ON A CLASSICAL FIBONACCI IDENTITY OF AURIFEUILLE

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Abstract. In this paper, we present identities that are analogous to a classical Fibonacci identity of Aurifeuille. Aurifeuille's identity gives certain factors of L_{5n}, n odd. The analogues of Aurifeuille's identity that we present involve pairs of sequences that generalize the Fibonacci and Lucas sequences.

1. Introduction

To present the results in this paper, we require two pairs of integer sequences. For all n, we define the first pair of sequences by

$$\begin{aligned} u_n &= u_n(r) = ru_{n-1} + u_{n-2}, u_0 = 0, u_1 = 1, \\ v_n &= v_n(r) = rv_{n-1} + v_{n-2}, v_0 = 2, v_1 = r, \end{aligned} \tag{1.1}$$

where r is an integer. We say that $r^2 + 4$ is the *discriminant* of the pair of sequences (1.1). For $r = 1$, $u_n = F_n$, and $v_n = L_n$, the Fibonacci and Lucas numbers, respectively. The second pair of integer sequences is defined, for all integers n, by

$$\begin{aligned} U_n &= U_n(r,k) = v_{2k+1}U_{n-1} + U_{n-2}, U_0 = 0, U_1 = 1, \\ V_n &= V_n(r,k) = v_{2k+1}V_{n-1} + V_{n-2}, V_0 = 2, V_1 = v_{2k+1}. \end{aligned} \tag{1.2}$$

The sequences (1.2) rely for their definition upon the sequences (1.1). Specifically, the multiplier v_{2k+1} is known only after a value of r is stipulated for the sequences (1.1). When $k = 0$, the sequences (1.2) reduce to the sequences (1.1).

An old and beautiful identity, attributed by Maxey Brooke [1] to Aurifeuille (1879), states that

$$L_{5n} = L_n \left(L_{2n} + 5F_n + 3\right)\left(L_{2n} - 5F_n + 3\right), n \text{ odd}. \tag{1.3}$$

Identity (1.3) gives certain factors of L_{5n} when n is odd. In order to set the tone of our main results, we restate (1.3) with the use of the sequences (1.1), and include a generalization, (1.5), that we proved in [3] with $u_{2k+1} = F_{2k+1}$. We have the following theorem.

Theorem 1.1. *Let n be odd. For the pair of sequences (1.1), take $r = 1$. For the pair of sequences (1.2), take $p = v_{2k+1}$. Then*

$$v_{5n} = v_n \left(\left(v_{2n} + 3\right)^2 - 5^2 u_n^2\right), \tag{1.4}$$

$$V_{5n} = V_n \left(\left(V_{2n} + 3\right)^2 - 5^2 u_{2k+1}^2 U_n^2\right). \tag{1.5}$$

Again, (1.4) is simply a restatement of (1.3), where the last two factors are written as the difference of two squares. Notice that for $k = 0$, $u_{2k+1} = 1$, $U_n = u_n$, $V_n = v_n$, and (1.5) reduces to (1.4). Some identities in the sequel are rather lengthy. Therefore, to save space, we present all identities in a manner that is analogous to (1.4) and (1.5).

It is natural to ask if there are odd primes q, other than 5, for which L_{qn} factors in a manner analogous to the right side of (1.3). We have not been able to find such primes. However, in

Section 2, we show that such primes do exist for sequences other than the Fibonacci/Lucas sequences. In Section 3, we state a conjecture based on the patterns in our main results, and in Section 4 we provide a sample proof.

2. The Main Results

Our main results involve the two pairs of integer sequences (1.1) and (1.2), and are analogous to Theorem 1.1. Specifically, for an odd prime q, we choose a value of r so that v_{qn} and V_{qn} factor in a manner analogous to the right sides of (1.4) and (1.5), respectively. The theorem that follows addresses the case where $q = 13$.

Theorem 2.1. *Let n be odd. For the pair of sequences (1.1), take $r = 3$. Then*

$$v_{13n} = v_n \left((v_{6n} + 7v_{4n} + 15v_{2n} + 19)^2 - 13^2 (u_{5n} + 3u_{3n} + 5u_n)^2 \right), \tag{2.1}$$

$$V_{13n} = V_n \left((V_{6n} + 7V_{4n} + 15V_{2n} + 19)^2 - 13^2 u_{2k+1}^2 (U_{5n} + 3U_{3n} + 5U_n)^2 \right). \tag{2.2}$$

In (2.1), the last two factors are written as a difference of two squares. The first square is a linear combination of terms, with even subscripts, drawn from $\{v_n\}$, while the second square is a linear combination of terms, with odd subscripts, drawn from $\{u_n\}$. The situation is similar for (2.2), and the analogy with the results in Theorem 1.1 is clear.

By virtue of Theorem 1.1 (Theorem 2.1), we say that v_{5n} (v_{13n}) *admits* an Aurifeuille identity, and that V_{5n} $(V_{13n}$) admits a *family* (parametrized by k) of Aurifeuille identities.

In the theorems that follow, we give only the family of Aurifeuille identities associated with the sequences (1.2). In each case, the first member (corresponding to $k = 0$) of the family in question is the Aurifeuille identity associated with the sequences (1.1).

Theorem 2.2. *Let n be odd. For the pair of sequences (1.1), take $r = 8$. Then*

$$\begin{aligned} V_{17n} = V_n \Big(&(V_{8n} + 9V_{6n} + 11V_{4n} - 5V_{2n} - 15)^2 \\ &- 17^2 2^2 u_{2k+1}^2 (U_{7n} + 3U_{5n} + U_{3n} - 3U_n)^2 \Big). \end{aligned} \tag{2.3}$$

Theorem 2.3. *Let n be odd. For the pair of sequences (1.1), take $r = 5$. Then*

$$\begin{aligned} V_{29n} = V_n \Big(&(V_{14n} + 15V_{12n} + 33V_{10n} + 13V_{8n} + 15V_{6n} + 57V_{4n} + 45V_{2n} + 19)^2 \\ &- 29^2 u_{2k+1}^2 (U_{13n} + 5U_{11n} + 5U_{9n} + U_{7n} + 7U_{5n} + 11U_{3n} + 5U_n)^2 \Big). \end{aligned} \tag{2.4}$$

We have discovered more Aurifeuille identities analogous to those presented in the theorems of this section. Before stating some of these identities, we present a conjecture on the existence and form of such identities. We do this in the next section.

3. A Conjecture on the Existence and Form of Aurifeuille Identities

Theorems 1.1, 2.1, and 2.3 give Aurifeuille identities admitted by v_{qn} for $q = 5, 13$, and 29, respectively. In each case, for the stated value of r, $q = r^2 + 4$ is the discriminant of the sequences (1.1). In Theorem 2.2, $r^2 + 4 = 17 \times 2^2$, so that $q = 17$ is the square free part of the discriminant in question. Indeed, any odd prime $q \equiv 1 \pmod 4$ is the square free part of the discriminant of a pair of integer sequences (1.1). This is guaranteed by a well-known result concerning a particular class of second order Diophantine equations. Specifically, if $q \equiv 1 \pmod 4$ is prime, the Diophantine equation

$$r^2 - qs^2 = -4 \tag{3.1}$$

has solutions in integers r and s. This result, which we require for the conjecture that follows, is an immediate consequence of Theorem 107 in the book of Nagell [4]. Among all solutions with positive r and s, if $r_0 + s_0\sqrt{q}$ is least, then (r_0, s_0) is called the *fundamental solution* of (3.1).

Conjecture 3.1. *Suppose $q \equiv 1 \pmod 4$ is a prime. Let (r_0, s_0) be the fundamental solution of the Diophantine equation $r^2 - qs^2 = -4$. For the pair of sequences (1.1) take $r = r_0$. Then there exist integers $a_i, 0 \le i \le (q-1)/4$, and $b_i, 1 \le i \le (q-1)/4$, such that*

$$V_{qn} = V_n\left(\left(a_0 + \sum_{i=1}^{(q-1)/4} a_i V_{2in}\right)^2 - q^2 s_0^2 u_{2k+1}^2 \left(\sum_{i=1}^{(q-1)/4} b_i U_{(2i-1)n}\right)^2\right), n \text{ odd.} \tag{3.2}$$

For instance, in Theorem 2.2, $q = 17$, and $(r_0, s_0) = (8, 2)$. Also, with the subscripts in increasing order, the a_i are -15, -5, 11, 9, 1, and the b_i are -3, 1, 3, 1.

We now give two additional Aurifeuille identities that we have discovered. Due to the length of these identities, it is convenient to state them with the notation in Conjecture 3.1. In each case, we state the a_i and b_i with subscripts in increasing order.

Theorem 3.2. *Let $q = 37$, and take $(r_0, s_0) = (12, 2)$. Then 3.2 is true if the a_i are $627, 579, 477, 397, 349, 285, 183, 79, 19, 1$, and the b_i are $101, 87, 71, 61, 53, 39, 21, 7, 1$.*

Theorem 3.3. *Let $q = 41$, and take $(r_0, s_0) = (64, 10)$. Then 3.2 is true if the a_i are $-31, -65, -23, 11, 15, 35, 7, 49, 67, 21, 1$, and the b_i are $-7, -9, 1, 1, 5, 3, 3, 11, 7, 1$.*

4. A Sample Proof

It is impractical to prove our theorems with the method that we employ in [3] to prove Theorem 1.1. Instead, we have managed to prove each of our theorems with the *Verification Theorem* of Dresel [2, page 171]. Although Dresel's Theorem is stated for Fibonacci/Lucas identities, it also applies to the sequences in the present paper. To illustrate, we prove Theorem 2.2.

In order to use Dresel's Theorem, we first need to express (2.3) as a *homogeneous* identity in the variables n and k. As Dresel explains, since $(-1)^n = (\alpha\beta)^n$, where α and β are the roots of $x^2 - px - 1 = 0$, then $(-1)^n$ is of degree 2 in the variable n. Therefore, $(-1)^{2n}$ is of degree 4 in the variable n, and so on. The same is true for the variable k. Now put $L(k, n) = (-1)^{2k} V_{17n}$, and set $R(k, n)$ equal to

$$\begin{aligned} V_n\Big((-1)^{2k}&\left(V_{8n} - 9(-1)^n V_{6n} + 11(-1)^{2n} V_{4n} + 5(-1)^{3n} V_{2n} - 15(-1)^{4n}\right)^2 \\ &+ 17^2 2^2 u_{2k+1}^2 (-1)^n \left(U_{7n} - 3(-1)^n U_{5n} + (-1)^{2n} U_{3n} + 3(-1)^{3n} U_n\right)^2\Big). \end{aligned}$$

We now prove that

$$L(k, n) - R(k, n) = 0, \tag{4.1}$$

for *all* values of k and n.

Since (4.1) is an identity that is homogeneous of degree 4 in the variable k, to prove it with the Verification Theorem of Dresel, we need only verify its validity for five distinct values of k. Accordingly, we write down the cases that correspond to $k = 1, 2, 3, 4, 5$. Now, each of these five cases is an identity that is homogeneous of degree 18 in the variable n. Therefore, to prove any one of the five cases of (4.1), we need only verify its validity for eighteen distinct values of n; say $n = 1, 2, \ldots, 18$. With the use of the computer algebra system *Mathematica* 8.0, we managed to perform these 5×18 verifications with two nested "For" loops, thereby proving the validity of (4.1) for all k and n. Finally, the validity of (4.1) for all k and n implies the validity of (2.3) for odd n. This proves Theorem 2.2.

5. Concluding Comments and a Possible Direction for Research

We have managed to write down results, analogous to those presented in our theorems, for $q = 53, 61, 73, 89,$ and 97. Since the identities in question are quite lengthy, we have chosen not to present them here.

If $q \equiv 3 \pmod 4$ is prime, it is easily shown that (3.1) has no solutions in integers r and s. One simply considers the four cases corresponding to the parities of r and s. Accordingly, we have been unable to find any prime $q \equiv 3 \pmod 4$ such that v_{qn} admits an Aurifeuille identity. However, we cannot prove that no such prime exists.

Since our method of discovery was computational, we can provide no information about any patterns in the a_i and b_i of Conjecture 3.1. Therefore, an investigation into any such patterns may provide a direction for future research.

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