INTEGER VALUES OF GENERATING FUNCTIONS FOR THE FIBONACCI AND RELATED SEQUENCES

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Abstract. It is known that the generating function of the Fibonacci sequence, $F(x) = \sum F_i x^i = x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots$, attains an integer value if $x = F_i/F_{i+1}$ for any non-negative integer i. It has been conjectured that those values constitute *all* rational numbers, in the interval of convergence of F, that result in $F(x) \in \mathbb{Z}$. In this paper we prove this conjecture. We also extend these results to the class of sequences satisfying the recursion relation $R_{i+2} = aR_{i+1} + bR_i$ with initial values $(R_0, R_1) = (0, 1)$, where a and b are positive integers satisfying $b \mid a$.

Introduction

Let a and b be positive integers. Consider a sequence $\{R_i\}_{i\in\mathbb{N}}$ given by $R_{i+2} = aR_{i+1} + bR_i$ and some initial values (R_0, R_1). The generating function for $\{R_i\}$ is given by

$$R(x) = \sum_{i=0}^{\infty} R_i x^i.$$

Define

$$\phi = \frac{a + \sqrt{a^2 + 4b}}{2} \quad \text{and} \quad \psi = \frac{a - \sqrt{a^2 + 4b}}{2}.$$

Then $I = (\psi/b, -\psi/b)$ is the interval of convergence for $R(x)$, on which we have

$$R(x) = \frac{R_0(1 - ax) + R_1 x}{1 - ax - bx^2}.$$

If $(a, b, R_0, R_1) = (1, 1, 0, 1)$ or $(a, b, R_0, R_1) = (1, 1, 2, 1)$, then $\{R_i\}$ becomes the Fibonacci sequence $\{F_i\}$ or the Lucas sequence $\{L_i\}$, respectively. Let $F(x)$ and $L(x)$ denote the respective generating functions for these sequences. It was shown in [2] that, for any $i \in \mathbb{N}$, $F(x)$ is an integer when $x = F_{2i}/F_{2i+1}$, and $L(x)$ is an integer when $x = F_{2i}/F_{2i+1}$ or $x = L_{2i+1}/L_{2i+2}$. A question was then posed asking whether these values constitute *all* rational numbers in $I = (\frac{1-\sqrt{5}}{2}, -\frac{1-\sqrt{5}}{2})$ for which the respective generating function is an integer. In this paper we demonstrate an affirmative answer to this question. We also extend our results to sequences for which $(R_0, R_1) = (0, 1)$ and a, b are positive integers, but we find that it is necessary to stipulate that b is a factor of a.

Generalized Fibonacci Sequences

In this section, we set $(R_0, R_1) = (0, 1)$ and require a, b to be positive integers. Our goal is to establish the following result.

Theorem 1.1. *Suppose b divides a. The rational numbers $x \in I$ for which $R(x)$ is an integer are precisely those of the form $x = R_{2i}/R_{2i+1}$ where $i \in \mathbb{N}$.*

The proof will be given at the end of this section. First we lay out some framework and preliminary results. We begin by pointing out some basic properties of $\{R_i\}$. A basic induction argument reveals the following closed form formula for R_r in terms of matrices:

$$\begin{pmatrix} R_{r+1} & R_r \\ R_r & R_{r-1} \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}^{r-1}.$$

Taking determinants gives a generalization of Cassini's identity:

$$R_{r-1}R_{r+1} = R_r^2 - (-b)^{r-1}. \tag{1.1}$$

Also, a straightforward induction argument demonstrates a generalization of Binet's formula:

$$R_r = \frac{\phi^r - \psi^r}{\phi - \psi}. \tag{1.2}$$

Moving forward, we impose the condition that b divides a and we establish several Lemmas which will drive the proof of the main theorem. One consequence of the condition $b \mid a$ is that $a^2 + 4b$ is not a square. Indeed, the equation $a^2 + 4b = c^2$ leads to $4b = b^2(c' + a')(c' - a')$ where $a = ba'$ and $c = bc'$. There are only three cases, $b = 1$, $b = 2$, $b = 4$, and none of them gives a solution. The necessity of the condition $b \mid a$ in Theorem 1.1 is demonstrated at the end of this section after the proof of Theorem 1.1.

Consider the set of all real numbers η of the form $\eta = (x + y\sqrt{a^2 + 4b})/(2|\beta|^{1/2})$, where x and y are integers and β is a factor of b. Define the *conjugate* $\bar{\eta}$ of any such η to be the number $\bar{\eta} = (x - y\sqrt{a^2 + 4b})/(2|\beta|^{1/2})$. Let $\mathbb{W}$ denote the set of real numbers η of the form just described which satisfy $\eta\bar{\eta} = \pm 1$.

Let $\hat{\phi} = \phi/\sqrt{b}$ and $\hat{\psi} = \psi/\sqrt{b}$. Then $\hat{\phi}, \hat{\psi} \in \mathbb{W}$. The following lemma generalizes Theorem 243 in [1].

Lemma 1.2. *$\hat{\phi}$ is the smallest element of $\mathbb{W}$ which is greater than* 1.

Proof. Suppose $\eta = (x + y\sqrt{a^2 + 4b})/(2|\beta|^{1/2}) \in \mathbb{W}$ satisfies

$$1 < \eta \leq \hat{\phi}. \tag{1.3}$$

Since $\eta\bar{\eta} = \pm 1$, we have that

$$-1 < \bar{\eta} < 1. \tag{1.4}$$

By adding and subtracting the two inequalities (1.3) and (1.4), we get

$$0 < \eta + \bar{\eta} < \hat{\phi} + 1 \qquad \text{and} \tag{1.5}$$

$$0 < \eta - \bar{\eta} < \hat{\phi} + 1. \tag{1.6}$$

Since $(a + 2\sqrt{b})^2 \leq 2(a^2 + 4b)$, it follows that $\hat{\phi} + 1 = (a + 2\sqrt{b} + \sqrt{a^2 + 4b})/(2\sqrt{b}) \leq (\sqrt{2} + 1)\sqrt{a^2 + 4b}/(2\sqrt{b})$. So, by (1.6), we have $0 < y\sqrt{b/|\beta|} < (\sqrt{2} + 1)/2 < \sqrt{2}$, which implies $y = 1 = b/|\beta|$ and $\eta = (x + \sqrt{a^2 + 4b})/(2|\beta|^{1/2})$. Combining (1.3) and (1.5), we get $0 < x \leq a$. Now, $\eta\bar{\eta} = \pm 1$ implies $x^2 = a^2 + 4b \pm 4|\beta|$, so we must have $x = a$ and $|\beta| = b$, i.e., $\eta = \hat{\phi}$. □

Our strategy in the results to come will involve factoring out any factors of b which are squares. We will then appeal to the following lemma which treats the square-free case.

Lemma 1.3. *Suppose b has no factors, other than* 1*, which are squares. Suppose $|\beta|$ is a factor of b. If $n \in \mathbb{N}$ is such that $(a^2 + 4b)n^2 + 4\beta$ is a square, then there exists $r \in \mathbb{N}$ such that $b^{\lfloor r/2 \rfloor} n = R_r$ and $\beta = (-b)^{r \pmod 2}$.*

Proof. Let n be a non-negative integer satisfying $(a^2+4b)n^2+4\beta = w^2$ for some positive integer w and a factor β of b. Then $\epsilon := (w+n\sqrt{a^2+4b})/(2|\beta|^{1/2})$ is an element of $\mathbb{W}$. Choose a positive integer r so that $\hat{\phi}^{r-1} < \epsilon \le \hat{\phi}^r$. Then $\epsilon\hat{\phi}^{1-r}$ satisfies $1 < \epsilon\hat{\phi}^{1-r} \le \hat{\phi}$. We will use Lemma 1.2 to deduce that $\epsilon\hat{\phi}^{1-r} = \hat{\phi}$, but we must first show that $\epsilon\hat{\phi}^{1-r} \in \mathbb{W}$.

Let $r_0 = r \pmod 2$. It is straightforward to verify that $\pm\hat{\phi}^{1-r} = \hat{\psi}^{r-1}$ can be written in the form $(x+y\sqrt{a^2+4b})/(2b^{(1-r_0)/2})$, where x and y are integers. So we have

$$\epsilon\hat{\psi}^{r-1} = \frac{wx+(a^2+4b)ny+(nx+wy)\sqrt{a^2+4b}}{4|\beta|^{1/2}b^{(1-r_0)/2}}.$$

Showing $\epsilon\hat{\phi}^{1-r} \in \mathbb{W}$ amounts to showing that the quantities $wx+(a^2+4b)ny$ and $nx+wy$ are each divisible by $2\beta^{1-r_0}$.

On the one hand $\hat{\psi}^r$ can be written in the form $(x'+y'\sqrt{a^2+4b})/(2b^{r_0/2})$, but on the other hand,

$$\hat{\psi}^r = \hat{\psi}^{r-1}\hat{\psi} = \frac{(ax-(a^2+4b)y)-(x-ay)\sqrt{a^2+4b}}{4b^{(1-r_0+1)/2}}.$$

Since a^2+4b is not a perfect square, it must be the case that the quantities $ax-(a^2+4b)y$ and $x-ay$ are each divisible by $2b^{1-r_0}$.

We know that $w^2-n^2(a^2+4b) = 4\beta$. Thus w and an must have the same parity. To show that $wx+(a^2+4b)ny$ and $nx+wy$ are both even, we consider four cases: (i) If w and n are both even then we are done. (ii) If w is even but n is odd, then a must be even, and so x must be even since $x-ay$ is even, and we are done. (iii) If w and n are both odd, then a is odd. Since $x-ay$ is even, x and y must have the same parity, and we are done. (iv) It is not possible that w is odd and n is even.

Now suppose $r_0 = 0$. We must show that $wx+(a^2+4b)ny$ and $nx+wy$ are each divisible by 2β. From $w^2-n^2(a^2+4b) = 4\beta$, we know $\beta \mid w^2$, and since β has no square factors, it must be the case that $\beta \mid w$. Since β divides both a and $x-ay$, we know that $\beta \mid x$. It follows that β divides $wx+(a^2+4b)ny$ and $nx+wy$. The case where β is odd is simple. The terms $wx+(a^2+4b)ny$ and $nx+wy$ must both be divisible by 2β since they are both even. Consider the alternative case where β is even. It is clear that 2β divides $wx+(a^2+4b)ny$. Let $\hat{x} = x/\beta$, $\hat{a} = a/\beta$, $\hat{b} = b/\beta$, and $\hat{w} = w/\beta$. Now all that is left to show is that $n\hat{x}+\hat{w}y$ is even. From $w^2-n^2(a^2+4b) = 4\beta$, we have $n^2\hat{a}a \equiv \hat{w}w \pmod 4$. We consider four cases, as above: (i) If $\hat{w}$ and n are both even, then we are done. (ii) Suppose $\hat{w}$ is even but n is odd. Then $\hat{a}a \equiv \hat{w}w \equiv 0 \pmod 4$. Since β has no square factors, $\beta \not\equiv 0 \pmod 4$. It follows that $\hat{a}$ is even. Since $x-ay$ is divisible by 2β, we know $\hat{x}-\hat{a}y$ is even. So $\hat{x}$ is even, and we are done. (iii) Suppose $\hat{w}$ and n are both odd. Then, since β has no square factors, $w \not\equiv 0 \pmod 4$. From $n^2\hat{a}a \equiv \hat{w}w \not\equiv 0 \pmod 4$, it follows that $\hat{a}$ is odd. Then since $\hat{x}-\hat{a}y$ is even, $\hat{x}$ and y have the same parity, so we are done. (iv) It is not possible that $\hat{w}$ is odd and n is even. Indeed, as mentioned in case (iii), $w \not\equiv 0 \pmod 4$, so $n^2\hat{a}a \equiv \hat{w}w \not\equiv 0 \pmod 4$. In summary, we have demonstrated in this and the previous paragraph that $wx+(a^2+4b)ny$ and $nx+wy$ are each divisible by $2\beta^{1-r_0}$.

We conclude that $\epsilon\hat{\phi}^{1-r} = \pm\epsilon\hat{\psi}^{r-1}$ has the form $(x''+y''\sqrt{a^2+4b})/(2|b/\beta|^{1/2})$ when $r_0 = 0$, or the form $(x''+y''\sqrt{a^2+4b})/(2|\beta|^{1/2})$ when $r_0 = 1$. Thus $\epsilon\hat{\phi}^{1-r}$ is in $\mathbb{W}$ and satisfies $1 < \epsilon\hat{\phi}^{1-r} \le \hat{\phi}$, so Lemma 1.2 tells us that $\epsilon\hat{\phi}^{1-r} = \hat{\phi}$. If $r_0 = 0$, then $a+\sqrt{a^2+4b} = |\beta|^{1/2}(x''+y''\sqrt{a^2+4b})$, which implies $|\beta|^{1/2} = y'' = 1$. If $r_0 = 1$, then $a+\sqrt{a^2+4b} = |b/\beta|^{1/2}(x''+y''\sqrt{a^2+4b})$, which implies $|b/\beta|^{1/2} = y'' = 1$. It follows that $|\beta| = b^{r_0}$. Moreover,

$\epsilon\hat{\phi}^{1-r} = \hat{\phi}$ implies that $\epsilon = \hat{\phi}^r$. Therefore we may write

$$\epsilon = \frac{w + n\sqrt{a^2+4b}}{2b^{r_0/2}} = \frac{1}{2}\left[(\hat{\phi}^r + \hat{\psi}^r) + \frac{\hat{\phi}^r - \hat{\psi}^r}{\sqrt{a^2+4b}}\sqrt{a^2+4b}\right].$$

It is easy to establish the fact that $b^{r_0/2}(\hat{\phi}^r + \hat{\psi}^r)$ is an integer z by induction. Binet's formula (1.2) gives $R_r = b^{(r-1)/2}(\hat{\phi}^r - \hat{\psi}^r)/(\hat{\phi} - \hat{\psi}) = b^{r/2}(\hat{\phi}^r - \hat{\psi}^r)/\sqrt{a^2+4b}$. Substituting these into the above relation gives

$$w + n\sqrt{a^2+4b} = z + b^{-r/2+r_0/2}R_r\sqrt{a^2+4b} = z + b^{-\lfloor r/2\rfloor}R_r\sqrt{a^2+4b}.$$

Since a^2+4b is not a square, it follows that $n = b^{-\lfloor r/2\rfloor}R_r$.

We have also shown that $|\beta| = b^{r_0} = b^{r \pmod 2}$, so the only possible values of β are ± 1 or $\pm b$. We will now show that, unless $b = 1$, the values $\beta = -1$ and $\beta = b$ are not possible. Suppose n were a number which made $(a^2+4b)n^2 + 4\beta$ a square for $\beta = -1$ or $\beta = b$. According to what we have shown above, there exists $r \in \mathbb{N}$ such that $n = b^{-\lfloor r/2\rfloor}R_r$. According to part (1) of Proposition 1.4 below, such a value of n makes $(a^2+4b)n^2 + 4\beta'$ a square, where $\beta' = -\beta$. Consider integers $p = [(a^2+4b)n^2 + 4\beta]^{1/2}$ and $q = [(a^2+4b)n^2 + 4\beta']^{1/2}$. Then $p^2 - q^2 = 8\beta$. Since b has no square factors, it follows from the definitions of p and q that β must divide both p and q. Put $\tilde{p} = p/\beta$ and $\tilde{q} = q/\beta$. Then $\tilde{p}^2 - \tilde{q}^2 = 8/\beta \in \mathbb{Z}^+$. If β is 8, 4, or 2, then it is easy to verify that there are no solutions $\tilde{p}, \tilde{q}$. If $\beta = \pm 1$, then the only solutions are $(\tilde{p}, \tilde{q}) = (3, 1)$ or $(\tilde{p}, \tilde{q}) = (1, 3)$ and each of these imply $a = b = n = 1$. In the $a = b = n = 1$ case, $n = 1 = R_1$ makes $(a^2+4b)n^2 + 4(-1)$ a square and $n = 1 = R_2$ makes $(a^2+4b)n^2 + 4(1)$ a square. So, in all possible cases, we have $\beta = (-b)^{r \pmod 2}$. □

The following result characterizes terms of the sequence $\{R_i\}$ as the set of non-negative integers which make $(a^2+4b)n^2 - 4b$ or $(a^2+4b)n^2 + 4$ a square. It generalizes Solution H-187 in [3].

Proposition 1.4. *Let n be a non-negative integer. Then the following two facts hold.*

(1) *If $b^{\lfloor r/2\rfloor}n = R_r$ for some $r \in \mathbb{N}$, then $(a^2+4b)n^2 + 4(-b)^{r \pmod 2}$ is a square.*
(2) *If $(a^2+4b)n^2 + 4(-b)^{r_0}$ is a square for some $r_0 \in \{0, 1\}$, then $b^{\lfloor r/2\rfloor}n = R_r$ for some $r \in \mathbb{N}$ satisfying $r \pmod 2 = r_0$.*

Proof. Suppose $b^{\lfloor r/2\rfloor}n = R_r$. Then from Cassini's identity (1.1) we have

$$\begin{aligned}(bR_{r-1} + R_{r+1})^2 &= (bR_{r-1} - R_{r+1})^2 + 4bR_{r-1}R_{r+1}\\ &= (a^2+4b)R_r^2 + (-1)^r 4b^r\\ &= (a^2+4b)(b^{\lfloor r/2\rfloor}n)^2 + (-1)^r 4b^r\\ &= b^{r-r_0}\left[(a^2+4b)n^2 + 4(-b)^{r_0}\right].\end{aligned} \tag{1.7}$$

A straightforward induction argument shows that $b^{\lfloor r/2\rfloor}$ is a factor of R_r for all r. It follows that $(a^2+4b)n^2 + 4(-b)^{r_0} = (bR_{r-1} + R_{r+1})^2/b^{r-r_0}$ is a square. This proves the first part of the proposition.

If b has no square factors other than 1, then the second part of the proposition follows from Lemma 1.3. Otherwise, let z be the largest positive integer such that z^2 is a factor of b. Let $\tilde{b} = b/z^2$ and $\tilde{a} = a/z$. Suppose n is a positive integer which makes $(a^2+4b)n^2 + 4(-b)^{r_0}$ a perfect square for some $r_0 \in \{0, 1\}$.

First suppose that $r_0 = 0$. Then $(\tilde{a}^2 + 4\tilde{b})(zn)^2 + 4$ is a perfect square. It follows from Lemma 1.3 that $\tilde{b}^{\lfloor r/2\rfloor}(zn) = \tilde{R}_r$, where r is an even positive integer and $\tilde{R}_r$ is an element of the sequence satisfying $\tilde{R}_{i+2} = \tilde{a}\tilde{R}_{i+1} + \tilde{b}\tilde{R}_i$ and $(\tilde{R}_0, \tilde{R}_1) = (0, 1)$. It is easy to verify that $R_r = z^{r-1}\tilde{R}_r$. It follows that $R_r = z^{r-1}\tilde{b}^{r/2}(zn) = b^{r/2}n$.

Next suppose that $r_0 = 1$. Then $(\tilde{a}^2 + 4\tilde{b})n^2 - 4\tilde{b}$ is a perfect square. It follows from 1.3 that $\tilde{b}^{\lfloor r/2\rfloor}n = \tilde{R}_r$ for some positive odd integer r. Since $R_r = z^{r-1}\tilde{R}_r$, we know $R_r = z^{r-1}\tilde{b}^{(r-1)/2}n = b^{(r-1)/2}n$. We have shown that $R_r = b^{\lfloor r/2\rfloor}n$ where $r \pmod 2 = r_0$. □

We are now in a position to prove the main theorem of the paper.

Proof of Theorem 1.1. First of all, notice that for any even integer $r \geq 0$ we have

$$R\left(\frac{R_r}{R_{r+1}}\right) = \frac{R_r R_{r+1}}{R_{r+1}^2 - aR_rR_{r+1} - bR_r^2} = \frac{R_rR_{r+1}}{b(R_{r+1}R_{r-1} - R_r^2)} = \frac{R_rR_{r+1}}{(-b)^r},$$

where the denominator was simplified in the last equality using (1.1). As mentioned in the proof of Proposition 1.4, R_r is a multiple of $b^{\lfloor r/2\rfloor}$ for any nonnegative integer r, so R_rR_{r+1} is a multiple of b^r. It follows that $R(R_r/R_{r+1})$ is an integer

Next assume $R(x) = k \in \mathbb{Z}$ for some rational number $x \in I = (\psi/b, -\psi/b)$. We must have $x \geq 0$ since $R(\psi/b) = -1/(2a)$ and $R'(x) = (1 + bx^2)/(1 - ax - bx^2)^2 > 0$. Solving $R(x) = k$ for x gives

$$x = \frac{1}{2bk}\left(-(ak+1) + \sqrt{(ak+1)^2 + 4bk^2}\right). \tag{1.8}$$

In order for $x \in \mathbb{Q}$, there must exist a positive integer c such that $4bk^2 = c^2 - (ak+1)^2 = [c-(ak+1)][c+(ak+1)]$. Proceeding as in the derivation of Euclid's formula for Pythagorean triples, we choose positive integers m and n such that

$$\frac{m}{n} = \frac{2bk}{c-(ak+1)} = \frac{c+(ak+1)}{2k}. \tag{1.9}$$

It follows that

$$\frac{c}{bk} = \frac{m^2+bn^2}{bmn} \quad \text{and} \quad \frac{ak+1}{bk} = \frac{m^2-bn^2}{bmn}.$$

Assume that m and n are chosen to be co-prime. Then

$$m^2 + bn^2 = \beta c \quad \text{and} \quad mn = \beta k \quad \text{and} \quad m^2 - bn^2 = \beta(ak+1), \tag{1.10}$$

where $\pm\beta = \gcd(m^2 \pm bn^2, bmn) = \gcd(b, m)$. Then we have $m^2 - amn - bn^2 = \beta$. Solving for m gives

$$2m = an + \sqrt{(a^2+4b)n^2 + 4\beta}.$$

Let ζ^2 be the largest square factor of β and let $\tilde{\beta} = \beta/\zeta^2$. Let z be the largest positive integer such that z^2 is a factor of b/ζ^2 and let $\tilde{b} = b/(\zeta^2z^2)$ and $\tilde{a} = a/(\zeta z)$. Then

$$2m = an + \zeta\sqrt{(\tilde{a}^2+4\tilde{b})(zn)^2 + 4\tilde{\beta}}.$$

Lemma 1.3 tells us that there exists a non-negative integer r such that $\tilde{b}^{\lfloor r/2\rfloor}zn = \tilde{R}_r$, where $\tilde{R}_r$ is given by $\tilde{R}_{r+2} = \tilde{a}\tilde{R}_{r+1} + \tilde{b}\tilde{R}_r$ with $(\tilde{R}_0, \tilde{R}_1) = (0, 1)$, and also $\tilde{\beta} = (-\tilde{b})^{r \pmod 2}$. Using (1.7) we see that

$$\sqrt{(\tilde{a}^2+4\tilde{b})(zn)^2 + 4\tilde{\beta}} = \tilde{b}^{-\lfloor r/2\rfloor}(\tilde{b}\tilde{R}_{r-1} + \tilde{R}_{r+1}) = \tilde{b}^{-\lfloor r/2\rfloor}(2\tilde{R}_{r+1} - \tilde{a}\tilde{R}_r).$$

It is easy to verify that $R_r = (z\zeta)^{r-1}\tilde{R}_r$. If r is even, we have $n = \zeta b^{-r/2}R_r$ and

$$2m = an + \zeta b^{-r/2}(2R_{r+1} - aR_r),$$

so $m = \zeta b^{-r/2}R_{r+1}$. If r is odd, we have $n = z^{-1}b^{(1-r)/2}R_r$ and

$$2m = an + z^{-1}b^{(1-r)/2}(2R_{r+1} - aR_r),$$

so $m = z^{-1}b^{(1-r)/2}R_{r+1}$. Therefore, regardless of the parity of r, substituting (1.10) into (1.8) shows that $x = n/m = R_r/R_{r+1}$.

Finally we show that the condition $x \in I$ requires that r is even. Observe that

$$\frac{\psi}{b} + \frac{R_r}{R_{r+1}} = \frac{\psi}{b} + \frac{\phi^r - \psi^r}{\phi^{r+1} - \psi^{r+1}} = \frac{\psi\phi^{r+1} - \psi\psi^{r+1} + b(\phi^r - \psi^r)}{b(\phi^{r+1} - \psi^{r+1})} = \frac{(\phi - \psi)\psi^{r+1}}{b(\phi^{r+1} - \psi^{r+1})},$$

where in the last equality we used the fact that $\phi\psi = -b$. By the definitions of ϕ and ψ, we have $\phi^s - \psi^s > |\phi|^s - |\psi|^s > 0$ for all $s \in \mathbb{N}$, so the sign of $\psi/b + R_r/R_{r+1}$ matches the sign of ψ^{r+1}, which is negative when r is even and positive when r is odd. Since $x \in I$, we must have $R_r/R_{r+1} < -\psi/b$. Thus, r is even. □

We conclude this section by pointing out that the requirement $b \mid a$ in Theorem 1.1 is necessary to ensure the rational solutions to $R(x) \in \mathbb{Z}$ are of the form $x = R_{2i}/R_{2i+1}$. To see this, suppose that $R(R_r/R_{r+1})$ is an integer when r is even. Since $R(R_2/R_3) = b^{-2}a(a^2 + b)$, there exists an integer k such that $a(a^2 + b) = kb^2$. Then, for any integer m, we have $(a^3 + mab) = kb^2 + (m - 1)ab$. Therefore, from the first equation in the proof of Theorem 1.1, we have

$$\begin{aligned} b^4R(R_4/R_5) &= a(a^2 + 2b)(a^4 + 3a^2b + b^2) \\ &= a(a^3 + 2ab)(a^3 + 3ab) + (a^3 + 2ab)b^2 \\ &= ak^2b^4 + kb^4 + 3ka^2b^3 + ab^3 + 2a^3b^2 \\ &= b^4(ak^2 + k) + b^2(3ka^2b + ab + 2a^3) \\ &= b^4(ak^2 + 2k) + b^2(a^3 + 3ka^2b) \\ &= b^4(ak^2 + 3k) + b^2(3ka - 1)ab. \end{aligned}$$

It follows that $a(3ka - 1)$ is divisible by b. From $a(a^2 + b) = kb^2$, we know that $b \mid a^3$, so $\gcd(b, 3ka - 1) = 1$. Therefore it must be the case that $b \mid a$.

Lucas and Pell-Lucas Sequences

In this final section, we touch on the situation where $(R_0, R_1) \neq (0, 1)$ by considering a couple examples. If $(a, b, R_0, R_1) = (1, 1, 0, 1)$ or $(a, b, R_0, R_1) = (1, 1, 2, 1)$, then $\{R_i\}$ becomes the Fibonacci sequence $\{F_i\}$ or Lucas sequence $\{L_i\}$, respectively. Let $L(x)$ denote the generating function for the Lucas sequence. The next proposition answers affirmatively the question posed in [2], asking whether $x = F_{2r}/F_{2r+1}$ and $x = L_{2r+1}/L_{2r+2}$ constitute the *only* rational numbers x which lead to $L(x) \in \mathbb{Z}$, provided we also include $x = -1/2$.

Theorem 1.5. *The rational numbers $x \in I$ for which $L(x)$ is an integer are precisely those of the form $-1/2$, F_{2i}/F_{2i+1}, or L_{2i+1}/L_{2i+2}, where $i \in \mathbb{N}$.*

Proof. Assume $x \in I$. Notice that $1 < L(x) < 3$ on the interval $(\psi, 0]$ and L is increasing on the interval $(0, -\psi)$. $L(x) = 2$ has the two solutions $x = -1/2$ and $x = 0 = F_0/F_1$, so we only need to show that x is a positive rational number for which $L(x)$ is a positive integer > 2 if and only if $x = L_{2i-1}/L_{2i}$ or $x = F_{2i}/F_{2i+1}$ for some positive integer i.

Suppose k is a positive integer > 2 and $x \in I \cap \mathbb{Q}$ is a positive solution to $L(x) = k$. Then

$$x = \frac{(1 - k) + \sqrt{(k - 1)^2 - 4k(2 - k)}}{2k}.$$

So $(k-1)^2 - 4k(2-k) = 5(k-1)^2 - 4$ is a square. It follows from the details of Solution H-187 in [3], or from Proposition 1.4 above, that $k-1$ is a Fibonacci number with an odd index. The index must be at least 3 because $k \geq 1$.

Now, since $L(x)$ is increasing on the interval $(0, -\psi)$, there exists at most one solution to $L(x) = k$. Thus, it suffices to show that for all $i \in \mathbb{Z}^+$

$$L\left(L_{2i-1}/L_{2i}\right) = F_{4i-1} + 1 \text{ and } L\left(F_{2i}/F_{2i+1}\right) = F_{4i+1} + 1. \tag{1.11}$$

The first equality follows from $L\left(L_{2i-1}/L_{2i}\right) = F_{2i-1} \cdot L_{2i}$ [2]. By using the closed form of Lucas numbers $L_i = \phi^i + \psi^i$ and the identity $\phi\psi = -1$, we get

$$F_{2i-1} \cdot L_{2i} = \frac{(\phi^{2i-1} - \psi^{2i-1})(\phi^{2i} + \psi^{2i})}{\phi - \psi} = \frac{\phi^{4i-1} - \psi^{4i-1} - \psi + \phi}{\phi - \psi} = F_{4i-1} + 1.$$

The second equality follows from $L\left(F_{2i}/F_{2i+1}\right) = F_{2i+1}(F_{2i+1} + F_{2i-1})$ [2]. This leads to

$$F_{2i+1}(F_{2i+1} + F_{2i-1}) = \frac{(\phi^{4i+1} - \psi^{4i+1})(\phi - \psi) + (\phi - \psi)^2}{(\phi - \psi)^2} = F_{4i+1} + 1.$$

□

This result can be adapted to treat the Pell and Pell-Lucas sequences given by $(a, b, R_0, R_1) = (2, 1, 0, 1)$ and $(a, b, R_0, R_1) = (2, 1, 2, 2)$, and denoted $\{P_i\}$ and $\{Q_i\}$, respectively. Let $Q(x)$ denote the generating function for the Pell-Lucas sequence. Then we have the following proposition, whose proof we omit since it is virtually identical to the proof of Theorem 1.5.

Theorem 1.6. *The rational numbers $x \in I$ for which $Q(x)$ is a positive integer are the ones of the form $x = P_{2i}/P_{2i+1}$ or $x = Q_{2i+1}/Q_{2(i+1)}$ where $i \in \mathbb{N}$.*

It is natural to ask whether these results can be generalized to any initial values (R_0, R_1) and any positive integers a, b with $b \mid a$. At this level of generality, proving statements analogous to those above presents difficulties we have not been able to overcome. One may also question whether the results of the previous section can be extended to hold for non-positive integers a or b. The answer is yes, even though both the results and the proofs would need to be modified slightly. One would need to account for sign changes in the terms R_r, and consequently reconsider the situations in which $R_r/R_{r+1} \in I$, as well as deal with the cases in which ϕ and ψ are complex. For example, if $a > 0$, $b < 0$, and $a^2 + 4b > 0$, the interval of convergence I is $(\psi/b, -\psi/b)$, but if $a < 0$, $b < 0$, and $a^2 + 4b > 0$, it is $(-\phi/b, \phi/b)$. Also, for both of the two cases, all R_i/R_{i+1}'s fall into I and result in $R(R_i/R_{i+1}) \in \mathbb{Z}$. So, Theorem 1.1 would need to be adjusted accordingly. The adjustments needed to address these issues appear to be purely technical and we do not believe they bring anything novel to the table.

References

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