POLYNOMIAL EXTENSIONS OF THE LUCAS AND GINSBURG IDENTITIES REVISITED

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Abstract. We extend the charming identity [10]

$$g_{n+k}^3 - (-1)^k l_k g_n^3 + (-1)^k g_{n-k}^3 = \begin{cases} f_k f_{2k} f_{3n}, & \text{if } g_r = f_r; \\ (x^2+4) f_k f_{2k} l_{3n}, & \text{if } g_r = l_r; \end{cases}$$

to Jacobsthal, Vieta, and Chebyshev polynomial families. We then deduce the corresponding Jacobsthal and Jacobsthal-Lucas numeric identities.

1. INTRODUCTION

The extended *Gibonacci polynomials* $g_n(x)$ are defined by the recurrence $g_{n+2}(x) = a(x)g_{n+1}(x) + b(x)g_n(x)$, where x is an arbitrary complex variable; $a(x), b(x), g_0(x)$ and $g_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $g_0(x) = 2$ and $g_1(x) = x$, $g_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 8, 9].

In particular, *Pell polynomials* $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [7, 9].

Let $a(x) = 1$ and $b(x) = x$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $g_0(x) = 2$ and $g_1(x) = 1$, $g_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [4, 5, 11]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$.

Suppose $a(x) = x$ and $b(x) = -1$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = V_n(x)$, the nth *Vieta polynomial*; and when $g_0(x) = 2$ and $g_1(x) = x$, $g_n(x) = v_n(x)$, the nth *Vieta-Lucas polynomial* [6, 11, 16].

Let $a(x) = 2x$ and $b(x) = -1$. When $g_0(x) = 1$ and $g_1(x) = x$, $g_n(x) = T_n(x)$, the nth *Chebyshev polynomial of the first kind*; and when $g_0(x) = 1$ and $g_1(x) = 2x$, $g_n(x) = U_n(x)$, the nth *Chebyshev polynomial of the second kind* [6, 9, 11, 15].

2. Links Among the Subfamilies

The Fibonacci, Pell, and Jacobsthal polynomials, and Chebyshev polynomials of the second kind are closely connected; and so are the Lucas, Pell-Lucas, and Jacobsthal-Lucas polynomials, and Chebyshev polynomials of the first kind [6, 11, 16, 17]:

$$\begin{array}{rclcrcl} J_n(x) & = & x^{(n-1)/2} f_n(1/\sqrt{x}) & \qquad & j_n(x) & = & x^{n/2} l_n(1/\sqrt{x}) \\ V_n(x) & = & i^{n-1} f_n(-ix) & & v_n(x) & = & i^n l_n(-ix) \\ V_n(x) & = & U_{n-1}(x/2) & & v_n(x) & = & 2T_n(x/2), \end{array}$$

where $i = \sqrt{-1}$.

In the interest of brevity and convenience, we will omit the argument in the functional notation, when there is *no* ambiguity; so g_n will mean $g_n(x)$.

3. Additional Polynomial Extensions

In [10], the author extended the well-known delights $F_{n+1}^3+F_n^3-F_{n-1}^3 = F_{3n}$ [2, 8, 10, 12, 14], $L_{n+1}^3 + L_n^3 - L_{n-1}^3 = 5L_{3n}$ [10, 12], and $F_{n+2}^3 - 3F_n^3 + F_{n-2}^3 = 3F_{3n}$ [3, 10, 14] to Fibonacci and Lucas polynomials:

$$g_{n+k}^3 - (-1)^k l_k g_n^3 + (-1)^k g_{n-k}^3 = \begin{cases} f_k f_{2k} f_{3n}, & \text{if } g_r = f_r; \\ (x^2+4) f_k f_{2k} l_{3n}, & \text{if } g_r = l_r; \end{cases} \tag{3.1}$$

we then extracted their Pell and Pell-Lucas counterparts, and the corresponding numeric versions.

Using the links among the gibonacci subfamilies, we now extend this charming identity to the Jacobsthal, Vieta, and Chebyshev subfamilies. Again, in the interest of brevity, we will highlight the key steps in the messy algebra involved in the first two cases, and leave the third case for the curious-minded to confirm.

3.1. **Jacobsthal and Jacobsthal-Lucas Extensions.**

We now establish the Jacobsthal and Jacobsthal-Lucas extensions of identity (3.1):

$$g_{n+k}^3 - (-x)^k g_n^3 + (-x)^{3k} g_{n-k}^3 = \begin{cases} g_k g_{2k} g_{3n}, & \text{if } g_r(x) = J_r(x); \\ (4x+1) J_k(x) J_{2k}(x) g_{3n}, & \text{if } g_r(x) = j_r(x). \end{cases} \tag{3.2}$$

To begin with, since $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$, we first replace x with $1/\sqrt{x}$ in the identity

$$f_{n+k}^3 - (-1)^k l_k f_n^3 + (-1)^k f_{n-k}^3 = f_k f_{2k} f_{3n}; \tag{3.3}$$

and then multiply both sides of the resulting equation with $x^{3(n+k-1)/2}$. This yields

$$\begin{aligned} &\left[x^{(n+k-1)/2} f_{n+k}\right]^3 - (-1)^k x^k \left[x^{k/2} l_k\right] \left[x^{(n-1)/2} f_n\right]^3 + (-1)^k x^{3k} \left[x^{(n-k-1)/2} f_{n-k}\right]^3 \\ &= \left[x^{(k-1)/2} f_k\right] \left[x^{(2k-1)/2} f_{2k}\right] \left[x^{(3n-1)/2} f_{3n}\right] J_{n+k}^3(x) - (-x)^k j_k(x) J_n^3(x) + (-x)^{3k} J_{n-k}^3(x) \\ &= J_k(x) J_{2k}(x) J_{3n}(x), \end{aligned} \tag{3.4}$$

where $f_n = f_n(1/\sqrt{x})$ and $l_n = l_n(1/\sqrt{x})$.

Likewise, replace x with $1/\sqrt{x}$ in the identity

$$l_{n+k}^3 - (-1)^k l_k l_n^3 + (-1)^k l_{n-k}^3 = (4x+1) J_k(x) J_{2k}(x) l_{3n}; \tag{3.5}$$

multiply both sides of the resulting equation with $x^{3(n+k)/2}$. Since $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$, we then get

$$\begin{aligned} &\left[x^{(n+k)/2} l_{n+k}\right]^3 - (-1)^k x^k \left[x^{k/2} l_k\right] \left[x^{n/2} l_n\right]^3 + (-1)^k x^{3k} \left[x^{(n-k)/2} l_{n-k}\right]^3 \\ &= (4x+1) \left[x^{(k-1)/2} f_k\right] \left[x^{(2k-1)/2} f_{2k}\right] \left[x^{(3n)/2} l_{3n}\right] j_{n+k}^3(x) - (-x)^k j_k(x) j_n^3(x) + (-x)^{3k} j_{n-k}^3(x) \\ &= (4x+1) J_k(x) J_{2k}(x) j_{3n}(x), \end{aligned} \tag{3.6}$$

where $f_n = f_n(1/\sqrt{x})$ and $l_n = l_n(1/\sqrt{x})$.

Combining (3.4) and (3.6), we get identity (3.2), as desired. □

In particular, we have

$$g_{n+1}^3 + xg_n^3 - x^3 g_{n-1}^3 = \begin{cases} g_{3n}, & \text{if } g_r(x) = J_r(x); \\ (4x+1)g_{3n}, & \text{if } g_r(x) = j_r(x); \end{cases} \tag{3.7}$$

$$g_{n+2}^3 - (2x+1)x^2 g_n^3 + x^6 g_{n-2}^3 = \begin{cases} g_{3n}, & \text{if } g_r(x) = J_r(x); \\ (2x+1)(4x+1)g_{3n}, & \text{if } g_r(x) = j_r(x). \end{cases} \tag{3.8}$$

For example,

$$\begin{aligned} j_5^3 - (2x+1)x^2 j_3^3 + x^6 j_1^3 &= 72x^6 + 294x^5 + 405x^4 + 264x^3 + 89x^2 + 15x + 1 \\ &= (2x+1)(4x+1)j_9(x). \end{aligned}$$

It follows from identities (3.7) and (3.8) that

$$G_{n+1}^3 + 2G_n^3 - 8G_{n-1}^3 = \begin{cases} G_{3n}, & \text{if } G_r = J_r; \\ 9G_{3n}, & \text{if } G_r = j_r; \end{cases}$$

$$G_{n+2}^3 - 20G_n^3 + 64G_{n-2}^3 = \begin{cases} 5G_{3n}, & \text{if } G_r = J_r; \\ 45G_{3n}, & \text{if } G_r = j_r. \end{cases}$$

For example, $J_{11}^3 + 2J_{10}^3 - 8J_9^3 = 683^3 + 2 \cdot 341^3 - 8 \cdot 171^3 = 357,913,941 = J_{30}$, and $j_{10}^3 - 20j_8^3 + 64j_6^3 = 1025^3 - 20 \cdot 257^3 + 64 \cdot 65^3 = 754,974,765 = 45j_{24}$.

Next we investigate the implications of identity (3.1) to the Vieta family.

3.2. **Vieta Extensions.** Replace x with $-ix$ in identity (3.3) and then multiply both sides of the ensuing equation with $i^{3(n+k-1)}$. Since $V_n(x) = i^{n-1} f_n(-ix)$ and $v_n(x) = i^n l_n(-ix)$, this gives

$$\begin{aligned} &\left[i^{n+k-1} f_{n+k}\right]^3 - (-1)^k i^{2k} \left[i^k l_k\right] \left[i^{n-1} f_n\right]^3 + (-1)^k i^{6k} \left[i^{n-k-1} f_{n-k}\right]^3 \\ &= \left[i^{k-1} f_k\right] \left[i^{2k-1} f_{2k}\right] \left[i^{3n-1} f_{3n}\right] V_{n+k}^3 - v_k V_n^3 + V_{n-k}^3 \\ &= V_k V_{2k} V_{3n}, \end{aligned} \tag{3.9}$$

where $f_n = f_n(-ix)$ and $l_n = l_n(-ix)$.

Similarly, identity (3.5) yields

$$v_{n+k}^3 - v_k v_n^3 + v_{n-k}^3 = (x^2 - 4)V_k V_{2k} v_{3n}. \tag{3.10}$$

Identities (3.9) and (3.10) now give the Vieta extensions of identity (3.1):

$$g_{n+k}^3 - v_k g_n^3 + g_{n-k}^3 = \begin{cases} g_k g_{2k} g_{3n}, & \text{if } g_r = V_r; \\ (x^2-4)V_k V_{2k} g_{3n}, & \text{if } g_r = v_r. \end{cases} \tag{3.11}$$

□

Consequently, we have

$$g_{n+1}^3 - xg_n^3 + g_{n-1}^3 = \begin{cases} xg_{3n}, & \text{if } g_r = V_r; \\ x(x^2-4)g_{3n}, & \text{if } g_r = v_r; \end{cases}$$

$$g_{n+2}^3 - (x^2-2)g_n^3 + g_{n-2}^3 = \begin{cases} x(x^3-2x)g_{3n}, & \text{if } g_r = V_r; \\ x(x^2-4)(x^3-2x)g_{3n}, & \text{if } g_r = v_r. \end{cases}$$

For example,

$$\begin{aligned} v_6^3 - (x^2-2)v_4^3 + v_2^3 &= x^{18} - 18x^{16} + 134x^{14} - 532x^{12} + 1209x^{10} \\ &\quad - 1562x^8 + 1058x^6 - 300x^4 + 16x^2 \\ &= x(x^2-4)(x^3-2x)v_{12}. \end{aligned}$$

Finally, we turn to the implications of identity (3.1) to the Chebyshev family.

3.3. **Chebyshev Extensions.** Since $V_n(2x) = U_{n-1}(x)$ and $v_n(2x) = 2T_n(x)$, it follows from identity (3.11) that

$$g_{n+k}^3 - 2T_k g_n^3 + g_{n-k}^3 = \begin{cases} g_{k-1}g_{2k-1}g_{3n+2}, & \text{if } g_r = U_r; \\ (x^2-1)U_{k-1}U_{2k-1}g_{3n}, & \text{if } g_r = T_r. \end{cases} \tag{3.12}$$

This implies

$$g_{n+1}^3 - 2xg_n^3 + g_{n-1}^3 = \begin{cases} 2xg_{3n+2}, & \text{if } g_r = U_r; \\ 2x(x^2-1)g_{3n}, & \text{if } g_r = T_r; \end{cases}$$

$$g_{n+2}^3 - 2(2x^2-1)g_n^3 + g_{n-2}^3 = \begin{cases} 8x^2(2x^2-1)g_{3n+2}, & \text{if } g_r = U_r; \\ 8x^2(x^2-1)(2x^2-1)g_{3n}, & \text{if } g_r = T_r. \end{cases}$$

For example,

$$\begin{aligned} U_5^3 - 2(2x^2-1)U_3^3 + U_1^3 &= 32768x^{15} - 98304x^{13} + 114688x^{11} - 65536x^9 \\ &\quad + 18816x^7 - 2432x^5 + 96x^3 \\ &= 8x^2(2x^2-1)U_{11}. \end{aligned}$$

4. Acknowledgment

The author would like to thank the reviewer for his/her thoughtful comments and suggestions for improving the original version.

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MSC2010: 11B37, 11B39

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