ON A GENERALIZED PELL EQUATION STUDIED BY EULER AND SADEK

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Abstract. In this paper, we follow the path set forth by Euler and Sadek in their study of a generalized Pell equation. Euler and Sadek effectively studied the positive rational solutions of the Pell equation in question. Here, we study the positive integer solutions of this equation. Indeed, by imposing additional conditions on certain parameters, we demonstrate that the positive rational solutions produced by a certain system of Euler and Sadek turn out to be positive integer solutions. Furthermore, the positive integer solutions produced by this system are the only positive integer solutions of the Pell equation in question.

1. Introduction

The standard Fibonacci and Lucas numbers have starting values $(F_0, F_1) = (0, 1)$, and $(L_0, L_1) = (2, 1)$, respectively. The identity

$$L_n^2 - 5F_n^2 = 4(-1)^n, \tag{1.1}$$

which occurs in [3, p. 56], inspires the two Pell Diophantine equations

$$x^2 - 5y^2 = 4 \text{ and } x^2 - 5y^2 = -4. \tag{1.2}$$

All the integer solutions, $x > 0$ and $y > 0$, of the first equation in (1.2) are (L_{2n}, F_{2n}), $n \geq 1$. All the integer solutions, $x > 0$ and $y > 0$, of the second equation in (1.2) are (L_{2n-1}, F_{2n-1}), $n \geq 1$.

Over the years, various methods have been employed to solve the Diophantine equations in (1.2). Long and Jordan [6] solve both the equations in (1.2) with the use of continued fractions. Later, Lind [5] solves these equations by working in the quadratic field $Q\left(\sqrt{5}\right)$. Then, in a letter to the editor, Ferguson [2] simultaneously solves these equations with a clever method of descent.

Motivated by a desire to generalize the first of the equations in (1.2), Euler and Sadek [1] consider the Pell equation

$$x^2 - dy^2 = a^2, \tag{1.3}$$

in which a and d are positive integers, and where $a^2 + d = b^2$, for some positive integer b. Clearly $b > a > 0$. In this paper, we consider only those a and b for which $\sqrt{d}$ is irrational. These constraints on the parameters a, b, and d are assumed throughout this paper, and henceforth we do not restate them.

We take an *integer* solution (x, y) of (1.3) to be a solution in which both x and y are integers. We take a *rational* solution (x, y) of (1.3) to be a solution in which both x and y are rational. An integer solution is a rational solution, but a rational solution is not necessarily an integer solution. A positive solution is one where $x > 0$ and $y > 0$. Throughout this paper, we always indicate the type of solution that we are considering. As is customary in this topic, we refer to the solution (x, y), or to the solution $x + y\sqrt{d}$, interchangeably.

For $n \geq 1$, Euler and Sadek [1] define

$$\begin{aligned} x_{n+1} &= \frac{bx_n + dy_n}{a}, \\ y_{n+1} &= \frac{x_n + by_n}{a}, \end{aligned} \tag{1.4}$$

with the smallest positive solution $(x_1, y_1) = (b, 1)$. Given the constraints on a, b, d, they effectively prove [1, p. 243] that all (x_n, y_n) generated by (1.4) are positive rational solutions of (1.3). For instance, with $a = 3$ and $b = 4$, the first three solutions generated by (1.4) are $(4, 1)$, $\left(\frac{23}{3}, \frac{8}{3}\right)$, and $\left(\frac{148}{9}, \frac{55}{9}\right)$. Indeed, without further constraints on the parameters in (1.3), the paper of Euler and Sadek is effectively a study of the positive rational solutions of (1.3) that are produced by (1.4).

Our first task in this paper is to show that (1.4) produces only positive integer solutions of (1.3) when $a|(2b)$. We demonstrate this in Section 3, where we express these solutions in terms of two second order recurring sequences. We define these second order recurring sequences, and prove certain identities that we require, in Section 2.

Our second task is to show that, when $a|(2b)$, the positive integer solutions produced by (1.4) are the *only* solutions of (1.3). We proceed by considering two cases. In Section 4 we consider $a|(2b)$ with a even, and in Section 5 we consider $a|(2b)$ with a odd.

2. Two Second Order Recurring Sequences

For an integer $p > 2$, we require the sequences U_n and V_n defined, for all integers n, by

$$\begin{aligned} U_n &= U_n(p) = pU_{n-1} - U_{n-2}, U_0 = 0, U_1 = 1, \\ V_n &= V_n(p) = pV_{n-1} - V_{n-2}, V_0 = 2, V_1 = p. \end{aligned} \tag{2.1}$$

Let α and β denote the two distinct real roots of $x^2 - px + 1 = 0$. Then the closed forms (the Binet forms) for U_n and V_n are

$$\begin{aligned} U_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta}, \\ V_n &= \alpha^n + \beta^n. \end{aligned}$$

For the remainder of this paper, we take $p = \frac{2b}{a}$ in (2.1), where a and b are as in the definition of (1.3). Accordingly, for this value of p, we see from the recurrence for U_n, that

$$aU_{n+1} - bU_n = bU_n - aU_{n-1}, \tag{2.2}$$

for all n.

We now proceed by induction to prove the identity

$$aV_n = 2\left(bU_n - aU_{n-1}\right), n \geq 1, \tag{2.3}$$

which links the sequences U_n and V_n. By substitution, we see that (2.3) is true for $n = 1$ and $n = 2$. Now assume that, for some positive integer $k \geq 2$, (2.3) is true for all $1 \leq n \leq k$. From the recurrence for V_n, we have

$$\begin{aligned} aV_{k+1} &= 2bV_k - aV_{k-1} \\ &= \frac{2b}{a}(aV_k) - (aV_{k-1}) \\ &= \frac{4b}{a}(bU_k - aU_{k-1}) - 2(bU_{k-1} - aU_{k-2}) \text{ by (2.3)} \\ &= \frac{4b}{a}(aU_{k+1} - bU_k) - 2(aU_k - bU_{k-1}) \text{ by (2.2).} \end{aligned}$$

Finally, we use the recurrence for U_n to replace U_{k-1} in the line above by $\frac{2b}{a}U_k - U_{k+1}$, then expand the entire line to obtain $2(bU_{k+1} - aU_k)$. This proves (2.3).

Notice that since $b > a$, $p = \frac{2b}{a} > 2$, so that the terms in both the sequences (2.1) are positive for $n \geq 1$.

3. The Condition $a|(2b)$ and Positive Integer Solutions

Our first lemma states that the condition $a|(2b)$ guarantees that (1.4) generates only positive integer solutions of (1.3). Furthermore, this lemma expresses these solutions in terms of the integer sequences U_n and V_n.

Lemma 3.1. *The system (1.4) generates only positive integer solutions of (1.3) when $a|(2b)$. With $p = \frac{2b}{a}$ in (2.1), these solutions are given by*

$$(x_n, y_n) = (bU_n - aU_{n-1}, U_n) = \left(\frac{aV_n}{2}, U_n\right), n \geq 1. \tag{3.1}$$

Proof. Suppose $a|(2b)$. Then, with $p = \frac{2b}{a}$, both the sequences in (2.1) are integer sequences. Furthermore, it is easily verified that

$$\begin{aligned} (x_1, y_1) &= (bU_1 - aU_0, U_1) = (b, 1), \\ (x_2, y_2) &= (bU_2 - aU_1, U_2) = \left(\frac{2b^2}{a} - a, \frac{2b}{a}\right), \\ (x_3, y_3) &= (bU_3 - aU_2, U_3) = \left(\frac{4b^3}{a^2} - 3b, \frac{4b^2}{a^2} - 1\right). \end{aligned}$$

We now proceed by induction. Suppose that, for some integer $k \geq 1$,

$$(x_k, y_k) = (bU_k - aU_{k-1}, U_k).$$

Then, with the use of (1.4), we have

$$\begin{aligned} x_{k+1} &= \frac{bx_k + dy_k}{a} \\ &= \frac{b(bU_k - aU_{k-1}) + (b^2 - a^2)U_k}{a} \\ &= \frac{b(aU_{k+1} - bU_k) + (b^2 - a^2)U_k}{a} \qquad \text{by (2.2)} \\ &= bU_{k+1} - aU_k. \end{aligned}$$

Similarly, $y_{k+1} = U_{k+1}$. Finally, we complete the proof of Lemma 3.1 with the use of (2.3). □

It remains for us to show that *all* the positive integer solutions of (1.3) are generated by the system (1.4). We achieve this by considering separately the cases a even, and a odd.

4. The Case Where $a|(2b)$ with a Even

Let $c > 2$ be an integer, and consider the Diophantine equation

$$x^2 - \left(c^2 - 4\right) y^2 = 4. \tag{4.1}$$

We begin with a known result that occurs as Theorem 8 in [4]. We point out that, in [4, p. 134], there is an error in the definition of the sequence g_n, which is our sequence V_n. In the notation of the present paper, Theorem 8 in [4] is the following theorem.

Theorem 4.1. *The only positive integer solutions of (4.1) are* $(V_n(c), U_n(c))$, $n \geq 1$.

We now consider the case where $a|(2b)$ with a even. Accordingly, set $a = 2m$ for m a positive integer. Then $a|(2b) \Rightarrow m|b \Rightarrow b = mc$, for c a positive integer. Notice that $b > a \Rightarrow c > 2$. Then the Diophantine equation (1.3) becomes

$$x^2 - m^2 \left(c^2 - 4\right) y^2 = 4m^2. \tag{4.2}$$

Our next lemma gives a connection between the solutions of (4.1) and the solutions of (4.2).

Lemma 4.2. *There is a one-to-one correspondence between the positive integer solutions of (4.1), and the positive integer solutions of (4.2).*

Proof. Suppose (x_0, y_0) is a positive integer solution of (4.1). Then $x_0^2 - \left(c^2 - 4\right) y_0^2 = 4$, so that $m^2 x_0^2 - m^2 \left(c^2 - 4\right) y_0^2 = 4m^2$. That is, (mx_0, y_0) is a positive integer solution of (4.2).

Now suppose (x_0, y_0) is a positive integer solution of (4.2). Substitution gives $x_0^2 - m^2 \left(c^2 - 4\right) y_0^2 = 4m^2$, in which $m^2 | \left(x_0^2\right) \Rightarrow m | x_0$. But then $\left(\frac{x_0}{m}\right)^2 - \left(c^2 - 4\right) y_0^2 = 4$, which implies that $\left(\frac{x_0}{m}, y_0\right)$ is a positive integer solution of (4.1). This completes the proof of Lemma 4.2. □

Based on Theorem 4.1 and Lemma 4.2, all the positive integer solutions of (4.2) are $(mV_n(c), U_n(c))$, $n \geq 1$. Expressed in terms of a and b, these solutions are $\left(\frac{a}{2} V_n\left(\frac{2b}{a}\right), U_n\left(\frac{2b}{a}\right)\right)$, $n \geq 1$. These solutions are of the same form as those presented in Lemma 3.1. This settles the case where $a|(2b)$ with a even.

5. The Case Where $a|(2b)$ with a Odd

Following Euler and Sadek, assume that there exists a positive integer solution (u, v) of (1.4) that lies between two successive positive integer solutions, (x_n, y_n) and (x_{n+1}, y_{n+1}), generated by (1.4). That is, assume

$$x_n + y_n \sqrt{d} < u + v\sqrt{d} < x_{n+1} + y_{n+1}\sqrt{d}. \tag{5.1}$$

Then, following the algebraic path set forth by Euler and Sadek, we arrive at

$$a < \frac{(ux_n - dvy_n)}{a} + \frac{(vx_n - uy_n)\sqrt{d}}{a} < b + \sqrt{d}, \tag{5.2}$$

which is Euler and Sadek's equation (2.5).

Euler and Sadek effectively demonstrate that

$$\frac{(ux_n - dvy_n)}{a} + \frac{(vx_n - uy_n)\sqrt{d}}{a} \tag{5.3}$$

is a positive rational solution of (1.3) that is less than the smallest positive integer solution $b + \sqrt{d}$. Under the assumption that $a|(2b)$ with a odd, we now proceed to demonstrate that (5.3) is actually a positive integer solution.

Because (u, v) and (x_n, y_n) are positive integer solutions of (1.3), we have

$$u^2 - dv^2 = a^2, \text{and} \tag{5.4}$$

$$x_n^2 - dy_n^2 = a^2. \tag{5.5}$$

Since $a|(2b)$ with a odd, $a|b$. Let $b = ae$, where e is a positive integer. Then $d = b^2 - a^2 = a^2\left(e^2 - 1\right)$, so that

$$\frac{ux_n - dvy_n}{a} = \frac{ux_n - a^2\left(e^2-1\right)vy_n}{a} = \frac{ux_n}{a} - a\left(e^2-1\right)vy_n. \tag{5.6}$$

Now, with the use of (5.4), we have

$$\frac{u^2x_n^2}{a^2} = \left(\frac{dv^2 + a^2}{a^2}\right)x_n^2, \tag{5.7}$$

in which $d = a^2\left(e^2 - 1\right)$. Equation (5.7) shows that $\frac{u^2x_n^2}{a^2}$ is an integer, and so $\frac{ux_n}{a}$ is an integer. Then from (5.6) we see that the leftmost term in (5.3) is an integer.

Next, we square the coefficient of $\sqrt{d}$ in (5.3). Then, keeping in mind that $d = a^2\left(e^2 - 1\right)$, we use (5.4) and (5.5) to substitute for u^2 and x_n^2. The fractional part in the resulting expression is $\frac{-2uvx_ny_n}{a^2}$. Again, squaring this expression, and substituting for u^2 and x_n^2, we see that the resulting expression is an integer. This proves that the coefficient of $\sqrt{d}$ in (5.3) is an integer.

Keeping in mind that (5.3) is known to be a positive rational solution of (1.3), we have demonstrated that (5.3) is a positive integer solution of (1.3) that is less than the smallest positive integer solution $b + \sqrt{d}$. This contradiction shows that, under the assumption that $a|(2b)$ with a odd, all the positive integer solutions of (1.3) are generated by (1.4).

6. A Summary and Concluding Comments

In the theorem that follows, we summarize our conclusions concerning the positive integer solutions of (1.3) that are generated by (1.4).

Theorem 6.1. *Suppose $a|(2b)$. Then the system (1.4) generates only positive integer solutions of (1.3). With $p = \frac{2b}{a}$ in (2.1), these solutions are given by*

$$(x_n, y_n) = (bU_n - aU_{n-1}, U_n) = \left(\frac{aV_n}{2}, U_n\right), n \geq 1. \tag{6.1}$$

Furthermore, the solutions in (6.1) are the only positive integer solutions of (1.3).

With $a = 2$ and $b = 3$, the Diophantine equation (1.3) becomes $x^2 - 5y^2 = 4$, and the sequences (2.1) are

$$\begin{aligned} U_n &= 3U_{n-1} - U_{n-2}, U_0 = 0, U_1 = 1, \\ V_n &= 3V_{n-1} - V_{n-2}, V_0 = 2, V_1 = 3. \end{aligned} \tag{6.2}$$

In (6.2), $U_n = F_{2n}$, and $V_n = L_{2n}$, so that the solutions given in (6.1) become $(x_n, y_n) = (L_{2n}, F_{2n}), n \geq 1$. This agrees with the result stated in the introduction.

In (6.1), it is easy to prove by induction that $U_{-n} = -U_n$, and $V_{-n} = V_n$, for all integers n. This means that if we allow negative integer values of n, then (6.1) produces all the integer solutions of (1.3) that lie in the first and fourth quadrants.

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