t-SION OF TWO POLYNOMIAL SEQUENCES AND FACTORIZATION PROPERTIES

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ABSTRACT. Certain second-order recurrence sequences (G_n) and (H_n) give the coefficients for sequences P and Q of polynomials in $\mathbb{R}[x]$. The t-sion of P and Q, denoted by $P \circ_t Q$, is then defined so as to generalize both the fusion and fission of P and Q. Specifically, $P \circ_t Q$ is the fusion of P and Q if $t = 1$ and the fission if $t = -1$. Choosing Q in a certain manner derived from P gives a sequence $\widetilde{P}$ for which $P \circ_t \widetilde{P}$ is the self t-sion of P. Explicit formulas are obtained for the polynomials in $P \circ_t \widetilde{P}$.

1. INTRODUCTION

Let A denote a positive real number, and consider the second-order recurrence sequence given by $G_0 = 0$, $G_1 = 1$, and

$$G_n = AG_{n-1} + G_{n-2}, \qquad n \geq 2.$$

The companion sequence $(H_n)_{n=0}^{\infty}$ of $(G_n)_{n=0}^{\infty}$ has the initial values $H_0 = 2$, $H_1 = A$, and

$$H_n = AH_{n-1} + H_{n-2}, \qquad n \geq 2.$$

Let $D = A^2 + 4$. It is known that the zeros α and β of the common characteristic polynomial $c(x) = x^2 - Ax - 1$ of the sequences $(G_n)_{n=0}^{\infty}$ and $(H_n)_{n=0}^{\infty}$ are distinct real numbers, say $\alpha > \beta$, and that

$$G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad H_n = \alpha^n + \beta^n. \tag{1.1}$$

Also, $\alpha + \beta = A$, $\alpha\beta = -1$, and $(\alpha - \beta)^2 = D$.

In this paper, we investigate certain polynomials whose coefficients come from the two sequences above, thus generalizing results in Kimberling [1] in two ways. First, the present work involves the two sequences $(G_n)_{n=0}^{\infty}$ and $(H_n)_{n=0}^{\infty}$, whereas in [1], only the Fibonacci sequence is studied. Second, the operations of fusion and fission in [1] are shown to be special cases of a single operation.

Let t denote an arbitrary integer, let

$$\omega(x) = \omega_u x^u + \omega_{u-1} x^{u-1} + \cdots + \omega_1 x + \omega_0,$$

be a polynomial in $\mathbb{R}[x]$, and let $Q = (q_n(x))_{n=0}^{\infty}$ be a sequence in $\mathbb{R}[x]$. For $u + t \geq 0$ the *(Q, t)-step of* $\omega(x)$ is the polynomial in $\mathbb{R}[x]$ defined by

$$h_t(\omega(x)) = \omega_u q_{u+t}(x) + \omega_{u-1} q_{u-1+t}(x) + \cdots + \omega_\tau q_{\tau+t}(x),$$

where

$$\tau = \begin{cases} 0, & \text{if } t \geq 0; \\ |t|, & \text{if } t < 0, \end{cases} \tag{1.2}$$

otherwise let $h_t(\omega(x))$ be the zero polynomial. Now taking another sequence $P = (p_n(x))_{n=0}^{\infty}$ in $\mathbb{R}[x]$, we define the *t-sion of P by Q*, denoted by $P \circ_t Q$, as the sequence $R = (r_n(x))_{n=0}^{\infty}$ of polynomials

$$r_n(x) = h_t(p_n(x)).$$

For $t = 1$ and $t = -1$, the sequences $P \circ_t Q$ are the fusion and fission of P and Q, respectively, as defined in [1].

The subsequence $(r_n(x))_{n=0}^{k}$ can be represented as follows. Let

$$D_k = \max_{n=0\ldots k}\{\deg(p_n(x))\} - \tau,$$

where τ has been defined in (1.2). Clearly, it suffices to assume that $D_k \geq 0$. The nth row $(1 \leq n \leq k+1)$ of the matrix $\mathcal{P}_{k,t} \in \mathbb{R}^{(k+1)\times(D_k+1)}$ includes the coefficients of

$$p_{n-1}(x) = p_{n-1,u}x^u + p_{n-1,u-1}x^{u-1} + \cdots + p_{n-1,1}x + p_{n-1,0},$$

and their positions from right to left, starting with the coefficient of the term of least degree, are given by

$$[0 \quad \ldots \quad 0 \quad p_{n-1,u} \quad p_{n-1,u-1} \quad \ldots \quad p_{n-1,\tau+1} \quad p_{n-1,\tau}].$$

(Every entry is zero if $u + t < 0$.) We also define the matrix $\mathcal{Q}_{k,t} \in \mathbb{R}^{(D_k+1)\times(D_k'+1)}$, where

$$D_k' = \max_{j=0\ldots D_k}\{\deg(q_{\tau+t+j}(x))\},$$

corresponding to $\mathcal{P}_{k,t}$, but with two differences. First, the rows, by starting at the bottom of the matrix, consist of the coefficients of the polynomials $q_{\tau+t}(x)$, $q_{\tau+t+1}(x)$, $\ldots$, $q_{\tau+t+D_k}(x)$, respectively. Second, the last column consists entirely of constants. Obviously,

$$\mathcal{P}_{k,t}\mathcal{Q}_{k,t} = \mathcal{R}_{k,t} \in \mathbb{R}^{(k+1)\times(D_k'+1)}.$$

If one assumes that $\deg(p_n(x)) = \deg(q_n(x)) = n$, then, except for trivial cases, for $k \geq \tau$ we find

$$D_k = \begin{cases} k, & \text{if } t \geq 0; \\ k - \tau = k + t, & \text{if } t < 0, \end{cases} \qquad \text{and} \qquad D_k' = k + t.$$

Now let $(a_n)_{n=0}^{\infty}$ be a fixed sequence of nonzero real numbers. The rest of this paper is restricted to these pairs of polynomials:

$$p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$
$$q_n(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n.$$

We shall rewrite Q as $\widetilde{P}$ in order to match $(q_n(x))$ with $(p_n(x))$. We call $P \circ_t \widetilde{P}$ the *self t-sion of P*.

It is easy to see that

$$\mathcal{P}_{k,t}(i,j) = \begin{cases} 0, & \text{if } i + j \leq k+1, \\ a_{k+1-j}, & \text{if } i + j > k+1, \end{cases} \qquad \begin{array}{l} 1 \leq i \leq k+1; \\ 1 \leq j \leq D_k + 1, \end{array}$$

$$\mathcal{Q}_{k,t}(i,j) = \begin{cases} 0, & \text{if } i > j, \\ a_{j-i}, & \text{if } i \leq j, \end{cases} \qquad \begin{array}{l} 1 \leq i \leq D_k + 1; \\ 1 \leq j \leq D_k' + 1. \end{array}$$

2. First Type: $a_n = G_{n+1}$

In this section, we develop representations for $r_n(x)$ in the case that $a_n = G_{n+1}$, noting that $G_0 = 0$. Assume first that $t \geq 0$. Expanding the product of the matrices

$$\mathcal{P}_{k,t} = \left[\begin{array}{ccccccccc}
 & & & & & & & & G_1 \\
 & & & & & & & G_2 & G_1 \\
 & & & & & & \iddots & \vdots & \vdots \\
 & & & & & \iddots & & \vdots & \vdots \\
 & & & & \iddots & & & \vdots & \vdots \\
0 & \dots & 0 & G_n & \dots & G_{n-j} & \dots & G_2 & G_1 \\
 & & \iddots & & & & & \vdots & \vdots \\
 & G_k & & & \dots & & & G_2 & G_1 \\
G_{k+1} & G_k & & & \dots & & & G_2 & G_1
\end{array}\right] \tag{2.1}$$

and

$$\mathcal{Q}_{k,t} = \left[\begin{array}{ccccccccc:ccc}
G_1 & G_2 & & & \dots & & & G_k & G_{k+1} & G_{k+2} & \dots & G_{k+t+1} \\
 & G_1 & & & \dots & & & G_{k-1} & G_k & G_{k+1} & \dots & G_{k+t} \\
 & & \ddots & & & & & \vdots & \vdots & \vdots & & \vdots \\
0 & \dots & 0 & G_1 & \dots & G_j & \dots & G_{n-1} & G_n & G_{n+1} & \dots & G_{n+t} \\
 & & & & \ddots & & & \vdots & \vdots & \vdots & & \vdots \\
 & & & & & \ddots & & \vdots & \vdots & \vdots & & \vdots \\
 & & & & & & \ddots & \vdots & \vdots & \vdots & & \vdots \\
 & & & & & & & G_1 & G_2 & G_3 & \dots & G_{t+2} \\
 & & & & & & & & G_1 & G_2 & \dots & G_{t+1}
\end{array}\right] \tag{2.2}$$

shows, for $1 \leq n \leq k+1$, that

$$\begin{aligned}
r_{n-1}(x) = {} & G_n \left((G_1 x^{n+t-1} + G_2 x^{n+t-2} + \cdots + G_n x^t) + (G_{n+1} x^{t-1} + \cdots + G_{n+t})\right) \\
& + G_{n-1} \left((G_1 x^{n+t-2} + \cdots + G_{n-1} x^t) + (G_n x^{t-1} + \cdots + G_{n-1+t})\right) + \\
& \vdots \\
& + G_{n-j} \left((G_1 x^{n+t-1-j} + \cdots + G_{n-j} x^t) + (G_{n+1-j} x^{t-1} + \cdots + G_{n-j+t})\right) + \\
& \vdots \\
& + G_1 \left((G_1 x^t) + (G_2 x^{t-1} + \cdots + G_{t+1})\right).
\end{aligned}$$

Hence,

$$\begin{aligned} r_{n-1}(x) = & \left(G_nG_1\right)x^{n+t-1} + \left(G_nG_2 + G_{n-1}G_1\right)x^{n+t-2} + \\ & \vdots \\ & + \left(G_n^2 + G_{n-1}^2 + \cdots + G_1^2\right)x^t \\ & - \\ & + \left(G_nG_{n+1} + G_{n-1}G_n + \cdots + G_1G_2\right)x^{t-1} + \\ & \vdots \\ & + \left(G_nG_{n+t} + G_{n-1}G_{n+t-1} + \cdots + G_1G_{t+1}\right). \end{aligned}$$

Applying Corollary 4.2 (of Lemma 4.1, in Section 4), we have

$$\begin{aligned} r_{n-1}(x) = & \frac{H_{n+2} - H_{n-2}}{AD}x^{n+t-1} + \frac{H_{n+3} - H_{n-1}}{AD}x^{n+t-2} + \frac{H_{n+4} - H_{n-4}}{AD}x^{n+t-3} + \cdots \\ & + \frac{H_{2n+1} - (-1)^n}{AD}x^t \\ & - \\ & + \frac{H_{2n+2} - H_{1+(-1)^n}}{AD}x^{t-1} + \frac{H_{2n+3} - H_{2+(-1)^n}}{AD}x^{t-2} + \cdots + \frac{H_{2n+t+1} - H_{t+(-1)^n}}{AD}. \end{aligned}$$

Finally, by Lemma 4.4, we conclude that

$$\begin{aligned} Ar_{n-1}(x) = & \left(G_nG_2x^{n+t-1} + G_{n+1}G_2x^{n+t-2}\right) + \left(G_nG_4x^{n+t-3} + G_{n+1}G_4x^{n+t-4}\right) + \cdots \\ & + \begin{cases} \left(G_nG_nx^{t+1} + G_{n+1}G_nx^t\right), & \text{if } n \equiv 0 \pmod 2, \\ G_{n+1}G_nx^t, & \text{if } n \equiv 1 \pmod 2, \end{cases} \\ & - \\ & + \begin{cases} G_{n+2}G_nx^{t-1} + G_{n+3}G_nx^{t-2} + \cdots + G_{n+t+1}G_n, & \text{if } n \equiv 0 \pmod 2, \\ G_{n+1}G_{n+1}x^{t-1} + G_{n+2}G_{n+1}x^{t-2} + \cdots + G_{n+t}G_{n+1}, & \text{if } n \equiv 1 \pmod 2. \end{cases} \end{aligned}$$

Hence, we have proved the following theorem.

Theorem 2.1. *If $n \geq 2$ is even, then*

$$\begin{aligned} Ar_{n-1}(x) = & \left(G_nx + G_{n+1}\right)\left(G_2x^{n+t-2} + G_4x^{n+t-4} + \cdots + G_nx^t\right) \\ & + G_n\left(G_{n+2}x^{t-1} + G_{n+3}x^{t-2} + \cdots + G_{n+t+1}\right), \end{aligned}$$

if $n \geq 1$ is odd, then

$$\begin{aligned} Ar_{n-1}(x) = & \left(G_nx + G_{n+1}\right)\left(G_2x^{n+t-2} + G_4x^{n+t-4} + \cdots + G_{n-1}x^{t+1}\right) \\ & + G_{n+1}\left(G_nx^t + G_{n+1}x^{t-1} + \cdots + G_{n+t}\right). \end{aligned}$$

If $A = 1$ and $t = 1$, then Theorem 2.1 gives Theorem 4.1 in [1]. Choosing $A = 1$ and $t = -1$ gives Theorem 4.2 of [1].

Assume now that $t < 0$. Then expanding the product of

$$\mathcal{P}_{k,t} = \left[\begin{array}{ccccccccc}
0 & & & & \cdots & & & & 0 \\
\vdots & & & & \vdots & & & & \vdots \\
0 & & & & \cdots & & & & 0 \\
\hline
 & & & & & & & & G_{\tau+1} \\
 & & & & & & & G_{\tau+2} & G_{\tau+1} \\
 & & & & & & \iddots & \vdots & \vdots \\
 & & & & & \iddots & & \vdots & \vdots \\
 & & & & \iddots & & & \vdots & \vdots \\
0 & \cdots & 0 & G_n & \cdots & G_{n-j} & \cdots & G_{\tau+2} & G_{\tau+1} \\
 & & \iddots & & & & & \vdots & \vdots \\
 & G_k & & & \cdots & & & G_{\tau+2} & G_{\tau+1} \\
G_{k+1} & G_k & & & \cdots & & & G_{\tau+2} & G_{\tau+1}
\end{array}\right] \tag{2.3}$$

and

$$\mathcal{Q}_{k,t} = \left[\begin{array}{ccccccccc}
G_1 & G_2 & & & \cdots & & & G_{k-\tau} & G_{k+1-\tau} \\
 & G_1 & & & \cdots & & & G_{k-1-\tau} & G_{k-\tau} \\
 & & \ddots & & & & & \vdots & \vdots \\
0 & \cdots & 0 & G_1 & \cdots & G_j & \cdots & G_{n-1-\tau} & G_{n-\tau} \\
 & & & & \ddots & & & \vdots & \vdots \\
 & & & & & \ddots & & \vdots & \vdots \\
 & & & & & & \ddots & \vdots & \vdots \\
 & & & & & & & G_1 & G_2 \\
 & & & & & & & & G_1
\end{array}\right] \tag{2.4}$$

in the manner already shown for $t \geq 0$ (we omit the details), gives the following theorem.

Theorem 2.2. *If $n - \tau \geq 1$ is even, then*

$$Ar_{n-1}(x) = (G_n x + G_{n+1})\left(G_2 x^{n-\tau-2} + G_4 x^{n-\tau-4} + \cdots + G_{n-\tau}\right);$$

if $n - \tau \geq 1$ is odd, then

$$Ar_{n-1}(x) = x\,(G_n x + G_{n+1})\left(G_2 x^{n-\tau-3} + G_4 x^{n-\tau-5} + \cdots + G_{n-\tau-1}\right) + G_n G_{n-\tau+1}.$$

3. Second Type: $a_n = H_n$

Here we represent the polynomials $r_n(x)$ in the case that $a_n = H_n$. Replacing G_{i+1} by H_i in the matrices (2.1) and (2.2) if $t \geq 0$, and in the matrices (2.3) and (2.4) if $t < 0$, gives matrices for the polynomials having coefficients in the sequence $(H_n)_{n=0}^{\infty}$. The method and calculations are nearly identical to those in the proof of Theorem 2.1.

The following two theorems are our final results. We prove only the first result, as the proof of the second is quite similar.

Theorem 3.1. *Let $t \geq 0$. If $n \geq 2$ is even, then*

$$\begin{aligned}
Ar_{n-1}(x) = A&\left((2H_{n-1}x^{n+t-1} + H_{n-2}x^{n+t-2} + \cdots + 2H_1 x^{t+1} + H_0 x^t) + (H_1 x^{t-1} + \cdots + H_t)\right) \\
&+ \left(H_n x^2 + DG_n x + H_n\right)\left(G_2 x^{n+t-4} + G_4 x^{n+t-6} + \cdots + G_{n-2}x^t\right) + H_n G_n x^t \\
&+ H_{n-1}\left(H_{n+1}x^{t-1} + \cdots + H_{n+t}\right).
\end{aligned}$$

If $n \geq 1$ is odd, then

$$\begin{aligned} Ar_{n-1}(x) = A & \left((2H_{n-1}x^{n+t-1} + H_{n-2}x^{n+t-2} + \cdots + H_1x^{t+1} + 2H_0x^t) + (2H_1x^{t-1} + \cdots + 2H_t)\right) \\ & + \left(H_nx^2 + DG_nx + H_n\right)\left(G_2x^{n+t-4} + G_4x^{n+t-6} + \cdots + G_{n-3}x^{t+1}\right) \\ & + H_nG_{n-1}x^{t+1} + DG_nG_{n-1}x^t \\ & + DG_{n-1}\left(G_{n+1}x^{t-1} + \cdots + G_{n+t}\right). \end{aligned}$$

Theorem 3.2. *Let $t < 0$. If $n - \tau \geq 1$ is even, then*

$$\begin{aligned} Ar_{n-1}(x) = A & \left(2H_{n-1}x^{n-\tau-1} + H_{n-2}x^{n-\tau-2} + 2H_{n-3}x^{n-\tau-3} + \cdots + H_\tau\right) \\ & + \left(H_nx^2 + DG_nx + H_n\right)\left(G_2x^{n-\tau-4} + G_4x^{n-\tau-6} + \cdots + G_{n-\tau-2}\right) \\ & + G_{n-\tau}H_n. \end{aligned}$$

If $n - \tau$ is odd, then

$$\begin{aligned} Ar_{n-1}(x) = A & \left(2H_{n-1}x^{n-\tau-1} + H_{n-2}x^{n-\tau-2} + 2H_{n-3}x^{n-\tau-3} + \cdots + 2H_\tau\right) \\ & + \left(H_nx^2 + DG_nx + H_n\right)\left(G_2x^{n-\tau-4} + G_4x^{n-\tau-6} + \cdots + G_{n-\tau-3}x\right) \\ & + G_{n-\tau-1}\left(H_nx + DG_n\right). \end{aligned}$$

Proof. (*Theorem 3.1.*)

$$\begin{aligned} r_{n-1}(x) = H_{n-1} & \left((H_0x^{n+t-1} + H_1x^{n+t-2} + \cdots + H_{n-1}x^t) + (H_nx^{t-1} + \cdots + H_{n+t-1})\right) \\ & + H_{n-2}\left((H_0x^{n+t-2} + \cdots + H_{n-2}x^t) + (H_{n-1}x^{t-1} + \cdots + H_{n-2+t})\right) + \\ & \vdots \\ & + H_{n-j}\left((H_0x^{n+t-j} + \cdots + H_{n-j}x^t) + (H_{n+1-j}x^{t-1} + \cdots + H_{n+t-j})\right) + \\ & \vdots \\ & + H_0\left((H_0x^t) + (H_1x^{t-1} + \cdots + H_t)\right). \end{aligned}$$

Thus,

$$\begin{aligned} r_{n-1}(x) = & \left(H_{n-1}H_1\right)x^{n+t-1} + \left(H_{n-1}H_1 + H_{n-2}H_0\right)x^{n+t-2} \\ & + \left(H_{n-1}H_2 + H_{n-2}H_1 + H_{n-3}H_0\right)x^{n+t-3} \\ & + \left(H_{n-1}H_3 + H_{n-2}H_2 + H_{n-3}H_1 + H_{n-4}H_0\right)x^{n+t-4} + \\ & \vdots \\ & + \left(H_{n-1}^2 + H_{n-2}^2 + \cdots + H_0^2\right)x^t \\ & ------------------------------ \\ & + \left(H_{n-1}H_n + H_{n-2}H_{n-1} + \cdots + H_0H_1\right)x^{t-1} + \\ & \vdots \\ & + \left(H_{n-1}H_{n+t-1} + H_{n-2}H_{n+t-2} + \cdots + H_0H_t\right). \end{aligned}$$

Applying Lemma 4.3 gives

$$Ar_{n-1}(x) = 2AH_{n-1}x^{n+t-1} + (H_{n+1} - H_{n-1} + AH_{n-2})x^{n+t-2} + (H_{n+2} - H_{n-2} + 2AH_{n-3})x^{n+t-3}$$
$$+ (H_{n+3} - H_{n-3} + AH_{n-4})x^{n+t-4} + \cdots$$
$$+ \left(H_{2n-2} - H_2 + \frac{3+(-1)^{n-2}}{2}AH_1\right)x^{t+1} + \left(H_{2n-1} - H_1 + \frac{3+(-1)^{n-1}}{2}AH_0\right)x^t$$
$$- -$$
$$+ \left(H_{2n} - H_2 + \frac{3+(-1)^{n-1}}{2}AH_1\right)x^{t-1} + \cdots + \left(H_{2n+t-1} - H_{t+1} + \frac{3+(-1)^{n-1}}{2}AH_0\right).$$

Let

$$h_0(x) = (2H_{n-1}x^{n+t-1} + H_{n-2}x^{n+t-2} + \cdots + 2H_1x^{t+1} + H_0x^t) + (H_1x^{t-1} + \cdots + H_t),$$
$$h_1(x) = (2H_{n-1}x^{n+t-1} + H_{n-2}x^{n+t-2} + \cdots + H_1x^{t+1} + 2H_0x^t) + (2H_1x^{t-1} + \cdots + 2H_t).$$

If n is even, we apply Lemmata 4.4 and 4.5 to find

$$\begin{aligned} Ar_{n-1}(x) = {} & Ah_0(x) \\ & + H_n(G_0+G_2)x^{n+t-2} + DG_nG_2x^{n+t-3} + H_n(G_2+G_4)x^{n+t-4} + DG_nG_2x^{n+t-5} \\ & + \cdots + DG_nG_{n-2}x^{t+1} + H_n(G_n+G_{n-2})x^t \\ & - \\ & + H_{n+1}(G_n+G_{n-2})x^{t-1} + \cdots + H_{n+t}(G_n+G_{n-2})x^{t-1}. \end{aligned}$$

Separating the appropriate parts completes the proof for even n.

Suppose now that n is odd, and apply again Lemmata 4.4 and 4.5 to obtain

$$\begin{aligned} Ar_{n-1}(x) = {} & h_1(x) \\ & + H_n(G_0+G_2)x^{n+t-2} + DG_nG_2x^{n+t-3} + H_n(G_2+G_4)x^{n+t-4} + DG_nG_2x^{n+t-5} \\ & + \cdots + H_n(G_{n-1}+G_{n-3})x^{t+1} + DG_nG_{n-1}x^t \\ & - \\ & + DG_{n+1}G_{n-1}x^{t-1} + \cdots + DG_{n+t}G_{n-1}x^t, \end{aligned}$$

which leads immediately to a proof for odd n. □

4. Lemmata

We introduce

$$\varepsilon_k = \begin{cases} 0, & \text{if } k \equiv 0 \pmod 2; \\ 1, & \text{if } k \equiv 1 \pmod 2). \end{cases}$$

Lemma 4.1. *Let $s \geq 0$ be an integer. Then*

$$\sum_{i=1}^{k} G_iG_{i+s} = \frac{H_{2k+s+1} - H_{s+1}}{AD} + \frac{\varepsilon_k H_s}{D}.$$

Proof. Similar to the proof of the next lemma. □

Corollary 4.2. *The sum above is*

$$\frac{H_{2k+s+1} - H_{s+1}}{AD} \quad \text{or} \quad \frac{H_{2k+s+1} - H_{s-1}}{AD}$$

if k is even or odd, respectively. Indeed, in (4.1), we have $H_n = 0$ if k is even and $H_n = H_s/D$ if k is odd.

Lemma 4.3. *Let $s \geq 0$ be an integer. Then*

$$\sum_{i=1}^{k} H_i H_{i+s} = \frac{H_{2k+s+1} - H_{s+1}}{A} - \varepsilon_k H_s$$

and

$$\sum_{i=0}^{k} H_i H_{i+s} = \frac{H_{2k+s+1} - H_{s+1}}{A} + \frac{3 + (-1)^k}{2} H_s.$$

Proof. We will apply the identitites $\alpha\beta = -1$, $\alpha^2 - 1 = A\alpha$, and $\beta^2 - 1 = A\beta$.

$$\begin{aligned}\sum_{i=1}^{k} H_i H_{i+s} &= \sum_{i=1}^{k} (\alpha^i + \beta^i)(\alpha^{i+s} + \beta^{i+s}) = \sum_{i=1}^{k} \alpha^{2i+s} + \beta^{2i+s} + (\alpha\beta)^i(\alpha^i + \beta^i) \\ &= \alpha^{s+2} \sum_{i=0}^{k-1} (\alpha^2)^i + \beta^{s+2} \sum_{i=0}^{k-1} (\beta^2)^i - H_s \sum_{i=0}^{k-1} (-1)^i \\ &= \alpha^{s+2} \frac{\alpha^{2k} - 1}{\alpha^2 - 1} + \beta^{s+2} \frac{\beta^{2k} - 1}{\beta^2 - 1} - H_s \frac{(-1)^k - 1}{-2} \\ &= \frac{\alpha^{2k+s+1} + \beta^{2k+s+1}}{A} - \frac{\alpha^{s+1} + \beta^{s+1}}{A} + H_s \frac{(-1)^k - 1}{2},\end{aligned}$$

and the proof is complete. □

Lemma 4.4. *Let $x \geq y$ be non-negative integers. Then*

$$H_{x+y} - H_{x-y} = \begin{cases} DG_x G_y, & \text{if } y \equiv 0 \pmod 2; \\ H_x H_y, & \text{if } y \equiv 1 \pmod 2. \end{cases}$$

Proof. This is an easy consequence of (1.1). □

Lemma 4.5. *If x is a positive integer, then $H_x = G_{x-1} + G_{x+1}$.*

Proof. Use the explicit formulae (1.1). □

References

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