It is, for example, a direct consequence of the identity

$$(z-1)(z-2)\cdots(z-p)=\sum_{\ell=0}^{p}\binom{p}{\ell}z^{\ell}B_{p-\ell}^{(p+1)} \tag{2.11}$$

which appears (unnumbered) in [6, p. 149].

3. The Proof via Generating Function

The expressions given in (2.4) and (2.10), namely

$$\tilde{B}_n^{(p)}=\frac{(-1)^{p-1}}{(p-1)!}\frac{n!}{(n-p)!}\beta^{n-p+1}(1+\beta)(2+\beta)\cdots(p-1+\beta),\quad n\ge p$$

and

$$\tilde{B}_n^{(p)}=\frac{s_{p-n}^{(p)}}{\binom{p-1}{n}},\quad 0\le n\le p-1,$$

are now used to compute the generating function

$$G(z)=\sum_{n=0}^{\infty}\tilde{B}_n^{(p)}\frac{z^n}{n!} \tag{3.1}$$

and to show that it coincides with the generating function of the generalized Bernoulli numbers (1.1), proving that the numbers $\tilde{B}_n^{(p)}$ defined by identities (2.4) and (2.10) are indeed the generalized Bernoulli numbers $B_n^{(p)}$.

Split the sum as $G(z)=G_1(z)+G_2(z)$, where

$$G_1(z)=\sum_{n=0}^{p-1}\tilde{B}_n^{(p)}\frac{z^n}{n!}\quad\text{and}\quad G_2(z)=\sum_{n=p}^{\infty}\tilde{B}_n^{(p)}\frac{z^n}{n!}. \tag{3.2}$$

Observe that

$$\begin{aligned}G_2(z)&=\sum_{n=p}^{\infty}\frac{(-1)^{p-1}}{(p-1)!}\frac{n!}{(n-p)!}\beta^{n-p+1}(1+\beta)\cdots((p-1)+\beta)\frac{z^n}{n!}\\&=\frac{(-1)^{p-1}}{(p-1)!}\beta(1+\beta)\cdots(p-1+\beta)\sum_{n=p}^{\infty}\frac{n!}{(n-p)!}\beta^{n-p}\frac{z^n}{n!}\\&=\frac{(-1)^{p-1}}{(p-1)!}(-1)^p\sum_{k=1}^{p}s_k^{(p)}(-1)^k z^p f_k(z)\end{aligned}$$

with

$$f_k(z) = \sum_{n=p}^{\infty} \frac{B_{n-p+k}}{(n-p)!(n-p+k)} z^{n-p}. \tag{3.3}$$

A $(k-1)$st antiderivative of $f_k(z)$, denoted by $g_k(z)$, is

$$\begin{aligned} g_k(z) &= \sum_{n=p}^{\infty} \frac{B_{n-p+k}}{(n-p+k)!} z^{n-p+k-1} \\ &= z^{-1} \sum_{\ell=k}^{\infty} \frac{B_\ell}{\ell!} z^\ell \\ &= \frac{1}{z} \left[\frac{z}{e^z - 1} - \sum_{\ell=0}^{k-1} \frac{B_\ell}{\ell!} z^\ell \right], \end{aligned}$$

therefore,

$$\begin{aligned} f_k(z) &= \left(\frac{d}{dz}\right)^{k-1} \frac{1}{e^z-1} - \left(\frac{d}{dz}\right)^{k-1} \frac{1}{z} \\ &= \left(\frac{d}{dz}\right)^{k-1} \frac{1}{e^z-1} + \frac{(-1)^k (k-1)!}{z^k}. \end{aligned}$$

This gives

$$\begin{aligned} G_2(z) &= -\frac{z^p}{(p-1)!} \sum_{k=1}^{p} s_k^{(p)} (-1)^k f_k(z) \\ &= -\frac{z^p}{(p-1)!} \sum_{k=1}^{p} s_k^{(p)} (-1)^k \left(\frac{d}{dz}\right)^{k-1} \left[\frac{1}{e^z-1}\right] - \frac{z^p}{(p-1)!} \sum_{k=1}^{p} s_k^{(p)} \frac{(k-1)!}{z^k}. \end{aligned}$$

On the other hand,

$$\begin{aligned} G_1(z) &= \sum_{n=0}^{p-1} \tilde{B}_n^{(p)} \frac{z^n}{n!} \\ &= \sum_{n=0}^{p-1} \frac{s_{p-n}^{(p)}}{\binom{p-1}{n}} \frac{z^n}{n!} \\ &= \frac{1}{(p-1)!} \sum_{n=0}^{p-1} s_{p-n}^{(p)} (p-1-n)! z^n \\ &= \frac{1}{(p-1)!} \sum_{k=1}^{p} s_k^{(p)} (k-1)! z^{p-k}. \end{aligned}$$

This sum cancels the second term in the expression for $G_2(z)$. Hence,

$$G(z) = G_1(z) + G_2(z) = -\frac{z^p}{(p-1)!} \sum_{k=1}^{p} s_k^{(p)} (-1)^k \left(\frac{d}{dz}\right)^{k-1} \left[\frac{1}{e^z-1}\right]. \tag{3.4}$$

Using (2.9) gives

$$G(z) = -\frac{(-z)^p}{(p-1)!} \left((p-1) + \frac{d}{dz}\right) \cdots \left(1 + \frac{d}{dz}\right) \left[\frac{1}{e^z-1}\right]. \tag{3.5}$$

The next lemma simplifies this expression. Its proof by induction is elementary, so it is omitted.

Lemma 3.1. *For $n \geq 1$, the identity*

$$\frac{(-1)^n}{n!}\left(n+\frac{d}{dz}\right)\left(n-1+\frac{d}{dz}\right)\cdots\left(1+\frac{d}{dz}\right)\frac{1}{e^z-1}=\frac{1}{(e^z-1)^{n+1}} \tag{3.6}$$

holds.

Substituting in (3.5) produces

$$G(z)=-\frac{(-z)^p}{(p-1)!}\frac{(p-1)!}{(-1)^{p-1}}\frac{1}{(e^z-1)^p}=\left(\frac{z}{e^z-1}\right)^p, \tag{3.7}$$

which is the generating function of the generalized Bernoulli numbers. This proves both Lucas's formula for $B_n^{(p)}$ given in (2.4) with $n \geq p$ and in (2.10) for $0 \leq p \leq n-1$.

4. Lucas's Formula via Recurrences

The generalized Bernoulli numbers $B_n^{(p)}$ satisfy the recurrence

$$pB_n^{(p+1)}=(p-n)B_n^{(p)}-pnB_{n-1}^{(p)}. \tag{4.1}$$

Lucas's formula for $B_n^{(p)}$ is now established by showing that the numbers $\tilde{B}_n^{(p)}$ defined by

$$\tilde{B}_n^{(p)}=\frac{(-1)^{p-1}}{(p-1)!}\frac{n!}{(n-p)!}\beta^{n-p+1}(1+\beta)(2+\beta)\cdots(p-1+\beta), \quad n \geq p$$

satisfy the same recurrence.

Start with

$$(p-n)\tilde{B}_n^{(p)}-pn\tilde{B}_{n-1}^{(p)}=(p-n)\frac{(-1)^{p-1}n!}{(p-1)!(n-p)!}\beta^{n-p}\prod_{k=0}^{p-1}(k+\beta)-$$
$$pn\frac{(-1)^{p-1}n!}{(p-1)!(n-p-1)!}\beta^{n-1-p}\prod_{k=0}^{p-1}(k+\beta),$$

and write it as

$$\begin{aligned}(p-n)\tilde{B}_n^{(p)}-pn\tilde{B}_{n-1}^{(p)}&=\frac{(-1)^{p-1}n!}{(p-1)!(n-p-1)!}\beta^{n-1-p}\left[-\prod_{k=0}^{p-1}(k+\beta)-p\beta\prod_{k=0}^{p-1}(k+\beta)\right]\\&=\frac{(-1)^{p}n!}{(p-1)!(n-p-1)!}\beta^{n-1-p}(p+\beta)\prod_{k=0}^{p-1}(k+\beta)\\&=p\frac{(-1)^p}{p!}\frac{n!}{(n-p-1)!}\beta^{n-1-p}\prod_{k=0}^{p}(k+\beta)\\&=p\tilde{B}_n^{(p+1)}.\end{aligned}$$

To conclude the proof, it suffices to check that the initial conditions match. This is clear, since

$$\tilde{B}_n^{(1)}=\frac{n!}{(n-1)!}\beta^n=n\beta^n=n\frac{B_n}{n}=B_n. \tag{4.2}$$

This establishes Lucas's formula for the generalized Bernoulli numbers.

5. Polynomial Versions of Lucas's Formulas

To conclude this part of the paper about Lucas's formula, we provide here without proof its polynomial version. The generalized Bernoulli polynomials are defined by the generating function

$$\sum_{n=0}^{\infty} B_n^{(p)}(x)\frac{z^n}{n!} = \left(\frac{z}{e^z-1}\right)^p e^{xz}.$$

Theorem 5.1. *For $n \geq p$, the generalized Bernoulli polynomials are given by*

$$B_n^{(p)}(x) = \frac{(-1)^{p-1}}{(p-1)!}\frac{n!}{(n-p)!}\beta^{n-p+1}(x)\left(\beta(x)+1-x\right)\cdots\left(\beta(x)+p-1-x\right)$$

where, in symbolic notation,

$$\beta^n(x) = \frac{B_n(x)}{n}.$$

A proof can be obtained, for example, using the recurrence technique as in the previous section, given that the generalized Bernoulli polynomials satisfy the recurrence

$$pB_{n+1}^{(p+1)}(x) = (n+1)(x-p)B_n^{(p)}(x) + (p-n-1)B_{n+1}^{(p)}(x).$$

For completeness, we also provide an equivalent formula for the generalized Euler polynomials, defined by the generating function

$$\sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{z^n}{n!} = \left(\frac{2}{e^z+1}\right)^p e^{xz}.$$

Theorem 5.2. *For $n \geq p$, the generalized Euler polynomials are given by*

$$E_n^{(p)}(x) = \frac{2^{p-1}}{(p-1)!}(E+x)^n(E+1)\dots(E+p-1)$$

where, in symbolic notation,

$$E^n = E_n^{(1)}(0) = E_n, \quad (E+x)^n = E_n^{(1)}(x).$$

This identity can be checked using the recurrence

$$E_n^{(p+1)}(x) = \frac{2}{p}E_{n+1}^{(p)}(x) + \frac{2}{p}(p-x)E_n^{(p)}(x).$$

6. A New Approach to Dilcher's Formula

This section analyzes the sum

$$S_N(n) := \sum \binom{2n}{2j_1,\, 2j_2,\, \dots, 2j_N} B_{2j_1}B_{2j_2}\cdots B_{2j_N}, \tag{6.1}$$

using Lucas's expression for the generalized Bernoulli numbers $B_n^{(p)}$. An alternative formulation is presented.

Proposition 6.1. *The sum $S_N(n)$ is given by*

$$S_N(n) = \sum_{k=0}^{N} \frac{(2n)!}{(2n-k)!}2^{-k}\binom{N}{k}B_{2n-k}^{(N-k)} \tag{6.2}$$

for $2n > N$.

Proof. The umbral method [8] shows that the sum $S_N(n)$ is given by

$$S_N(n) = \frac{1}{2^N} \sum_{\epsilon_i = \pm 1} \left(\epsilon_1 B_1 + \cdots + \epsilon_N B_N\right)^{2n}. \tag{6.3}$$

Introduce the notation

$$Y_{2n}^{(M,N)} = \left(-B_1 - \cdots - B_M + B_{M+1} + \cdots + B_N\right)^{2n} \tag{6.4}$$

where there are M minus signs and $N-M$ plus signs. Thus,

$$S_N(n) = \frac{1}{2^N} \sum_{M=0}^{N} \binom{N}{M} Y_{2n}^{(M,N)}. \tag{6.5}$$

The next step uses the famous umbral identity

$$f(-B) = f(B) + f'(0) \tag{6.6}$$

(see Section 2 of [3] for details) to obtain

$$Y_{2n}^{(M,N)} = Y_{2n}^{(M-1,N)} + 2nY_{2n-1}^{(M-1,N-1)}. \tag{6.7}$$

This may be written as

$$Q_{2n}^{(M)} = Q_{2n}^{(M-1)} + 2nQ_{2n-1}^{(M-1)}, \tag{6.8}$$

where $Q_j^M = Y_j^{(M,P+j)}$ and $P = N-2n$. Then (6.8) is easily solved to produce

$$Q_{2n}^{(M)} = \sum_{k=0}^{M} \binom{M}{k} \frac{(2n)!}{(2n-k)!} Q_{2n-k}^{(0)}. \tag{6.9}$$

Since the initial condition is

$$Q_{2n-k}^{(0)} = Y_{2n-k}^{(0,N-k)} = B_{2n-k}^{(N-k)}, \tag{6.10}$$

it follows that

$$Y_{2n}^{(M,N)} = \sum_{k=0}^{M} \binom{M}{k} \frac{(2n)!}{(2n-k)!} B_{2n-k}^{(N-k)}. \tag{6.11}$$

Substituting in (6.5) yields

$$\begin{aligned}
S_N(n) &= \frac{1}{2^N} \sum_{M=0}^{N} \binom{N}{M} Y_{2n}^{(M,N)} \\
&= \frac{1}{2^N} \sum_{M=0}^{N} \binom{N}{M} \sum_{k=0}^{M} \binom{M}{k} \frac{(2n)!}{(2n-k)!} B_{2n-k}^{(N-k)} \\
&= \frac{1}{2^N} \sum_{k=0}^{N} \frac{(2n)!}{(2n-k)!} B_{2n-k}^{(N-k)} \sum_{M=0}^{N} \binom{M}{k}\binom{N}{M}.
\end{aligned}$$

Now use the basic identity (see [4, 3.118])

$$\sum_{M=0}^{N} \binom{M}{k}\binom{N}{M} = \sum_{M=k}^{N} \binom{M}{k}\binom{N}{M} = 2^{N-k}\binom{N}{k} \tag{6.12}$$

to obtain the result. □

Lucas's identity for generalized Bernoulli numbers is now used to obtain a second expression for the sum $S_N(n)$.

Proposition 6.2. *For $2n > N$, the sum $S_N(n)$ is given by*

$$S_N(n) = \frac{(2n)!}{(2n-N)!}\beta^{2n-N+1}\sum_{\ell=0}^{N-1}\binom{N}{\ell+1}\frac{(-1)^{\ell}}{2^{N-1-\ell}}\frac{(\beta+1)_{\ell}}{\ell!}. \tag{6.13}$$

Proof. Using the Pochhammer symbol

$$(\beta+1)_{p-1} = \frac{\Gamma(\beta+p)}{\Gamma(\beta+1)} = (\beta+1)\cdots(\beta+p-1), \tag{6.14}$$

Lucas's formula (2.4) is stated in the form

$$B_n^{(p)} = \frac{(-1)^{p-1}}{(p-1)!}\frac{n!}{(n-p)!}\beta^{n-p+1}(\beta+1)_{p-1}. \tag{6.15}$$

Using Proposition 6.1 and $B_n^{(0)} = \delta_n$ so that $B_{2n-N}^{(0)} = 0$ since $2n > N$, it follows that

$$\begin{aligned} S_N(n) &= \sum_{k=0}^{N-1}\frac{(2n)!}{(2n-k)!}2^{-k}\binom{N}{k}\frac{(-1)^{N-k-1}}{(N-k-1)!}\frac{(2n-k)!}{(2n-N)!}\beta^{2n-N+1}(\beta+1)_{N-k-1} \\ &= \frac{(2n)!}{(2n-N)!}\beta^{2n-N+1}\sum_{k=0}^{N-1}2^{-k}\binom{N}{k}\frac{(-1)^{N-k-1}}{(N-k-1)!}(\beta+1)_{N-k-1} \end{aligned}$$

that reduces to the stated form. $\square$

To obtain a hypergeometric form of the sum $S_N(n)$, observe that

$$N(1-N)_{\ell} = (-1)^{\ell}\frac{N!}{(N-\ell-1)!} \tag{6.16}$$

and $(2)_{\ell} = (\ell+1)!$ give

$$(-1)^{\ell}\binom{N}{\ell+1} = N\frac{(1-N)_{\ell}}{(2)_{\ell}}, \tag{6.17}$$

and the following result follows from Proposition 6.2.

Proposition 6.3. *For $2n > N$, the hypergeometric form of the sum $S_N(n)$ is given by*

$$S_N(n) = \frac{(2n)!}{(2n-N)!}\beta^{2n-N+1}2^{1-N}N\ {}_2F_1\left(\left.\begin{matrix}1-N, & 1+\beta \\ & 2\end{matrix}\right| 2\right). \tag{6.18}$$

The final form of the sum $S_N(n)$ involves the Meixner-Pollaczek polynomials defined by (see [7, 15.9.10])

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!}e^{in\phi}{}_2F_1\left(\left.\begin{matrix}-n, & \lambda+ix \\ & 2\lambda\end{matrix}\right| 1-e^{-2i\phi}\right). \tag{6.19}$$

Choosing $\lambda = 1$ and $\phi = \pi/2$ gives the next result.

Theorem 6.4. *For $2n > N$, the sum $S_N(n)$ is given by*

$$S_N(n) = \frac{(2n)!}{(2n-N)!}\frac{1}{(2i)^{N-1}}\beta^{2n-N+1}P_{N-1}^{(1)}\left(-i\beta;\frac{\pi}{2}\right). \tag{6.20}$$

Some examples are presented next.

Example 6.5. *The Meixner-Pollaczek polynomial*

$$P_2^{(1)}\left(x;\frac{\pi}{2}\right) = 2x^2 - 1 \tag{6.21}$$

gives

$$\begin{aligned} S_3(n) &= \frac{(2n)!}{(2n-3)!} \times (-1/4)\beta^{2n-2}(-2\beta^2 - 1) \\ &= \frac{(2n)(2n-1)(2n-2)}{4}\left[2\frac{B_{2n}}{2n} + \frac{B_{2n-2}}{2n-2}\right] \\ &= (2n-1)(n-1)B_{2n} + \tfrac{1}{2}n(2n-1)B_{2n-2}, \end{aligned}$$

which coincides with [2, eq. (2.6)].

Example 6.6. *The Meixner-Pollaczek of degree* 3 *is*

$$P_3^{(1)}\left(x;\frac{\pi}{2}\right) = \frac{4}{3}(-2x + x^3) \tag{6.22}$$

that produces

$$\begin{aligned} S_4(n) &= \frac{(2n)!}{(2n-4)!}\frac{1}{(2i)^3}\beta^{2n-3}\frac{4}{3}(2i\beta + i\beta^3) \\ &= -\tfrac{1}{3}(2n-1)(n-1)(2n-3)B_{2n} - \tfrac{1}{3}(2n)(2n-1)(2n-3)B_{2n-2}, \end{aligned}$$

which coincides with [2, eq. (2.7)].

The next step is to establish a correspondence between the *Dilcher coefficients* $b_k^{(N)}$ in (1.6) and the coefficients $p_k^{(n)}$ in

$$P_n^{(1)}(x;\frac{\pi}{2}) = \sum_{k=0}^{n} p_k^{(n)} x^k, \tag{6.23}$$

the Meixner-Pollaczek polynomials. In particular, it is shown that the recurrence (1.7) is a consequence of the classical three-term recurrence for orthogonal polynomials.

Theorem 6.7. *For* $2n > N$, *the coefficients* $b_k^{(N)}$ *defined in* (1.6) *and the coefficients* $p_k^{(n)}$ *of the Meixner-Pollaczek polynomial* $P_n^{(1)}(x;\frac{\pi}{2})$ *are related by*

$$b_k^{(N)} = \frac{(-1)^{N-1-k}}{2^{N-1}} p_{N-1-2k}^{(N-1)}. \tag{6.24}$$

The recurrence relation (1.7) *is equivalent to the three-term recurrence*

$$(n+1)P_{n+1}^{(1)}\left(x;\frac{\pi}{2}\right) - 2xP_n^{(1)}\left(x;\frac{\pi}{2}\right) + (n+1)P_{n-1}^{(1)}\left(x;\frac{\pi}{2}\right) = 0 \tag{6.25}$$

satisfied by the Meixner-Pollaczek polynomials.

Proof. The Meixner-Pollaczek polynomials are orthogonal, hence they satisfy a three-term recurrence. The specific form for this family in (6.25) appears in [7, Chapter 18]. In terms of its coefficients $p_k^{(n)}$ this is expressed as

$$(n+1)p_k^{(n+1)} - 2p_{k-1}^{(n)} + (n+1)p_k^{(n-1)} = 0. \tag{6.26}$$

Comparing the two expressions for $S_N(n)$ in (1.6) and (6.20) gives (6.24). This is equivalent to

$$p_\ell^{(N-1)} = 2^{N-1} i^{N-1+\ell} b_{\frac{1}{2}(N-1-\ell)}^{(N)}. \tag{6.27}$$

Replacing in (6.26) and simplifying yields (1.7). □

Theorem 2 in [2], stated below, may be proven along the same lines of the proof of Theorem 2.2. Details are omitted.

Theorem 6.8. *If* $2n \leq N-1$*, then*

$$\begin{aligned} S_N(n) &= (-1)^n \frac{(2n)!(N-2n-1)!}{2^{N-1}} p_{N-2n-1}^{(N-1)} \\ &= (-1)^{N-1}(2n)!(N-2n-1)! b_n^{(N)}. \end{aligned} \tag{6.28}$$

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