CLOSED FORMS FOR FINITE SUMS OF WEIGHTED PRODUCTS OF THE SINE AND COSINE FUNCTIONS

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Abstract. In this paper, we present closed forms for eight finite sums of weighted products of the sine/cosine functions. In each finite sum that we define, the number of factors in the summand is governed by the size of the integer parameter $j \geq 1$, and can be made as large as we please.

As a consequence of one of our main results, it follows that

$$\sum_{i=1}^{n}(2\cos 1)^{i-1}\sin(i+1) = (2\cos 1)^{n}\sin n.$$

Here the weight term in the summand is $(2\cos 1)^{i-1}$.

1. Introduction

In this paper, we give closed forms for eight finite sums of weighted products of the sine/cosine functions. We believe that each of our results is new. Four of the finite sums that we consider contain only the integer parameter $j \geq 1$. In addition to the parameter j, the remaining four finite sums that we consider contain a second parameter, k. In each of the eight finite sums, the parameter j governs the length of the product in the summand, and can be arbitrarily large.

The idea for the present paper comes from the recent paper [1], where we present closed forms for seven families of finite sums of weighted products of generalized Fibonacci numbers. For instance, in [1] we give the finite sum

$$\sum_{i=1}^{n} 2^{i-1} F_i F_{i+1} F_{i+2}^2 F_{i+3} = \frac{1}{5} \times 2^n F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}, \tag{1.1}$$

which is a special case of one of our main results in [1]. In (1.1), the weight term in the summand is 2^{i-1}. By analogy, as a special case of (3.3) in the present paper, we have

$$\begin{aligned}&\sum_{i=1}^{n}(2\cos 3)^{i-1}\cos i\cos(i+1)\cos(i+5)\\ &= (2\cos 3)^{n}\cos n\cos(n+1)\cos(n+2) - \cos 1\cos 2,\end{aligned}$$

which appears in Section 4. Here, the weight term is $(2\cos 3)^{i-1}$.

In Section 2, we define the eight finite sums of weighted products of the sine/cosine functions that are the topic of this paper. In Section 3, we give the closed forms of these eight sums, and provide a sample proof. In Section 4, we give several special cases of our main results.

2. The Finite Sums

Throughout this paper, the upper limit of summation is an integer $n \geq 1$. For all the sums that we define, the parameter $j \geq 1$ is taken to be an integer. We now define the eight finite sums of weighted products whose closed forms we give in the next section. The four finite sums defined immediately below contain only the parameter j. These finite sums are

$$S_1(n,j) = \sum_{i=1}^{n} (2\cos j)^{i-1} \sin i \cdots \sin(i+j-2)\sin(i+2j-1),$$
$$S_2(n,j) = \sum_{i=1}^{n} (-1)^i (2\cos j)^{i-1} \sin(3i) \cdots \sin(3(i+j-2))\sin(3i+2j-3),$$
$$S_3(n,j) = \sum_{i=1}^{n} (2\cos j)^{i-1} \cos i \cdots \cos(i+j-2)\cos(i+2j-1),$$
$$S_4(n,j) = \sum_{i=1}^{n} (-1)^i (2\cos j)^{i-1} \cos(3i) \cdots \cos(3(i+j-2))\cos(3i+2j-3).$$

The four finite sums that follow contain the additional parameter k. To avoid zero denominators and indeterminate forms, we impose restrictions on k. In S_5 and S_7, $k \neq 0$ is assumed to be a rational number with $jk+1 \neq 0$. In S_6 and S_8, $k \neq 0$ is assumed to be a rational number with $jk-1 \neq 0$.

$$S_5(n,j,k) = \sum_{i=1}^{n} \left(\frac{\sin(jk+1)}{\sin 1}\right)^{i-1} \sin(ki) \cdots \sin(k(i+j-2))\sin(k(i+j-1)+1),$$
$$S_6(n,j,k) = \sum_{i=1}^{n} (-1)^i \left(\frac{\sin(jk-1)}{\sin 1}\right)^{i-1} \sin(ki) \cdots \sin(k(i+j-2))\sin(k(i+j-1)-1),$$
$$S_7(n,j,k) = \sum_{i=1}^{n} \left(\frac{\sin(jk+1)}{\sin 1}\right)^{i-1} \cos(ki) \cdots \cos(k(i+j-2))\cos(k(i+j-1)+1),$$
$$S_8(n,j,k) = \sum_{i=1}^{n} (-1)^i \left(\frac{\sin(jk-1)}{\sin 1}\right)^{i-1} \cos(ki) \cdots \cos(k(i+j-2))\cos(k(i+j-1)-1).$$

Each of the finite sums that we define above has a so-called weight term. For instance, for S_1 the weight term is $(2\cos j)^{i-1}$. Excluding the weight term, the length of the product that defines the summand in each of the sums S_i, $1 \leq i \leq 8$, is j. It is easy to write down each summand when $j \geq 2$. For instance, when $j = 2$ the summand of S_1 is $(2\cos j)^{i-1} \sin i \sin(i+3)$.

When $j = 1$, the summand of each of the S_i is to be interpreted as the product of the weight term and the rightmost factor. For instance, for $j = 1$ the summand of S_2 is to be interpreted as $(-1)^i(2\cos j)^{i-1}\sin(3i-1)$.

3. The Closed Forms and a Sample Proof

In this section, we give the closed forms for each of the finite sums defined in Section 2. We present our main results in two theorems. To present our results, we employ some conventional notation. Specifically, we take the running variable i to be the dummy variable so that, for instance, $[\cos(ki)]_m^n$ is taken to mean $\cos(kn) - \cos(km)$. At the end of this section, we also provide a sample proof. In our first theorem, j is the only parameter.

Theorem 3.1. *Let $j \geq 1$ be an integer. Then*

$$S_1(n,j) = (2\cos j)^n \sin n \cdots \sin(n+j-1), \tag{3.1}$$

$$S_2(n,j) = \frac{(-1)^n (2\cos j)^n \sin(3n) \cdots \sin(3(n+j-1))}{2\cos(2j)+1}, \tag{3.2}$$

$$S_3(n,j) = \left[(2\cos j)^i \cos i \cdots \cos(i+j-1)\right]_0^n, \tag{3.3}$$

$$S_4(n,j) = \frac{\left[(-1)^i (2\cos j)^i \cos(3i) \cdots \cos(3(i+j-1))\right]_0^n}{2\cos(2j)+1}. \tag{3.4}$$

In our second theorem, the parameters are j and k.

Theorem 3.2. *Let $j \geq 1$ be an integer. Then, with the restrictions imposed on k in the definitions of S_5, S_6, S_7, and S_8, we have*

$$S_5(n,j,k) = \frac{\sin^n(jk+1)\sin(kn)\cdots\sin(k(n+j-1))}{\sin^{n-1}1\sin(jk)}, \tag{3.5}$$

$$S_6(n,j,k) = \frac{(-1)^n\sin^n(jk-1)\sin(kn)\cdots\sin(k(n+j-1))}{\sin^{n-1}1\sin(jk)}, \tag{3.6}$$

$$S_7(n,j,k) = \frac{\sin 1}{\sin(jk)}\left[\frac{\sin^i(jk+1)\cos(ki)\cdots\cos(k(i+j-1))}{\sin^i 1}\right]_0^n, \tag{3.7}$$

$$S_8(n,j,k) = \frac{\sin 1}{\sin(jk)}\left[\frac{(-1)^i\sin^i(jk-1)\cos(ki)\cdots\cos(k(i+j-1))}{\sin^i 1}\right]_0^n. \tag{3.8}$$

Each of (3.1)–(3.8) can be proved in the same manner. To illustrate the method, we now give a proof of (3.5). To proceed, we require two identities from elementary trigonometry. These identities are

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}, \tag{3.9}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right). \tag{3.10}$$

A key identity that we require for the proof of (3.5) is

$$\sin(jk+1)\sin(k(n+j)) - \sin 1\sin(kn) = \sin(jk)\sin(k(n+j)+1), \tag{3.11}$$

which is true for all real numbers j, k, and n. To prove (3.11), we apply (3.9) to each product on the left of (3.11), and then apply (3.10) to the result. We leave the details to the interested reader.

Denote the right side of (3.5) by $r(n,j,k)$. Then the difference $r(n+1,j,k) - r(n,j,k)$ is given by

$$\frac{\sin^n(jk+1)\sin(k(n+1))\cdots\sin(k(n+j-1))d(n,j,k)}{\sin(jk)\sin^n 1}, \tag{3.12}$$

where, for convenience, we denote the left side of (3.11) by $d(n,j,k)$. By (3.11), we see that (3.12) reduces to

$$\frac{\sin^n(jk+1)\sin(k(n+1))\cdots\sin(k(n+j-1))\sin(k(n+j)+1)}{\sin^n 1},$$

which is $S_5(n+1,j,k) - S_5(n,j,k)$. Therefore, we have established that

$$r(n+1,j,k) - r(n,j,k) = S_5(n+1,j,k) - S_5(n,j,k). \tag{3.13}$$

By (3.13), to complete the proof of (3.5) it is enough to prove that

$$r(1,j,k) = S_5(1,j,k). \tag{3.14}$$

Recalling our discussion in the last two paragraphs of Section 2, we see that both sides of (3.14) reduce to $\sin(k+1)$ when $j=1$. For $j \geq 2$, both sides of (3.14) reduce to $\sin k \cdots \sin(k(j-1))\sin(jk+1)$. This establishes (3.14), and completes the proof of (3.5).

4. Special Cases of (3.1)–(3.8)

In this section, we consider certain special cases of (3.1)–(3.8). Here, we need to keep in mind the last two paragraphs of Section 2.

Let $j=1$. Then (3.1) and (3.2) become, respectively,

$$\sum_{i=1}^{n}(2\cos 1)^{i-1}\sin(i+1) = (2\cos 1)^n \sin n, \tag{4.1}$$

$$\sum_{i=1}^{n}(-1)^i(2\cos 1)^{i-1}\sin(3i-1) = \frac{(-1)^n(2\cos 1)^n\sin(3n)}{2\cos 2+1}. \tag{4.2}$$

Let $j=1$ and $k=2$. Then (3.5) and (3.7) become, respectively,

$$\sum_{i=1}^{n}\left(\frac{\sin 3}{\sin 1}\right)^{i-1}\sin(2i+1) = \frac{\sin^n 3\sin(2n)}{\sin^{n-1}1\sin 2},$$

$$\sum_{i=1}^{n}\left(\frac{\sin 3}{\sin 1}\right)^{i-1}\cos(2i+1) = \frac{\sin 1}{\sin 2}\left(\frac{\sin^n 3\cos(2n)}{\sin^n 1}-1\right).$$

Let $j=3$. Then (3.3) becomes

$$\sum_{i=1}^{n}(2\cos 3)^{i-1}\cos i\cos(i+1)\cos(i+5)$$
$$= (2\cos 3)^n\cos n\cos(n+1)\cos(n+2) - \cos 1\cos 2.$$

With $j=3$, (3.4) becomes

$$\sum_{i=1}^{n}(-1)^i(2\cos 3)^{i-1}\cos(3i)\cos^2(3i+3)$$
$$= \frac{(-1)^n(2\cos 3)^n\cos(3n)\cos(3n+3)\cos(3n+6)-\cos 3\cos 6}{2\cos 6+1}.$$

Let $j=4$ and $k=1$. Then (3.6) becomes

$$\sum_{i=1}^{n}(-1)^i\left(\frac{\sin 3}{\sin 1}\right)^{i-1}\sin i\sin(i+1)\sin^2(i+2)$$
$$= \frac{(-1)^n\sin^n 3\sin n\sin(n+1)\sin(n+2)\sin(n+3)}{\sin^{n-1}1\sin 4}.$$

Finally for this section, with $j = 4$ and $k = 1$, (3.8) becomes

$$\begin{aligned}
&\sum_{i=1}^{n}(-1)^i\left(\frac{\sin 3}{\sin 1}\right)^{i-1}\cos i\cos(i+1)\cos^2(i+2)\\
&= \frac{\sin 1}{\sin 4}\left(\frac{(-1)^n\sin^n 3\cos n\cos(n+1)\cos(n+2)\cos(n+3)}{\sin^n 1} - \cos 1\cos 2\cos 3\right).
\end{aligned}$$

5. Concluding Comments

As part of our investigation, we searched for results in which the summand contains a run of squared terms. However, such results appear to be scarce. Indeed, we could find only two such results, and in each case the weight term is unity. Once more, keeping in mind the last two paragraphs of Section 2, these two results are

$$\begin{aligned}
\sum_{i=1}^{n}\sin^2 i\cdots\sin^2(i+j-2)\sin(2i+j-2) &= \frac{\sin^2 n\cdots\sin^2(n+j-1)}{\sin j},\\
\sum_{i=1}^{n}\cos^2 i\cdots\cos^2(i+j-2)\sin(2i+j-2) &= -\frac{1}{\sin j}\left[\cos^2 i\cdots\cos^2(i+j-1)\right]_0^n.
\end{aligned}$$

For $j = 1$, $j = 2$, and $j = 3$, the first of the sums in the array immediately above becomes, respectively,

$$\begin{aligned}
\sum_{i=1}^{n}\sin(2i-1) &= \frac{\sin^2 n}{\sin 1},\\
\sum_{i=1}^{n}\sin^2 i\sin(2i) &= \frac{\sin^2 n\sin^2(n+1)}{\sin 2},\\
\sum_{i=1}^{n}\sin^2 i\sin^2(i+1)\sin(2i+1) &= \frac{\sin^2 n\sin^2(n+1)\sin^2(n+2)}{\sin 3}.
\end{aligned}$$

In order to present this paper succinctly, we have chosen to present all our results in abbreviated form. We now indicate how our results can be expressed in their most general form. Let θ be any real number that is not a rational multiple of π. This condition on θ excludes the possibility of vanishing denominators. Then this entire paper can be generalized in the following manner: take *every* occurrence of sin and cos, and multiply the argument by θ. For instance, the generalized forms of the sums (4.1) and (4.2) are, respectively,

$$\begin{aligned}
\sum_{i=1}^{n}(2\cos\theta)^{i-1}\sin((i+1)\theta) &= (2\cos\theta)^n\sin(n\theta),\\
\sum_{i=1}^{n}(-1)^i(2\cos\theta)^{i-1}\sin((3i-1)\theta) &= \frac{(-1)^n(2\cos\theta)^n\sin(3n\theta)}{2\cos(2\theta)+1}.
\end{aligned}$$

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References

[1] R. S. Melham, *Closed forms for finite sums of weighted products of generalized Fibonacci numbers*, The Fibonacci Quarterly, **55.2** (2017), 99–104.

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