FIBONACCI AND LUCAS NUMBERS WHICH ARE ONE AWAY FROM THEIR PRODUCTS

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Abstract. We explicitly solve the Diophantine equations of the form

$$A_{n_1}A_{n_2}A_{n_3}\cdots A_{n_k} \pm 1 = B_m,$$

where (A_n) and (B_m) are the Fibonacci or Lucas sequences.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, and let $(L_n)_{n\geq 0}$ be the Lucas sequence given by the same recursive pattern but with the initial values $L_0 = 2$ and $L_1 = 1$.

Diophantine equations involving Fibonacci and Lucas numbers have been a popular area of research as collected in Guy's book [5] and in the historical section of [2] and [3]. See also [7, 8, 9], and [13] for some recent results on this topic. In this article, we are interested in solving the following Diophantine equations:

$$F_m = F_{n_1}F_{n_2}F_{n_3}\cdots F_{n_k} \pm 1, \tag{1.1}$$

$$F_m = L_{n_1}L_{n_2}L_{n_3}\cdots L_{n_k} \pm 1, \tag{1.2}$$

$$L_m = F_{n_1}F_{n_2}F_{n_3}\cdots F_{n_k} \pm 1, \tag{1.3}$$

$$L_m = L_{n_1}L_{n_2}L_{n_3}\cdots L_{n_k} \pm 1, \tag{1.4}$$

where $m \geq 0$, $k \geq 1$, and $0 \leq n_1 \leq n_2 \leq \cdots \leq n_k$.

Since $F_0 = 0$, $F_1 = F_2 = L_1 = 1$, we avoid some trivial solutions when $k \geq 2$ by assuming that $n_1 \geq 3$ in (1.1) and (1.3), and that $n_j \neq 1$ for any $j \in \{1, 2, \ldots, k\}$ in (1.2) and (1.4). Notice that (1.1), (1.2), (1.3), and (1.4) are actually equivalent to, respectively,

$$\begin{aligned} F_{n_1}^{a_1}F_{n_2}^{a_2}\cdots F_{n_\ell}^{a_\ell} \pm 1 &= F_m \\ L_{n_1}^{a_1}L_{n_2}^{a_2}\cdots L_{n_\ell}^{a_\ell} \pm 1 &= F_m \\ F_{n_1}^{a_1}F_{n_2}^{a_2}\cdots F_{n_\ell}^{a_\ell} \pm 1 &= L_m \\ L_{n_1}^{a_1}L_{n_2}^{a_2}\cdots L_{n_\ell}^{a_\ell} \pm 1 &= L_m \end{aligned}$$

where $m \geq 0$, $\ell \geq 1$, $0 \leq n_1 < n_2 < \cdots < n_\ell$, and $a_1, a_2, \ldots, a_\ell \geq 1$. For convenience, we sometimes go back and forth between the equations given in (1.1) to (1.4) and those which are equivalent to them such as the above.

Finally, we remark that similar equations are also considered by Pongsriiam [11] and partially by Szalay [13], where F_m and L_m in (1.1), (1.2), (1.3), and (1.4) are replaced by F_m^2 and L_m^2.

2. Preliminaries and Lemmas

Since one of our main tools in solving the above equations is the Primitive Divisor Theorem of Carmichael [4], we first recall some facts about it. Let α and β be algebraic numbers such that $\alpha+\beta$ and $\alpha\beta$ are nonzero coprime integers and $\alpha\beta^{-1}$ is not a root of unity. Let $(u_n)_{n\geq 0}$ be the sequence given by

$$u_0 = 0,\ u_1 = 1, \text{ and } u_n = (\alpha+\beta)u_{n-1} - (\alpha\beta)u_{n-2} \text{ for } n \geq 2.$$

Then Binet's formula for u_n is given by

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for } n \geq 0.$$

So if $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, then (u_n) is the Fibonacci sequence.

A prime p is said to be a primitive divisor of u_n if $p \mid u_n$ but p does not divide $u_1u_2\cdots u_{n-1}$. Then the primitive divisor theorem of Carmichael can be stated as follows.

Theorem 2.1. [Primitive Divisor Theorem of Carmichael [4]] *If α and β are real numbers and $n \neq 1, 2, 6$, then u_n has a primitive divisor except when $n = 12$, $\alpha+\beta = 1$ and $\alpha\beta = -1$. In particular, F_n has a primitive divisor for every $n \neq 1, 2, 6, 12$, and L_n has a primitive divisor for every $n \neq 1, 6$.*

There is a long history about primitive divisors and the most remarkable results in this topic are given by Bilu, Hanrot, and Voutier [1], by Stewart [12], and by Kunrui [6], but Theorem 2.1 is good enough in our situation.

In solving equation (1.2), it is useful to recall Pongsriiam's result [10] on the factorization of Fibonacci numbers as a product of Lucas numbers as follows.

Theorem 2.2. [10, Theorem 2] *A Fibonacci number F_m can be written as a product of Lucas numbers if and only if $m = 2^\ell$ or $m = 3\cdot 2^\ell$ for some $\ell \geq 0$. Furthermore, for each $\ell \geq 2$, there is a unique representation of F_{2^ℓ}, and exactly five representations of $F_{3\cdot 2^\ell}$ as a nontrivial unordered product of Lucas numbers:*

$$F_{2^\ell} = L_{2^{\ell-1}}L_{2^{\ell-2}}L_{2^{\ell-3}}\cdots L_2, \tag{2.1}$$

$$F_{3\cdot 2^\ell} = L_{3\cdot 2^{\ell-1}}L_{3\cdot 2^{\ell-2}}L_{3\cdot 2^{\ell-3}}\cdots L_{12}A, \quad \textit{where} \tag{2.2}$$

$$A = F_{12} = L_2L_2L_0L_0L_0L_0 = L_3L_2L_2L_0L_0 = L_6L_0L_0L_0 = L_6L_3L_0 = L_3L_3L_2L_2. \tag{2.3}$$

Here nontrivial product means that there is no $L_1 = 1$ as a factor, and unordered product means that the permutation between the factors is not counted as a distinct representation.

Remark 2.3. *If $\ell = 2$, then the product $L_{3\cdot 2^{\ell-1}}L_{3\cdot 2^{\ell-2}}L_{3\cdot 2^{\ell-3}}\cdots L_{12}$ appearing in (2.2) is empty and (2.2) becomes $F_{12} = A$, which can be written as a product of Lucas numbers as given in (2.3).*

We also need a factorization of $F_m \pm 1$ and $L_m \pm 1$. Recall that we can define F_n and L_n for a negative integer n by the formula

$$F_{-k} = (-1)^{k+1}F_k \text{ and } L_{-k} = (-1)^k L_k \text{ for } k \geq 0.$$

Then the following holds for all integers m, k.

$$F_mL_k = F_{m+k} + (-1)^k F_{m-k}. \tag{2.4}$$

The identity (2.4) can be proved using Binet's formula and straightforward algebraic manipulation as follows.

$$\begin{aligned} F_m L_k &= \left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right)\left(\alpha^k + \beta^k\right) \\ &= \frac{\alpha^{m+k} + \alpha^m \beta^k - \beta^m \alpha^k - \beta^{m+k}}{\alpha - \beta} \\ &= \frac{\alpha^{m+k} - \beta^{m+k}}{\alpha - \beta} + \frac{\alpha^m \left(-\frac{1}{\alpha}\right)^k - \beta^m \left(-\frac{1}{\beta}\right)^k}{\alpha - \beta} \\ &= F_{m+k} + (-1)^k F_{m-k}. \end{aligned}$$

We will particularly apply (2.4) in the following form.

Lemma 2.4. *For every $m \geq 1$, we have*

$$\text{(i)}\ \ F_m - 1 = \begin{cases} F_{\frac{m+2}{2}} L_{\frac{m-2}{2}}, & \text{if } m \equiv 0 \pmod 4; \\ F_{\frac{m-1}{2}} L_{\frac{m+1}{2}}, & \text{if } m \equiv 1 \pmod 4; \\ F_{\frac{m-2}{2}} L_{\frac{m+2}{2}}, & \text{if } m \equiv 2 \pmod 4; \\ F_{\frac{m+1}{2}} L_{\frac{m-1}{2}}, & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

$$\text{(ii)}\ \ F_m + 1 = \begin{cases} F_{\frac{m-2}{2}} L_{\frac{m+2}{2}}, & \text{if } m \equiv 0 \pmod 4; \\ F_{\frac{m+1}{2}} L_{\frac{m-1}{2}}, & \text{if } m \equiv 1 \pmod 4; \\ F_{\frac{m+2}{2}} L_{\frac{m-2}{2}}, & \text{if } m \equiv 2 \pmod 4; \\ F_{\frac{m-1}{2}} L_{\frac{m+1}{2}}, & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

Proof. This follows immediately from (2.4). For example, if m is even, replacing m by $\frac{m+2}{2}$ and k by $\frac{m-2}{2}$ in (2.4), we obtain

$$F_{\frac{m+2}{2}} L_{\frac{m-2}{2}} = F_m + (-1)^{\frac{m-2}{2}} F_2,$$

which is equal to $F_m - 1$ if $m \equiv 0 \pmod 4$ and is equal to $F_m + 1$ if $m \equiv 2 \pmod 4$. □

Next, we give a factorization of $L_m \pm 1$.

Lemma 2.5. *For every $m \geq 1$, we have*

$$\text{(i)}\ \ L_m - 1 = \begin{cases} L_{\frac{3m}{2}} / L_{\frac{m}{2}}, & \text{if } m \equiv 0 \pmod 4; \\ 5F_{\frac{m+1}{2}} F_{\frac{m-1}{2}}, & \text{if } m \equiv 1 \pmod 4; \\ F_{\frac{3m}{2}} / F_{\frac{m}{2}}, & \text{if } m \equiv 2 \pmod 4; \\ L_{\frac{m+1}{2}} L_{\frac{m-1}{2}}, & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

$$\text{(ii)}\ \ L_m + 1 = \begin{cases} F_{\frac{3m}{2}} / F_{\frac{m}{2}}, & \text{if } m \equiv 0 \pmod 4; \\ L_{\frac{m+1}{2}} L_{\frac{m-1}{2}}, & \text{if } m \equiv 1 \pmod 4; \\ L_{\frac{3m}{2}} / L_{\frac{m}{2}}, & \text{if } m \equiv 2 \pmod 4; \\ 5F_{\frac{m+1}{2}} F_{\frac{m-1}{2}}, & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

Proof. Similar to (2.4), this can be easily checked using Binet's formula and algebraic manipulation. □

3. Main Results

We begin this section by solving (1.2). Then we solve (1.4), (1.3), and (1.1), respectively. The solutions to each equation are a bit different but many of them can be obtained by a similar argument. In this case, we give a detailed proof for the first and a short proof for the others.

Theorem 3.1. *The Diophantine equation*

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} + 1 = F_m \tag{3.1}$$

with $m \geq 0$, $k \geq 1$, *and* $0 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ *has a solution if and only if*

$$m = 2^a - 1, 2^a + 1, 2^{a-1} + 2, 3 \cdot 2^a - 1, 3 \cdot 2^a + 1, 3 \cdot 2^a + 2,$$

for some $a \geq 2$. *In this case, the nontrivial solutions to* (3.1) *are given by*

(i) $L_1 + 1 = F_3$, $L_0^2 L_2 + 1 = L_2 L_3 + 1 = F_7$ *and for* $a \geq 4$,

$$L_2 L_4 L_8 \cdots L_{2^{a-3}} L_{2^{a-2}} L_{2^{a-1}-1} + 1 = F_{2^a - 1},$$

(ii) $L_0^2 + 1 = L_3 + 1 = F_5$ *and for* $a \geq 3$,

$$L_2 L_4 L_8 \cdots L_{2^{a-3}} L_{2^{a-2}} L_{2^{a-1}+1} + 1 = F_{2^a + 1},$$

(iii) $L_0 + 1 = F_4$, $L_4 + 1 = F_6$, $L_0 L_2^3 + 1 = L_2 L_6 + 1 = F_{10}$, *and for* $a \geq 5$,

$$L_2 L_4 L_8 \cdots L_{2^{a-4}} L_{2^{a-3}} L_{2^{a-2}+2} + 1 = F_{2^{a-1}+2},$$

(iv) $L_0^3 L_5 + 1 = L_0 L_3 L_5 + 1 = F_{11}$, *and for* $a \geq 3$,

$$A L_{12} L_{24} \cdots L_{3 \cdot 2^{a-2}} L_{3 \cdot 2^{a-1} - 1} + 1 = F_{3 \cdot 2^a - 1},$$

(v) $L_0^3 L_7 + 1 = L_0 L_3 L_7 + 1 = F_{13}$, *and for* $a \geq 3$,

$$A L_{12} L_{24} \cdots L_{3 \cdot 2^{a-2}} L_{3 \cdot 2^{a-1} + 1} + 1 = F_{3 \cdot 2^a + 1}, \quad \text{and}$$

(vi) $L_0^3 L_8 + 1 = L_0 L_3 L_8 + 1 = F_{14}$, *and for* $a \geq 3$,

$$A L_{12} L_{24} \cdots L_{3 \cdot 2^{a-2}} L_{3 \cdot 2^{a-1} + 2} + 1 = F_{3 \cdot 2^a + 2},$$

where $A = F_{12} = L_0^4 L_2^2 = L_0^2 L_2^2 L_3 = L_0^3 L_6 = L_0 L_3 L_6 = L_2^2 L_3^2$.

Here nontrivial solutions means either that $k = 1$ *or* $k \geq 2$ *and* $n_j \neq 1$ *for any* $j \in \{1, 2, \ldots, k\}$.

Remark 3.2. *If* $a = 3$, *the product* $L_{12} L_{24} \cdots L_{3 \cdot 2^{a-2}}$ *appearing in (iv) of Theorem 3.1 is empty. In this case, the equation*

$$A L_{12} L_{24} \cdots L_{3 \cdot 2^{a-2}} L_{3 \cdot 2^{a-1} - 1} + 1 = F_{3 \cdot 2^a - 1}$$

becomes

$$A L_{11} + 1 = F_{23}.$$

Similarly, if $a = 3$, *the last equations appearing in (v) and (vi) of Theorem 3.1 become* $AL_{13} = F_{25}$ *and* $AL_{14} = F_{26}$, *respectively.*

Proof of Theorem 3.1. Since the result can be easily checked for $1 \leq m \leq 14$, we assume throughout that $m \geq 15$.

Case 1: $m \equiv 1 \pmod 4$. Then by Lemma 2.4(i), we can write (3.1) as

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} = F_{\frac{m-1}{2}} L_{\frac{m+1}{2}}. \tag{3.2}$$

If $n_k = 0$, then the left-hand side of (3.2) is 2^k but by Theorem 2.1 the right-hand side of (3.2) has a prime divisor distinct from 2, a contradiction. So $n_k > 0$. By the well-known identity $F_{2n} = F_n L_n$, we can write (3.2) as

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_{k-1}} F_{2n_k} F_{\frac{m+1}{2}} = F_{\frac{m-1}{2}} F_{m+1} F_{n_k}. \tag{3.3}$$

Suppose for a contradiction that $m+1 > 2n_k$. By Theorem 2.1, there exists a prime p dividing F_{m+1} but does not divide F_ℓ for any $\ell < m+1$. Since $p \mid F_{m+1}$ and p does not divide F_{2n_k} and $F_{\frac{m+1}{2}}$, we see that $p \mid L_{n_i}$ for some $i = 1, 2, \ldots, k-1$. Then $p \mid L_{n_i} = \frac{F_{2n_i}}{F_{n_i}} \mid F_{2n_i}$ and $2n_i \leq 2n_{k-1} \leq 2n_k < m+1$, which is a contradiction. Similarly, the inequality $m+1 < 2n_k$ leads to a contradiction. So $m+1 = 2n_k$ and (3.3) is reduced to

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_{k-1}} = F_{\frac{m-1}{2}}.$$

We remark that this kind of argument will be used repeatedly throughout this article. Now we see that $F_{\frac{m-1}{2}}$ is a product of Lucas numbers, so we obtain by Theorem 2.2 that $\frac{m-1}{2} = 2^a$ or $3 \cdot 2^a$ for some $a \geq 0$. Since $m \geq 16$, $m = 2^b + 1$ or $3 \cdot 2^c + 1$, where $b \geq 4$ and $c \geq 3$. Theorem 2.2 also gives all representations of F_{2^a}, $F_{3 \cdot 2^a}$ for any $a \geq 2$. So for $m = 2^b + 1$, we obtain

$$F_{\frac{m-1}{2}} = F_{2^{b-1}} = L_{2^{b-2}} L_{2^{b-3}} \cdots L_4 L_2,$$

which means that $k = b - 1$, $n_1 = 2$, $n_2 = 4$, $n_3 = 8, \ldots, n_{k-1} = 2^{b-2}$, and $n_k = 2^{b-1} + 1$. For $m = 3 \cdot 2^c + 1$, we have

$$F_{\frac{m-1}{2}} = F_{3 \cdot 2^{c-1}} = L_{3 \cdot 2^{c-2}} L_{3 \cdot 2^{c-3}} \cdots L_{12} A,$$

where $A = F_{12} = L_2^2 L_0^4 = L_3 L_2^2 L_0^2 = L_6 L_0^3 = L_6 L_3 L_0 = L_3^2 L_2^2$. This gives five sets of solutions corresponding to $m = 3 \cdot 2^c + 1$.

Case 2: $m \equiv 2 \pmod 4$. By Lemma 2.4(i), (3.1) can be written as

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} = F_{\frac{m-2}{2}} L_{\frac{m+2}{2}}. \tag{3.4}$$

Similar to Case 1, we apply the identity $F_{2n} = F_n L_n$ and Theorem 2.1 to obtain $2n_k = m+2$ and (3.4) is reduced to

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_{k-1}} = F_{\frac{m-2}{2}}.$$

Similar to Case 1, we apply Theorem 2.2 to obtain

$$m = 2^b + 2 \text{ or } 3 \cdot 2^c + 2 \text{ for some } b \geq 4,\ c \geq 3,$$

and all representations of $F_{\frac{m-2}{2}}$ as a product of Lucas numbers.

Case 3: $m \equiv 3 \pmod 4$. By Lemma 2.4(i), (3.1) can be written as

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} = F_{\frac{m+1}{2}} L_{\frac{m-1}{2}}.$$

Similar to Case 1, applying Theorem 2.1 leads to $m - 1 = 2n_k$, and Theorem 2.2 gives

$$m = 2^b - 1 \text{ or } 3 \cdot 2^c - 1 \text{ for some } b \geq 4,\ c \geq 3,$$

and the required representations of $F_{\frac{m+1}{2}}$.

Case 4: $m \equiv 0 \pmod 4$. Similar to the other cases, we first apply Lemma 2.4(i), then use Theorems 2.1 and 2.2 to conclude that

$$2n_k = m - 2,\ m = 2^b - 2 \text{ or } 3 \cdot 2^c - 2 \text{ for some } b \geq 5,\ c \geq 3.$$

But this implies $m \equiv 2 \pmod 4$ contradicting the assumption $m \equiv 0 \pmod 4$.

Combining every case and the verification of small values $m \leq 14$, we obtain the desired result.

Theorem 3.3. *The Diophantine equation*

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} - 1 = F_m \tag{3.5}$$

with $m \geq 0$, $k \geq 1$, and $0 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ has a solution if and only if $0 \leq m \leq 8$, $m = 10$, $m = 2^a - 2$, or $m = 3 \cdot 2^{a-1} - 2$ for some $a \geq 4$.

In this case, the nontrivial solutions to (3.5) are given by

$$L_1 - 1 = F_0, \quad L_0 - 1 = F_1, \quad L_0 - 1 = F_2, \quad L_2 - 1 = F_3,$$
$$L_0^2 - 1 = L_3 - 1 = F_4, \quad L_0 L_2 - 1 = F_5, L_2^2 - 1 = F_6, \quad L_0 L_4 - 1 = F_7,$$
$$L_0 L_5 - 1 = F_8, \;\; L_0^3 L_4 - 1 = L_0 L_3 L_4 - 1 = F_{10}, \;\; L_0 L_2^3 L_4 - 1 = L_2 L_4 L_6 - 1 = F_{14},$$

and for $a \geq 4$

$$L_2 L_4 L_8 \cdots L_{2^{a-3}} L_{2^{a-2}} L_{2^{a-1}-2} - 1 = F_{2^a - 2},$$
$$A L_{12} L_{24} \cdots L_{3 \cdot 2^{a-4}} L_{3 \cdot 2^{a-3}} L_{3 \cdot 2^{a-2}-2} - 1 = F_{3 \cdot 2^{a-1}-2}, \tag{3.6}$$

where $A = L_0^4 L_2^2 = L_0^2 L_2^2 L_3 = L_0^3 L_6 = L_0 L_3 L_6 = L_2^2 L_3^2$. Here nontrivial solution means either that $k = 1$ or $k \geq 2$ and $n_j \neq 1$ for any $j \in \{1, 2, \ldots, k\}$. Also, the product $L_{12} L_{24} \cdots L_{3 \cdot 2^{a-3}}$ appearing in (3.6) is empty when $a = 4$ and (3.6) becomes

$$A L_{10} - 1 = F_{22}.$$

Proof. The argument is similar to that in Theorem 3.1, so we only give a short proof. We first directly check the result for $1 \leq m \leq 14$. Next we assume throughout that $m \geq 15$ and apply Lemma 2.4(ii) to write (3.5) in the form

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} = F_a L_b,$$

where $a, b \in \left\{\frac{m+2}{2}, \frac{m-2}{2}, \frac{m+1}{2}, \frac{m-1}{2}\right\}$. Then we use the identity $F_{2n} = F_n L_n$ and apply Theorem 2.1 to get $n_k = b$ and the above equation is reduced to

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_{k-1}} = F_a. \tag{3.7}$$

Applying Theorem 2.2 completes the process and we obtain the following.

If $m \equiv 1 \pmod 4$, then $2n_k = m - 1$, $m = 2^a - 1, 3 \cdot 2^b - 1$ for some $a \geq 4$, $b \geq 3$, which implies $m \equiv 3 \pmod 4$, a contradiction. So there is no solution in this case. Similarly, there is no solution when $m \equiv 0, 3 \pmod 4$.

For $m \equiv 2 \pmod 4$, we obtain $2n_k = m - 2$, $m = 2^a - 2, 3 \cdot 2^b - 2$ for some $a \geq 5$ and $b \geq 3$, which leads to the desired solution. □

Theorem 3.4. *The Diophantine equation*

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} + 1 = L_m \tag{3.8}$$

with $m \geq 0$, $k \geq 1$, and $0 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ has a solution if and only if $m = 0, 2, 4$ or $m \equiv 3 \pmod 4$. In this case, the nontrivial solutions to (3.8) are given by

$$L_1 + 1 = L_0, \quad L_0 + 1 = L_2, \quad L_2 + 1 = L_3, \quad L_0 L_2 + 1 = L_4,$$
$$L_0^2 L_4 + 1 = L_3 L_4 + 1 = L_7, \quad L_0 L_2^2 L_5 + 1 = L_5 L_6 + 1 = L_{11},$$

and an infinite family of solutions

$$L_{\frac{m-1}{2}} L_{\frac{m+1}{2}} + 1 = L_m$$

for every $m \geq 15$ with $m \equiv 3 \pmod 4$. Here nontrivial solution means either that $k = 1$ or $k \geq 2$ and $n_j \neq 1$ for any $j \in \{1, 2, \ldots, k\}$.

Proof. **Case 1**: $m \equiv 1 \pmod 4$. Then by Lemma 2.5(i), we can write (3.8) as

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} = 5 F_{\frac{m+1}{2}} F_{\frac{m-1}{2}}.$$

Since 5 does not divide any Lucas number, the above equation is not possible.
Case 2: $m \equiv 2 \pmod 4$. Again, we apply Lemma 2.5(i) to write (3.8) as

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} F_{\frac{m}{2}} = F_{\frac{3m}{2}}.$$

Suppose that $m \geq 10$. Similar to the proof of Theorem 3.1, we use the identity $F_{2n} = L_n F_n$ and apply Theorem 2.1 to obtain $\frac{3m}{2} = 2n_k$. This implies that $m \equiv 0 \pmod 4$, which contradicts the assumption that $m \equiv 2 \pmod 4$. So $m < 10$ and we only need to check the result for $m = 2, 6$. We see that $L_0 + 1 = L_2$ but $m = 6$ does not lead to a solution.
Case 3: $m \equiv 3 \pmod 4$. By Lemma 2.5(i), (3.8) can be written as

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} = L_{\frac{m+1}{2}} L_{\frac{m-1}{2}}.$$

We first assume that $m \geq 15$. Then by Theorem 2.1, $L_{\frac{m+1}{2}}$ and $L_{\frac{m-1}{2}}$ have primitive divisors, so we obtain $n_k = \frac{m+1}{2}$, $n_{k-1} = \frac{m-1}{2}$, $k = 2$. In this case, we obtain an infinite number of solutions given by

$$L_{\frac{m-1}{2}} L_{\frac{m+1}{2}} + 1 = L_m. \tag{3.9}$$

By Lemma 2.5(i), (3.9) also holds for $m < 15$. So we only need to check if there are other solutions to (3.8) when $m < 15$ and $m \equiv 3 \pmod 4$. We see that $L_2 + 1 = L_3$, $L_0 L_0 L_4 + 1 = L_3 L_4 + 1 = L_7$, $L_0 L_2 L_2 L_5 + 1 = L_5 L_6 + 1 = L_{11}$.
Case 4: $m \equiv 0 \pmod 4$. Suppose that $m \geq 5$. Similar to the other cases, we apply Lemma 2.5(i) to write (3.8) as

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} L_{\frac{m}{2}} = L_{\frac{3m}{2}}.$$

Applying Theorem 2.1 gives $3m = 2n_k$, and the above equation is reduced to

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_{k-1}} L_{\frac{m}{2}} = 1.$$

But the left-hand side of the above $\geq L_{\frac{m}{2}} \geq L_2 > 3$, a contradiction. So $m < 5$ and we only need to check the result for $m = 0, 4$. We see that $L_1 + 1 = L_0$ and $L_0 L_2 + 1 = L_4$. This completes the proof. □

Theorem 3.5. *The Diophantine equation*

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} - 1 = L_m \tag{3.10}$$

with $m \geq 0$, $k \geq 1$, and $0 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ has a solution if and only if $m = 0, 2, 4, 8$ or $m \equiv 1 \pmod 4$. In this case, the nontrivial solutions to (3.10) are given by

$$L_2 - 1 = L_0, \quad L_0 - 1 = L_1, \quad L_0^2 - 1 = L_3 - 1 = L_2, \quad L_0^3 - 1 = L_0 L_3 - 1 = L_4,$$
$$L_0^2 L_2 - 1 = L_2 L_3 - 1 = L_5, \quad L_0^4 L_2 - 1 = L_0^2 L_2 L_3 - 1 = L_2 L_3^2 - 1 = L_8,$$
$$L_4 L_5 - 1 = L_9, \quad L_0 L_2^2 L_7 - 1 = L_6 L_7 - 1 = L_{13},$$

and an infinite family of solutions

$$L_{\frac{m-1}{2}} L_{\frac{m+1}{2}} - 1 = L_m$$

for every $m \geq 17$ with $m \equiv 1 \pmod 4$. Here nontrivial solution means either that $k = 1$ or $k \geq 2$ and $n_j \neq 1$ for any $j \in \{1, 2, \ldots, k\}$.

Proof. The proof of this theorem is very similar to that of Theorem 3.4. The only main difference is that we apply Lemma 2.5(ii) instead of Lemma 2.5(i).
Case 1: $m \equiv 2 \pmod 4$. We first apply Lemma 2.5(ii). Then the rest of the argument in this case is the same as that in Case 4 of Theorem 3.4 and we only need to check the result when $m = 2$. This leads to

$$L_0 L_0 - 1 = L_2 \text{ and } L_3 - 1 = L_2.$$

Case 2: $m \equiv 3 \pmod 4$. This is the same as Case 1 of Theorem 3.4 where there is no solution.
Case 3: $m \equiv 1 \pmod 4$. The argument from Case 3 of Theorem 3.4 can be used here. This leads to the solutions given by

$$L_{\frac{m-1}{2}} L_{\frac{m+1}{2}} - 1 = L_m \text{ for } m \geq 17,$$

and for $m < 17$, we have

$$L_0 - 1 = L_1, \quad L_0 L_0 L_2 - 1 = L_2 L_3 - 1 = L_5,$$
$$L_4 L_5 - 1 = L_9, \quad L_0 L_2 L_2 L_7 - 1 = L_6 L_7 - 1 = L_{13}.$$

Case 4: $m \equiv 0 \pmod 4$. We first suppose that $m \geq 26$. Similar to the other cases, we apply Lemma 2.5(ii), the identity $F_{2n} = F_n L_n$, and Theorem 2.1 to obtain $\frac{3m}{2} = 2n_k$ and reduce (3.10) to

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_{k-1}} F_{\frac{m}{2}} = F_{n_k}.$$

Again, by the identity $F_{2n} = F_n L_n$, the above equation is

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_{k-2}} F_{2n_{k-1}} F_{\frac{m}{2}} = F_{n_k} F_{n_{k-1}}.$$

Note that $n_k = \frac{3m}{4} > \frac{m}{2} \geq 13$. So by Theorem 2.1, we obtain $n_k = 2n_{k-1}$ and the above equation is reduced to

$$L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_{k-2}} F_{\frac{m}{2}} = F_{n_{k-1}}. \tag{3.11}$$

If $\frac{m}{2} > n_{k-1}$, then the left-hand side of (3.11) is larger than the right-hand side, which is not the case. So $\frac{m}{2} \leq n_{k-1}$. But $n_{k-1} = \frac{n_k}{2} = \frac{3m}{8} < \frac{m}{2}$, a contradiction. So we only need to check the result when $m < 26$. We see that $m = 0, 4, 8$ lead to the desired solution and $m = 12, 16, 20, 24$ do not lead to a solution. This completes the proof. □

Theorem 3.6. *The Diophantine equation*

$$F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_k} + 1 = L_m \tag{3.12}$$

with $m \geq 0$, $k \geq 1$, *and* $0 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ *has a solution if and only if* $0 \leq m \leq 4$ *or* $m \equiv 1 \pmod 4$. *In this case, the nontrivial solutions to* (3.12) *are given by*

$$F_1 + 1 = L_0, \quad F_2 + 1 = L_0, \quad F_0 + 1 = L_1, \quad F_3 + 1 = L_2, \quad F_4 + 1 = L_3,$$
$$F_3 F_4 + 1 = L_4, \quad F_3 F_5 + 1 = L_5, \quad F_4 F_5^2 + 1 = L_9,$$
$$F_5 F_6 F_7 + 1 = F_3^3 F_5 F_7 + 1 = L_{13},$$

and an infinite family of solutions

$$F_5 F_{\frac{m-1}{2}} F_{\frac{m+1}{2}} + 1 = L_m$$

for every $m \geq 17$ *with* $m \equiv 1 \pmod 4$. *Here nontrivial solution means either that* $k = 1$ *or* $k \geq 2$ *and* $n_1 \geq 3$.

Proof. The proof of this result is similar to that of the other theorems. By applying Lemma 2.5(i), the identity $F_{2n} = F_n L_n$, and Theorem 2.1, we can obtain n_k in term of m and reduce (3.12) as follows.

Case 1: $m \equiv 0 \pmod 4$. If $m \geq 5$, then we obtain $3m = n_k$ and (3.12) is reduced to

$$F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_{k-1}} F_{\frac{3m}{2}} F_m = F_{\frac{m}{2}}.$$

Then the left-hand side of the above equation is $\geq F_{\frac{3m}{2}} > F_{\frac{m}{2}}$, a contradiction. So we only need to consider $m = 0, 4$.

Case 2: $m \equiv 1 \pmod 4$. Suppose $m \geq 27$. Then we obtain $n_k = \frac{m+1}{2}$. Repeating the argument, we obtain that $k = 3$, $n_3 = \frac{m+1}{2}$, $n_2 = \frac{m-1}{2}$, $n_1 = 5$. So we only need to check for additional solutions when $m = 1, 5, 9, 13, 17, 21, 25$.

Case 3: $m \equiv 2 \pmod 4$. If $m \geq 9$, then we obtain $\frac{3m}{2} = n_k$ and (3.12) is reduced to

$$F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_{k-1}} F_{\frac{m}{2}} = 1.$$

The left-hand side of the above is $\geq F_{\frac{m}{2}} > 1$, a contradiction. So we only need to consider $m = 2, 6$.

Case 4: $m \equiv 3 \pmod 4$. If $m \geq 14$, then we obtain $m + 1 = n_k$, $m - 1 = n_{k-1}$, and (3.12) is reduced to

$$F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_{k-2}} F_{\frac{m+1}{2}} F_{\frac{m-1}{2}} = 1.$$

But the left-hand side of the above equation $> F_{\frac{m+1}{2}} > 1$, contradiction. So we need to check the result when $m = 3, 7, 11$.

Combining every case, we only need to check the result when $0 \leq m \leq 7$ or $m = 9, 11, 13, 17, 21, 25$, which can be easily done. So the proof is complete. □

Theorem 3.7. *The Diophantine equation*

$$F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_k} - 1 = L_m \tag{3.13}$$

with $m \geq 0$, $k \geq 1$, and $0 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ has a solution if and only if $0 \leq m \leq 5$, $m = 8$, or $m \equiv 3 \pmod 4$. In this case, the nontrivial solutions to (3.13) are given by

$$\begin{aligned}
&F_4 - 1 = L_0, \quad F_3 - 1 = L_1, \quad F_3^2 - 1 = L_2, \quad F_5 - 1 = L_3,\\
&F_6 - 1 = F_3^3 - 1 = L_4, \quad F_3^2 F_4 - 1 = L_5, \quad F_3 F_4 F_5 - 1 = L_7,\\
&F_3 F_4 F_6 - 1 = F_3^4 F_4 - 1 = L_8, \quad F_5^2 F_6 - 1 = F_3^3 F_5^2 - 1 = L_{11},\\
&F_3 F_4^2 F_5 F_6 F_{11} - 1 = F_3^4 F_4^2 F_5 F_{11} - 1 = L_{23},
\end{aligned}$$

and an infinite family of solutions given by

$$F_5 F_{\frac{m-1}{2}} F_{\frac{m+1}{2}} - 1 = L_m$$

for $m \geq 15$ with $m \equiv 3 \pmod 4$. Here nontrivial solution means either that $k = 1$ or $k \geq 2$ and $n_1 \geq 3$.

Proof. The proof of this theorem is the same as that of Theorem 3.6. If m is congruent to $2, 3, 0$ or 1 modulo 4, respectively, then we can follow the argument in Case 1, Case 2, Case 3, or Case 4 of Theorem 3.6. We leave the details to the reader. □

Theorem 3.8. *The Diophantine equation*

$$F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_k} + 1 = F_m \tag{3.14}$$

with $m \geq 0$, $k \geq 1$, and $0 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ has a solution if and only if $m = 1, 2, 3, 4, 5, 7, 8, 10$. In this case, the nontrivial solutions to (3.14) are given by

$$F_0 + 1 = F_1, \quad F_0 + 1 = F_2, \quad F_1 + 1 = F_3, \quad F_2 + 1 = F_3, \quad F_3 + 1 = F_4,$$
$$F_3^2 + 1 = F_5, \quad F_3^2 F_4 + 1 = F_7, \quad F_3^2 F_5 + 1 = F_8, \quad \text{and} \quad F_3 F_4^3 + 1 = F_{10}.$$

Here nontrivial solution means either that $k = 1$ or $k \geq 2$ and $n_1 \geq 3$.

Proof. **Case 1**: $m \equiv 1 \pmod 4$ and $m \geq 12$. By Lemma 2.4(i), (3.14) can be written as

$$F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_k} = F_{\frac{m-1}{2}} L_{\frac{m+1}{2}}.$$

By the well-known identity $F_{2n} = F_n L_n$, the above is

$$F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_k} F_{\frac{m+1}{2}} = F_{\frac{m-1}{2}} F_{m+1}. \tag{3.15}$$

Then from (3.15) and Theorem 2.1, we obtain $n_k = m + 1$ and (3.15) is reduced to

$$F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_{k-1}} F_{\frac{m+1}{2}} = F_{\frac{m-1}{2}}. \tag{3.16}$$

The left-hand side of (3.16) is $\geq F_{\frac{m+1}{2}} > F_{\frac{m-1}{2}}$, so (3.16) is impossible. Thus there is no solution in this case.
Case 2: $m \equiv 2 \pmod 4$ and $m \geq 11$. Similar to Case 1, we apply Lemma 2.4(i), the identity $F_{2n} = F_n L_n$, and Theorem 2.1 to obtain that $n_k = m + 2$ and (3.14) is reduced to

$$F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_{k-1}} F_{\frac{m+2}{2}} = F_{\frac{m-2}{2}}.$$

Again, the left-hand side of the above is $> F_{\frac{m-2}{2}}$, a contradiction.
Case 3: $m \equiv 3 \pmod 4$ and $m \geq 14$. Similar to Case 1 and Case 2, (3.14) can be reduced to

$$F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_{k-1}} F_{\frac{m-1}{2}} = F_{\frac{m+1}{2}}.$$

Since $\left(F_{\frac{m-1}{2}}, F_{\frac{m+1}{2}}\right) = F_{\left(\frac{m-1}{2}, \frac{m+1}{2}\right)} = 1$, there exists a prime p such that $p \mid F_{\frac{m-1}{2}}$ but $p \nmid F_{\frac{m+1}{2}}$, which is a contradiction.
Case 4: $m \equiv 0 \pmod 4$ and $m \geq 15$. Similar to Case 3, there is no solution in this case.

From Case 1 to Case 4, we only need to find the solutions to (3.14) in the case $m \leq 12$, which can be easily done. This completes the proof. □

Theorem 3.9. *The Diophantine equation*

$$F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_k} - 1 = F_m \tag{3.17}$$

with $m \geq 0$, $k \geq 1$, and $0 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ has a solution if and only if $0 \leq m \leq 6$, or $m = 11, 13, 14$. In this case, the nontrivial solutions to (3.17) are given by

$$F_1 - 1 = F_0, \quad F_2 - 1 = F_0, \quad F_3 - 1 = F_1, \quad F_3 - 1 = F_2, \quad F_4 - 1 = F_3,$$
$$F_3^2 - 1 = F_4, \quad F_3 F_4 - 1 = F_5, \quad F_4^2 - 1 = F_6, \quad F_3 F_4^2 F_5 - 1 = F_{11},$$
$$F_3 F_4^2 F_7 - 1 = F_{13}, \quad F_3 F_4^2 F_8 - 1 = F_{14}.$$

Here nontrivial solution means either that $k = 1$ or $k \geq 2$ and $n_1 \geq 3$.

Proof. The proof of this theorem is similar to that of Theorem 3.8. We consider the equation according to the residue classes of m modulo 4. The only difference is that we apply Lemma 2.4(ii) instead of 2.4(i). Then we see that we only need to find a solution in the range $m \leq 14$. This leads to the desired result. □

4. Some Consequences

In this section, we give some results which follow immediately from our main theorems. We will use some of them in our next article.

Corollary 4.1.

(i) *The solutions to the Diophantine equation*

$$F_1F_2F_3\cdots F_n+1=F_m \tag{4.1}$$

with $m\geq 0$ *and* $n\geq 1$ *are given by*

$$F_1+1=F_3,\quad F_1F_2+1=F_3,\quad \textit{and}\quad F_1F_2F_3+1=F_4.$$

(ii) *The solutions to the Diophantine equation*

$$F_1F_2F_3\cdots F_n-1=F_m$$

with $m\geq 0$ *and* $n\geq 1$ *are given by*

$$F_1-1=F_0,\quad F_1F_2-1=F_0,\quad F_1F_2F_3-1=F_1,\quad F_1F_2F_3-1=F_2,\quad \textit{and}$$
$$F_1F_2F_3F_4-1=F_5.$$

Proof. It is easy to check the result when $n\leq 2$. For $n\geq 3$, (i) and (ii) are special cases of Theorem 3.8 and Theorem 3.9, respectively. □

Our results can be interpreted in terms of product sets and sumsets as well. Recall that for nonempty subsets A,B of $\mathbb{R}$ and $\alpha\in\mathbb{R}$, define

$$A+\alpha=\{a+\alpha\mid a\in A\},$$
$$A+B=\{a+b\mid a\in A, b\in B\},\quad \text{and}$$
$$AB=\{ab\mid a\in A, b\in B\}.$$

We also define

$$A^2=AA \text{ and } A^k=A^{k-1}A \text{ for } k\geq 3.$$

Now let

$$F=\{F_n\mid n\geq 0\} \text{ and } L=\{L_n\mid n\geq 0\}$$

be the sets of Fibonacci and Lucas numbers, respectively. Then $\bigcup_{k=1}^{\infty}F^k$ and $\bigcup_{k=1}^{\infty}L^k$ are the sets of all finite products of Fibonacci and Lucas numbers, respectively. Then we have the following result.

Corollary 4.2. *The following statements hold.*

(i) $F\cap\left(\bigcup_{k=1}^{\infty}F^k+1\right)=\{1,2,3,5,13,21,55\}$,

(ii) $F\cap\left(\bigcup_{k=1}^{\infty}F^k-1\right)=\{0,1,2,3,5,8,89,233,377\}$,

(iii) $L\cap\left(\bigcup_{k=1}^{\infty}L^k+1\right)=\{2,3,7\}\cup\{L_m\mid m\equiv 3 \pmod 4\}$,

(iv) $L\cap\left(\bigcup_{k=1}^{\infty}L^k-1\right)=\{2,3,7,47\}\cup\{L_m\mid m\equiv 1 \pmod 4\}$.

Proof. As mentioned earlier, $\bigcup_{k=1}^{\infty} F^k$ is the set of all finite products of Fibonacci numbers. So $\bigcup_{k=1}^{\infty} F^k + 1$ is the set

$$\{F_{n_1}F_{n_2}F_{n_3}\cdots F_{n_k}+1 \mid \ k \geq 1 \text{ and } 0 \leq n_1 \leq n_2 \leq \cdots \leq n_k\}.$$

So we can obtain (i) from Theorem 3.8. Similarly, the statements (ii), (iii), and (iv) follow immediately from Theorem 3.9, Theorem 3.4, and Theorem 3.5, respectively. □

Statements similar to Corollary 4.2 can be given for $F\cap\left(\bigcup_{k=1}^{\infty} L^k \pm 1\right)$ and $L\cap\left(\bigcup_{k=1}^{\infty} F^k \pm 1\right)$ as well. We leave the details to the reader.

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