POLYNOMIAL EXTENSIONS OF A DIMINNIE DELIGHT

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Abstract. As a neat application of Chebyshev polynomials of the first kind, we extend to Fibonacci polynomials a complex recurrence studied by C. R. Diminnie. We then explore the corresponding versions to Lucas, Pell, and Pell-Lucas polynomials, and extract the respective number-theoretic versions. In addition, we pursue two interesting recurrences with Fibonacci, Lucas, Pell, and Pell-Lucas implications.

1. Introduction

Gibonacci (generalized Fibonacci) polynomials $g_n(x)$ are defined by the recurrence $g_n(x) = xg_{n-1}(x) + g_{n-2}(x)$, where $g_1(x) = a$, $g_2(x) = b$, $a = a(x)$, $b = b(x)$, and $n \geq 3$. Clearly, $g_0(x) = b - ax$. When $a = 1$ and $b = x$, $g_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $a = x$ and $b = x^2+2$, $g_n(x) = l_n(x)$, the nth *Lucas polynomial*. In particular, $g_n(1) = G_n$, the nth *gibonacci number*; $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 5].

Table 1 shows the first six Fibonacci and Lucas polynomials.

Table 1: First Six Fibonacci and Lucas Polynomials

n	$f_n(x)$	$l_n(x)$
1	1	x
2	x	x^2+2
3	x^2+1	x^3+3x
4	x^3+2x	x^4+4x^2+2
5	x^4+3x^2+1	x^5+5x^3+5x
6	x^5+4x^3+3x	$x^6+6x^4+9x^2+2$

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1)$ and $2Q_n = q_n(1)$, respectively [4, 6].

Table 2 shows the first six Pell and Pell-Lucas polynomials, and Table 3 the first 10 Pell and Pell-Lucas numbers.

Table 2: First Six Pell and Pell-Lucas Polynomials

n	$p_n(x)$	$q_n(x)$
1	1	$2x$
2	$2x$	$4x^2+2$
3	$4x^2+1$	$8x^3+6x$
4	$8x^3+4x$	$16x^4+16x^2+2$
5	$16x^4+12x^2+1$	$32x^5+40x^3+10x$
6	$32x^5+32x^3+6x$	$64x^6+96x^4+36x^2+2$

Table 3: First 10 Pell and Pell-Lucas Numbers

n	1	2	3	4	5	6	7	8	9	10
P_n	1	2	5	12	29	70	169	408	985	2378
Q_n	1	3	7	17	41	99	239	577	1393	3363

1.1. **Binet-like Formulas.** Gibonacci polynomials can also be defined by the Binet-like formula

$$g_n = \frac{c\alpha^n - d\beta^n}{\alpha - \beta},$$

where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are solutions of the characteristic equation $t^2 - xt - 1 = 0, c = c(x) = a + (ax - b)\beta, d = d(x) = a + (ax - b)\alpha$, and $n \geq 0$. In particular,

$$f_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \alpha(x) = \dfrac{x + \Delta}{2}$ and $\beta = \beta(x) = \dfrac{x - \Delta}{2} = 0, \Delta = \Delta(x) = \sqrt{x^2 + 4}$, and $n \geq 0$ [1, 5]. Clearly, $\alpha\beta = -1$.

Likewise,

$$p_n(x) = \frac{\gamma^n - \delta^n}{\gamma - \delta},$$

$\gamma = \gamma(x) = x + D$ and $\delta = \delta(x) = x - D$ are the solutions of the equation $t^2 - 2xt - 1 = 0, D = D(x) = \sqrt{x^2 + 1}$, and $n \geq 0$ [4, 6]. Clearly, $\gamma\delta = -1$.

Chebyshev polynomials of the first kind $T_n(x)$ are defined by the recurrence $T_{n+2} = 2xT_{n+1}(x) - T_n(x)$, where $T_0(x) = 1, T_1(x) = x$, and $n \geq 0$ [6, 7]. Table 4 shows the Chebyshev polynomials $T_n(x)$, where $0 \leq n \leq 7$.

Table 4: Chebyshev polynomials $T_n(x)$

n	$T_n(x)$	n	$T_n(x)$
0	1	4	$8x^4 - 8x^2 + 1$
1	x	5	$16x^5 - 20x^3 + 5x$
2	$2x^2 - 1$	6	$32x^6 - 48x^4 + 18x^2 - 1$
3	$4x^3 - 3x$	7	$64x^7 - 112x^5 + 56x^3 - 7x$

To make our exposition simple, short, and elegant, we employ a slightly modified version of the polynomials $T_n(x)$. To this end, consider the polynomials $c_n(x)$, defined by the recurrence $c_n(x) = xc_{n-1}(x) - c_{n-2}(x)$, where $c_0(x) = 2, c_1(x) = x$, and $n \geq 2$. Then

$$\begin{aligned} c_2(x) &= x^2 - 2 & c_3(x) &= x^3 - 3x \\ c_4(x) &= x^4 - 4x^2 + 2 & c_5(x) &= \boxed{x^5 - 5x^3 + 5x} \\ c_6(x) &= x^6 - 6x^4 + 9x^2 - 2 & c_7(x) &= x^7 - 7x^5 + 14x^3 - 7x \\ &\vdots \end{aligned}$$

Clearly, $c_n(x) = 2T_n(x/2) = i^n l_n(-ix)$, where $i = \sqrt{-1}$. For example, $2T_5(x/2) = 2[16(x/2)^5 - 20(x/2)^3 + 5(x/2)] = x^5 - 5x^3 + 5x = c_5(x)$.

The polynomials $c_n(x)$ satisfy a charming property:

$$c_n\left(y + \frac{1}{y}\right) = y^n + \frac{1}{y^n}, \tag{1.1}$$

where $y \neq 0$ and $n \geq 0$; this follows by induction.

1.2. **A Diminnie Delight.** In 1994, C. R. Diminnie proposed the following spectacular problem [2]. Solve the recurrence

$$d_{n+1} = 5d_n(5d_n^4 - 5d_n^2 + 1), \tag{1.2}$$

where $d_0 = 1$ and $n \geq 0$. A few months later, A. Sinefakopoulos provided a beautiful solution to the problem [3, 8]: $d_n = F_{5^n}$.

We can extend this problem to Fibonacci polynomials. In the interest of brevity, clarity, and convenience, we drop the argument from the functional notation when omitting it causes *no* ambiguity. For example, g_n will mean $g_n(x)$.

2. Fibonacci Extensions

Solve the recurrence

$$a_{n+1} = a_n(\Delta^4 a_n^4 - 5\Delta^2 a_n^2 + 5), \tag{2.1}$$

where $a_n = a_n(x)$, $a_0 = 1$, and $n \geq 0$.

Then,

$$\begin{aligned} a_0 &= 1 \\ a_1 &= x^4 + 3x^2 + 1 \\ a_2 &= x^{24} + 23x^{22} + 231x^{20} + 1330x^{18} + 4845x^{16} + 11628x^{14} + 18564x^{12} \\ &\quad + 19448x^{10} + 12870x^8 + 5005x^6 + 1001x^4 + 78x^2 + 1 \\ a_3 &= x^{124} + 123x^{122} + 7381x^{120} + \cdots + 1, \end{aligned}$$

a polynomial of degree 124 and with 63 terms.

By looking at these four initial values of a_n, it does not appear to be easy to conjecture a formula for a_n. But, here is an interesting observation: $a_0 = f_{5^0}, a_1 = f_{5^1}$, and $a_2 = f_{5^2}$. This, coupled with the solution $d_n = F_{5^n}$ of recurrence (2.1), helps us conjecture that $a_n = f_{5^n}$, where $n \geq 0$.

To confirm this formula, we rely on the polynomials $c_n(x)$. To this end, first we establish a close relationship between a_{n+1} and c_5. Using recurrence (2.1), we have

$$\begin{aligned} \Delta a_{n+1} &= \Delta^5 a_n^5 - 5\Delta^3 a_n^3 + 5\Delta a_n \\ &= (\Delta a_n)^5 - 5(\Delta a_n)^3 + 5(\Delta a_n) \\ &= c_5(\Delta a_n). \end{aligned}$$

Consequently, we claim that the solution of the recurrence $\Delta a_{n+1} = c_5(\Delta a_n)$ is $a_n = f_{5^n}$.

More generally, we will now confirm that the solution of the recurrence

$$\Delta a_{n+1} = c_m(\Delta a_n) \tag{2.2}$$

is $a_n = f_{k \cdot m^n}$, where $a_0 = f_k, k$ and m are odd positive integers, $k \neq m, m \geq 3$ and $n \geq 0$.

Proof. Clearly, the formula is true when $n = 0$. Assume, it is true for an arbitrary integer $n \geq 0$. Since k and m are odd, by the Binet-like formula for $f_{k\cdot m^n}$, we then have

$$\begin{aligned}
\Delta a_{n+1} &= c_m(\Delta f_{k\cdot m^n}) \\
&= c_m\left(\alpha^{k\cdot m^n} - \beta^{k\cdot m^n}\right) \\
&= c_m\left(\alpha^{k\cdot m^n} + \frac{1}{\alpha^{k\cdot m^n}}\right) \\
&= \alpha^{k\cdot m^{n+1}} + \frac{1}{\alpha^{k\cdot m^{n+1}}} \\
&= \alpha^{k\cdot m^{n+1}} - \beta^{k\cdot m^{n+1}} \\
a_{n+1} &= f_{k\cdot m^{n+1}}.
\end{aligned}$$

So the formula works for $n + 1$ also. Thus, by induction, formula (2.2) works for all $n \geq 0$; that is, the solution of recurrence (2.2) is $a_n = f_{k\cdot m^n}$. □

For example, with $k = 3, m = 5$, and $a_0 = f_3 = x^2 + 1$, we have

$$\begin{aligned}
a_1 &= a_0(\Delta^4 a_0^4 - 5\Delta^2 a_0^2 + 5) \\
&= (x^2+1)[(x^2+4)^2(x^2+1)^4 - 5(x^2+4)(x^2+1)^2 + 5] \\
&= x^{14} + 13x^{12} + 66x^{10} + 165x^8 + 210x^6 + 126x^4 + 28x^2 + 1 \\
&= f_{3\cdot 5}.
\end{aligned}$$

In particular, the solution of recurrence (2.1) is $a_n = f_{5^n}$, where $n \geq 0$, as conjectured. Clearly, the solution of recurrence (1.2) follows from this.

Suppose we let $m = 3$ in recurrence (2.2). Since $c_3(x) = x^3 - 3x$,

$$\begin{aligned}
\Delta a_{n+1} &= c_3(\Delta a_n) \\
a_{n+1} &= \Delta^2 a_n^3 - 3a_n. \qquad (2.3)
\end{aligned}$$

The solution of this recurrence is $a_n = f_{k\cdot 3^n}$, where $a_0 = f_k$ and $n \geq 0$.

Likewise, when $m = 7$, we get

$$a_{n+1} = \Delta^6 a_n^7 - 7\Delta^4 a_n^5 + 14\Delta^2 a_n^3 - 7a_n; \qquad (2.4)$$

its solution is $a_n = f_{k\cdot 7^n}$, where $a_0 = f_k$ and $n \geq 0$.

Obviously, we can continue this procedure for any odd integer ≥ 9.

In particular, let $x = 1 = k$. Then the solutions of the recurrences $a_{n+1} = 5a_n^3 - 3a_n$ and $a_{n+1} = 125a_n^7 - 175a_n^5 + 70a_n^3 - 7a_n$ are $a_n = F_{3^n}$ and $a_n = F_{7^n}$, respectively.

As we can predict, the polynomial extension (2.1) has Pell consequences.

2.1. **Pell Extensions.** Let $b_n = b_n(x) = a_n(2x), b_0 = 1$, and $n \geq 0$. Then recurrences (2.3), (2.1), and (2.4) yield

$$b_{n+1} = b_n(4D^2 b_n^2 - 3) \qquad (2.5)$$

$$b_{n+1} = b_n(16D^4 b_n^4 - 20D^2 b_n^2 + 5) \qquad (2.6)$$

$$b_{n+1} = b_n(64D^6 b_n^6 - 112D^4 b_n^4 + 56D^2 b_n^2 - 7), \qquad (2.7)$$

respectively. The corresponding solutions are $b_n = p_{k\cdot 3^n}, b_n = p_{k\cdot 5^n}$, and $b_n = p_{k\cdot 7^n}$, respectively.

When $x = 1 = k$, these yield the solutions $b_n = P_{3^n}, b_n = P_{5^n}$, and $b_n = P_{7^n}$, respectively.

For example, $b_1 = 29 = P_{5^1}$; so $b_2 = 29(64 \cdot 29^4 - 40 \cdot 29^2 + 5) = 1,311,738,121 = P_{5^2}$, as expected.

3. Lucas Counterparts

Recall that the solutions of recurrences (1.2), (2.3), and (2.4) pivoted on the polynomial $c_m(x)$, where m is odd and ≥ 3. Interestingly, focusing on $c_m(x)$ with m even and ≥ 2 yields equally rewarding results.

For example, consider the recurrence

$$a_{n+1} = a_n^4 - 4a_n^2 + 2, \tag{3.1}$$

where $a_n = a_n(x), a_1 = l_{4e}, e$ is a positive integer such that $4 \nmid e$, and $n \geq 1$.

Clearly, $a_{n+1} = c_4(a_n)$. Using the Binet-like formula for $l_{e \cdot 4^n}$, property (1.1), and induction, we can show that $a_n = l_{e \cdot 4^n}$.

For example, let $e = 1$. Then $a_1 = l_4 = x^4 + 4x^2 + 2$, and

$$\begin{aligned} a_2 &= a_1^4 - 4a_1^2 + 2 \\ &= (x^4 + 4x^2 + 2)^4 - 4(x^4 + 4x^2 + 2)^2 + 2 \\ &= x^{16} + 16x^{14} + 104x^{12} + 352x^{10} + 660x^8 + 672x^6 + 336x^4 + 64x^2 + 2 \\ &= l_{4^2}. \end{aligned}$$

Similarly, the recurrences

$$a_{n+1} = a_n^2 - 2, \ a_1 = l_{2e} \ (2 \nmid e); \tag{3.2}$$

$$a_{n+1} = a_n^6 - 6a_n^4 + 9a_n^2 - 2, \ a_1 = l_{6e} \ (6 \nmid e) \tag{3.3}$$

yield the abbreviated recurrences $a_{n+1} = c_2(a_n)$ and $a_{n+1} = c_6(a_n)$, respectively, where $a_n = a_n(x)$. Correspondingly, we have $a_n = l_{e \cdot 2^n}$ and $a_n = l_{e \cdot 6^n}$, respectively.

In particular, let $x = 1 = e$. Then L_{2^n}, L_{4^n}, and L_{6^n} are the solutions of the recurrences (3.2), (3.1), and (3.3), respectively; M. Klamkin (1921–2004) found these solutions [8].

3.1. Pell-Lucas Byproducts.

Since $l_k(2x) = q_k(x)$, it follows from recurrences (3.2), (3.1), and (3.3) that

$$\begin{aligned} b_{n+1} &= b_n^2 - 2, \ b_1 = q_{2e} \ (2 \nmid e); \\ b_{n+1} &= b_n^4 - 4b_n^2 + 2, \ b_1 = q_{4e} \ (4 \nmid e); \\ b_{n+1} &= b_n^6 - 6b_4^4 + 9b_n^2 - 2, \ b_1 = q_{6e} \ (6 \nmid e), \end{aligned} \tag{3.4}$$

respectively, where $b_n = a_n(2x)$. The corresponding solutions are $b_n = q_{e \cdot 2^n}, b_n = q_{e \cdot 4^n}$, and $b_n = q_{e \cdot 6^n}$, respectively.

In particular, let $x = 1 = e$. Then $b_n = 2Q_{2^n}, b_n = 2Q_{4^n}$, and $b_n = 2Q_{6^n}$, respectively.

For example, consider recurrence (3.4), where $b_1 = 34 = 2Q_4$. Then $b_2 = 34^4 - 4 \cdot 34^2 + 2 = 1,331,714 = 2Q_{4^2}$.

4. Two Charming Recurrences

Next we study two equally delightful recurrences with Fibonacci and Pell implications.

4.1. **Recurrence A.** Consider the recurrence

$$x_{n+1} = x_n(\Delta^2 x_n^2 + 3), \tag{4.1}$$

where $x_0 = f_e, e$ is a positive even integer, and $n \geq 0$.

Suppose $e = 2$. Then $x_0 = f_2 = x$ and $x_1 = x[(x^2+4)x^2+3] = x^5 + 4x^3 + 3x = f_{2\cdot 3}$.

More generally, we conjecture that $x_n = f_{e\cdot 3^n}$, where $n \geq 0$. It is clearly true when $n = 0$. Assume it is true for an arbitrary integer $n \geq 0$. Then

$$\begin{aligned}
x_{n+1} &= \Delta^2 f_{e\cdot 3^n}^3 + 3f_{e\cdot 3^n} \\
\Delta x_{n+1} &= \left(\alpha^{e\cdot 3^n} - \beta^{e\cdot 3^n}\right)^3 + 3\left(\alpha^{e\cdot 3^n} - \beta^{e\cdot 3^n}\right) \\
&= \alpha^{e\cdot 3^{n+1}} - \beta^{e\cdot 3^{n+1}} - 3(\alpha\beta)^{e\cdot 3^n}\left(\alpha^{e\cdot 3^n} - \beta^{e\cdot 3^n}\right) + 3\left(\alpha^{e\cdot 3^n} - \beta^{e\cdot 3^n}\right) \\
&= \alpha^{e\cdot 3^{n+1}} - \beta^{e\cdot 3^{n+1}} \\
x_{n+1} &= f_{e\cdot 3^{n+1}}.
\end{aligned}$$

Thus, by induction, the conjecture works for all $n \geq 0$.

For example, let $e = 4$. Then $x_0 = f_4 = x^3 + 2x$. So

$$\begin{aligned}
x_1 &= x_0(\Delta^2 x_0^2 + 3) \\
&= (x^3+2x)[(x^2+4)(x^3+2x)^2+3] \\
&= x^{11} + 10x^9 + 36x^7 + 56x^5 + 35x^3 + 6x \\
&= f_{4\cdot 3^1}.
\end{aligned}$$

In particular, the solution of the recurrence $x_{n+1} = x_n(5x_n^2+3)$ is $x_n = F_{e\cdot 3^n}$, where $x_0 = F_e$ and $n \geq 0$.

4.2. **Pell Byproducts.** It follows from recurrence (4.1) that the solution of the recurrence $x_{n+1} = x_n(4D^2x_n^2 + 3)$ is $x_n = p_{e\cdot 3^n}$, where $x_0 = p_e$ and $n \geq 0$. In particular, the solution of the recurrence $x_{n+1} = x_n(8x_n^2+3)$ is $x_n = P_{e\cdot 3^n}$.

Next we study a similar recurrence which also has interesting consequences.

4.3. **Recurrence B.** Consider the recurrence

$$z_{n+2} = z_{n+1}(\Delta^2 z_n^2 + 2), \tag{4.2}$$

where $z_1 = f_{2k}, z_2 = f_{4k}, k$ is an odd positive integer, and $n \geq 1$.

When $k = 1$, $z_1 = f_2 = x$ and $z_2 = f_4 = x^3 + 2x$. So

$$\begin{aligned}
z_3 &= (x^3+2x)[(x^2+4)x^2+2] \\
&= x^7 + 6x^5 + 10x^3 + 4x \\
&= f_{2^3}.
\end{aligned}$$

More generally, it follows by induction that the solution of recurrence (4.2) is $z_n = f_{k\cdot 2^n}$, where $n \geq 1$.

For example, let $k = 3$. Then $z_1 = f_6 = x^5 + 4x^3 + 3x$ and $z_2 = f_{12} = x^{11} + 10x^9 + 36x^7 + 56x^5 + 35x^3 + 6x$. Consequently,

$$\begin{aligned} z_3 &= z_2(\Delta^2 z_1^2 + 2) \\ &= (x^{11} + 10x^9 + 36x^7 + 56x^5 + 35x^3 + 6x)[(x^2 + 4)(x^5 + 4x^3 + 3x)^2 + 2] \\ &= x^{23} + 22x^{21} + 210x^{19} + 1140x^{17} + 3876x^{15} + 8568x^{13} + 12376x^{11} \\ &\quad + 11440x^9 + 6435x^7 + 2002x^5 + 286x^3 + 12x \\ &= f_{3\cdot 2^3}. \end{aligned}$$

Suppose we let $x = 1$ in recurrence (4.2). Then the solution of the recurrence $z_{n+2} = z_{n+1}(5z_n^2 + 2)$ is $z_n = F_{k\cdot 2^n}$, where $z_1 = F_{2k}, z_2 = F_{4k}$, and $n \geq 0$.

Recurrence (4.1) also has Pell implications.

4.4. **Pell Consequences.** The solution of the recurrence $z_{n+2} = 2z_{n+1}(2D^2 z_n^2 + 1)$ is $z_n = p_{k\cdot 2^n}$, where $z_1 = p_{2k}, z_2 = p_{4k}$, and $n \geq 0$. Consequently, the solution of the recurrence $z_{n+2} = 2z_{n+1}(4z_n^2 + 1)$ is $z_n = P_{k\cdot 2^n}$, where $z_1 = P_{2k}, z_2 = P_{4k}$, and $n \geq 0$.

For example, let $k = 5$. Then $z_1 = P_{10} = 2,378$ and $z_2 = P_{20} = 15,994,428$. So $z_3 = 2z_2(4z_1^2 + 1) = 2 \cdot 15,994,428(4 \cdot 2378^2 + 1) = 723,573,111,879,672 = P_{40}$.

4.5. **Lucas Counterparts.** Interestingly, recurrences (4.1) and (4.2) have their own Lucas counterparts:

$$u_{n+1} = u_n(u_n^2 - 3), \tag{4.3}$$

where $u_0 = l_e$ and $n \geq 0$; and

$$v_{n+2} = v_{n+1}(v_n^2 - 2) - 2, \tag{4.4}$$

where $v_1 = l_{2k}, v_2 = l_{4k}$, and $n \geq 1$.

Their solutions are $u_n = l_{e\cdot 3^n}$ and $v_n = l_{k\cdot 2^n}$, respectively. Their proofs follow similarly, so we omit them.

For example, let $e = 4$. Then

$$\begin{aligned} u_1 &= l_4(l_4^2 - 3) \\ &= x^{12} + 12x^{10} + 54x^8 + 112x^6 + 105x^4 + 36x^2 + 2 \\ &= l_{l_{4\cdot 3^1}}; \end{aligned}$$

likewise,

$$\begin{aligned} v_3 &= l_{12}(l_6^2 - 2) - 2 \\ &= x^{24} + 24x^{22} + 252x^{20} + 1520x^{18} + 5814x^{16} + 14688x^{14} \\ &\quad + 24752x^{12} + 27456x^{10} + 19305x^8 + 8008x^6 + 1716x^4 + 144x^2 + 2 \\ &= l_{3\cdot 2^3}. \end{aligned}$$

4.6. **Pell-Lucas Versions.** It follows from recurrences (4.3) and (4.4) that the solutions of the recurrences

$$u_{n+1} = u_n(u_n^2 - 3),\ u_0 = q_e; \tag{4.5}$$

and

$$v_{n+2} = v_{n+1}(v_n^2 - 2) - 2,\ v_1 = q_{2k} \text{ and } v_2 = q_{4k} \tag{4.6}$$

are $u_n = q_{e\cdot 3^n}$ and $v_n = q_{k\cdot 2^n}$, respectively.

In particular, let $x = 1$. Then the solutions of the recurrences (4.3), (4.4), (4.5), and (4.6) are $u_n = L_{e\cdot 3^n}, v_n = L_{k\cdot 2^n}, u_n = 2Q_{e\cdot 3^n}$, and $v_n = 2Q_{k\cdot 2^n}$, respectively.

For example, when $k = 1$,

$$\begin{aligned} v_5 &= v_4(v_3^2 - 2) - 2 \\ &= 1,331,714(1154^2 - 2) - 2 \\ &= 2 \cdot 886,731,088,897 \\ &= 2Q_{2^5}. \end{aligned}$$

Finally, we invite Fibonacci enthusiasts to interpret combinatorially the recurrences investigated in the article.

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