CLOSED FORMS FOR FINITE SUMS OF WEIGHTED PRODUCTS OF GENERALIZED FIBONACCI NUMBERS

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Abstract. In this paper, we present closed forms for certain finite sums of weighted products of generalized Fibonacci numbers. Indeed, we present seven multi-parameter families of such finite sums, all of which we believe to be new. In each of these families, the number of factors in the summand is governed by the size of the integer parameter $j \geq 1$, and can be made as large as we please.

We present our main results in terms of sequences that generalize the Fibonacci/Lucas numbers. Consequently, each of our main results can be specialized to involve the Fibonacci/Lucas numbers. For instance, as a consequence of one of our main results, it follows that

$$\sum_{i=1}^{n} 2^{i-1} F_i F_{i+2} = 2^n F_n F_{n+1}.$$

Here the weight term in the summand is 2^{i-1}.

1. Introduction

We begin by establishing the notation for the sequences that feature in this paper. Let $a \geq 0$ and $b \geq 0$ be integers, with $(a,b) \neq (0,0)$. For p a positive integer, we define, for all integers n, the sequences $\{W_n\}$ and $\{\overline{W}_n\}$ by

$$W_n = pW_{n-1} + W_{n-2},\ W_0 = a,\ W_1 = b, \tag{1.1}$$

and

$$\overline{W}_n = W_{n-1} + W_{n+1}.$$

We leave to the reader the simple task of showing that

$$\overline{\overline{W}}_n = \left(p^2+4\right) W_n. \tag{1.2}$$

For $(a,b,p) = (0,1,1)$, we have $\{W_n\}=\{F_n\}$, and $\{\overline{W}_n\}=\{L_n\}$, which are the Fibonacci and Lucas sequences, respectively. Allowing p to be an arbitrary positive integer, and taking $(a,b) = (0,1)$, we write $\{W_n\}=\{U_n\}$, and $\{\overline{W}_n\}=\{V_n\}$, which are integer sequences that generalize the Fibonacci and Lucas numbers, respectively.

When $p = 1$, and a and b are arbitrary, we write $\{W_n\}=\{H_n\}$. Thus, $\{H_n\}$ and $\{\overline{H}_n\}$ satisfy the same recurrence as $\{F_n\}$, and are generalizations of $\{F_n\}$ and $\{L_n\}$, respectively.

Let α and β denote the two distinct real roots of $x^2 - px - 1 = 0$. Set $A = b - a\beta$ and $B = b - a\alpha$. Then the Binet forms for $\{W_n\}$ and $\{\overline{W}_n\}$ are, respectively,

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.3}$$

and

$$\overline{W}_n = A\alpha^n + B\beta^n. \tag{1.4}$$

Note that the closed forms for all the sequences that we consider in this paper can be obtained from (1.3) and (1.4).

The finite sum

$$\sum_{i=0}^{n} 2^i L_i = 2^{n+1} F_{n+1} \tag{1.5}$$

is identity 236 in Benjamin and Quinn [1]. In a recent paper, Sury [7] attracts much attention by proving (1.5) with the use of a simple polynomial identity. Kwong [4] then proves (1.5) quickly and elegantly with the use of generating functions. In a recent paper, Bhatnagar [2] lists several finite sums, put forward by other authors, as analogues of (1.5). One such formula that replaces 2^i by 3^i is

$$\sum_{i=0}^{n} 3^i \left(L_i + F_{i+1}\right) = 3^{n+1} F_{n+1}, \tag{1.6}$$

given by Marques [5] in a sightly different form.

Notice that in (1.6) one of the factors in the summand is not a product. The same can be said of similar formulas that Bhatnagar [2] lists in Section 1 of his paper. In this paper, we give seven multi-parameter analogues of (1.5), and in all cases each factor in the summand is a product of terms from generalized Fibonacci sequences. Indeed, the length of each product in question (see Section 2) is specified by the parameter j, which can be arbitrarily large.

In all the sums that we present in this paper, we take the lower limit of summation to be 1 instead of 0. This serves to make our formulas much more succinct. In each of our main results, the factors in the summand are drawn from the sequences $\{W_n\}$, $\left\{\overline{W}_n\right\}$, $\{H_n\}$, or $\left\{\overline{H}_n\right\}$. Accordingly, each of our main results can be specialized to the Fibonacci/Lucas numbers. Where possible, we express our results in terms of the sequences $\{W_n\}$ and $\left\{\overline{W}_n\right\}$, thus making these results as general as possible. However, some of our results apply only for the case $p = 1$, so we express these results in terms of $\{H_n\}$ and $\left\{\overline{H}_n\right\}$.

In Section 2, we define the seven finite sums of weighted products of generalized Fibonacci numbers that are the topic of this paper. In Section 3, we give the closed forms of these seven sums, and provide a sample proof. In Section 4, we give several special cases of our main results for the Fibonacci/Lucas numbers.

2. The Finite Sums

As stated in the introduction, for the remainder of this paper we take the lower limit of summation to be $i = 1$, and the upper limit of summation to be $n \geq 1$. Furthermore, $j \geq 1$ and $k \geq 1$ are assumed to be integers. We now define seven finite sums of weighted products whose closed forms we give in the next section. The first three finite sums involve sequences generated by the recurrence given in (1.1), in which $p \geq 1$ is an arbitrary integer. These finite sums are

$$S_1(n,j,k) = \sum_{i=1}^{n} U_{jk-1}^{i-1} W_{ki} \cdots W_{k(i+j-2)} W_{k(i+j-1)-1},$$

$$S_2(n,j,k) = \sum_{i=1}^{n} U_{jk+1}^{i-1} W_{ki} \cdots W_{k(i+j-2)} W_{k(i+j-1)+1},$$

$$S_3(n,j,k) = \sum_{i=1}^{n} V_{jk}^{i-1} W_{ki} \cdots W_{k(i+j-2)} W_{k(i+2j-1)}.$$

Next, we define four finite sums that involve sequences generated by the recurrence given in (1.1), in which $p = 1$. These finite sums are

$$S_4(n,j,k) = \sum_{i=1}^{n} F_{jk+2}^{i-1} H_{ki} \cdots H_{k(i+j-2)} H_{k(i+j-1)+2},$$

$$S_5(n,j,k) = \sum_{i=1}^{n} L_{jk+1}^{i-1} H_{ki} \cdots H_{k(i+j-2)} \overline{H}_{k(i+j-1)+1},$$

$$S_6(n,j,k) = \sum_{i=1}^{n} F_{jk-2}^{i-1} H_{ki} \cdots H_{k(i+j-2)} H_{k(i+j-1)-2},$$

$$S_7(n,j,k) = \sum_{i=1}^{n} L_{jk-1}^{i-1} H_{ki} \cdots H_{k(i+j-2)} \overline{H}_{k(i+j-1)-1}.$$

Each of the finite sums that we define above has a so-called weight term. For instance, for $S_1(n,j,k)$ the weight term is U_{jk-1}^{i-1}. Now, excluding the weight term, the length of the product in each of the finite sums $S_i(n,j,k)$, $1 \le i \le 7$, is j. When $j \ge 2$, it is easy to write down each summand. For example, when $j = 2$ the summand of $S_1(n,j,k)$ is $U_{2k-1}^{i-1} W_{ki} W_{k(i+1)-1}$.

When $j = 1$, the summand of each of the $S_i(n,j,k)$ is to be interpreted as the product of the weight term, and the *last* term in the product that defines the summand. For instance, for $j = 1$ the summand in $S_2(n,j,k)$ is to be interpreted as $U_{k+1}^{i-1} W_{ki+1}$.

3. The Closed Forms and a Sample Proof

In this section, we give the closed forms for each of the finite sums defined in Section 2. We present our main results in three theorems, where the parities of j and k need to be considered. To present our results, we employ some customary notation. Specifically, in all that follows, we take i to be the dummy variable. For instance, $[W_{ki}]_m^n$ means $W_{kn} - W_{km}$. At the end of this section we also provide a sample proof. Our first theorem applies to the sequence defined in (1.1).

Theorem 3.1. *Suppose $j \ge 1$ and $k \ge 1$ are such that the product jk is even. Then*

$$S_1(n,j,k) = \frac{1}{U_{jk}} \left[U_{jk-1}^{i} W_{ki} \cdots W_{k(i+j-1)} \right]_0^n, \tag{3.1}$$

$$S_2(n,j,k) = \frac{1}{U_{jk}} \left[U_{jk+1}^{i} W_{ki} \cdots W_{k(i+j-1)} \right]_0^n, \tag{3.2}$$

$$S_3(n,j,k) = \left[V_{jk}^{i} W_{ki} \cdots W_{k(i+j-1)} \right]_0^n. \tag{3.3}$$

Our next two theorems apply to the sequence defined in (1.1) only when $p = 1$.

Theorem 3.2. *Suppose $j \ge 1$ and $k \ge 1$ are such that the product jk is even. Then*

$$S_4(n,j,k) = \frac{1}{F_{jk}} \left[F_{jk+2}^{i} H_{ki} \cdots H_{k(i+j-1)} \right]_0^n, \tag{3.4}$$

$$S_5(n,j,k) = \frac{1}{F_{jk}} \left[L_{jk+1}^{i} H_{ki} \cdots H_{k(i+j-1)} \right]_0^n. \tag{3.5}$$

In our next theorem, j must be odd. Accordingly, excluding the weight term, the number of generalized Fibonacci factors in the summand is odd.

Theorem 3.3. *Suppose $j \geq 1$ and $k \geq 1$ are such that the product jk is odd. Then*

$$S_6(n,j,k) = \frac{1}{F_{jk}} \left[F_{jk-2}^i H_{ki} \cdots H_{k(i+j-1)}\right]_0^n, \tag{3.6}$$

$$S_7(n,j,k) = \frac{1}{F_{jk}} \left[L_{jk-1}^i H_{ki} \cdots H_{k(i+j-1)}\right]_0^n. \tag{3.7}$$

We conclude this section with a proof of (3.1). The method of proof that we employ can also be used to prove each of (3.2)–(3.7).

A key identity that we require for the proof of (3.1) is

$$U_{jk-1}W_{k(n+j)} - U_{jk}W_{k(n+j)-1} = (-1)^{jk}W_{kn}, \tag{3.8}$$

which is true for all integers j, k, and n. Upon substituting the Binet forms for the four terms on the left side of (3.8), then expanding and factoring, we obtain

$$\frac{-(\alpha\beta)^{jk-1}\left(A\alpha^{kn} - B\beta^{kn}\right)}{\alpha - \beta}, \tag{3.9}$$

where A and B are given in the introduction. Since $\alpha\beta = -1$, (3.8) follows from (3.9).

Denote the right side of (3.1) by $r(n,j,k)$. Then the difference $r(n+1,j,k) - r(n,j,k)$ is given by

$$\frac{U_{jk-1}^n W_{k(n+1)} \cdots W_{k(n+j-1)}\left(U_{jk-1}W_{k(n+j)} - W_{kn}\right)}{U_{jk}}.$$

Since jk is assumed to be even, by (3.8) we then have

$$\begin{aligned} r(n+1,j,k) - r(n,j,k) &= U_{jk-1}^n W_{k(n+1)} \cdots W_{k(n+j-1)} W_{k(n+j)-1} \\ &= S_1(n+1,j,k) - S_1(n,j,k). \end{aligned} \tag{3.10}$$

In light of (3.10), to complete the proof of (3.1) it is enough to prove that

$$r(1,j,k) = S_1(1,j,k). \tag{3.11}$$

We consider the cases $j = 1$ and $j \geq 2$ separately. For $j = 1$, to prove (3.11) we are required to prove that

$$U_{k-1}W_k - U_kW_{k-1} = W_0, \quad \text{for } k \text{ even}. \tag{3.12}$$

Next, we consider the case $j \geq 2$. Writing down both sides of (3.11), canceling terms that are common to both sides, and rearranging, we see that we are required to prove

$$U_{jk-1}W_{jk} - U_{jk}W_{jk-1} = W_0, \quad \text{for } jk \text{ even}. \tag{3.13}$$

Now both (3.12) and (3.13) follow from (3.8). This completes the proof of (3.1).

For the proof of each of (3.2)–(3.7), we require an identity that is analogous to (3.8). To assist the interested reader, we record these identities below. They are, respectively,

$$\begin{aligned} U_{jk+1}W_{k(n+j)} - U_{jk}W_{k(n+j)+1} &= (-1)^{jk}W_{kn}, \\ V_{jk}W_{k(n+j)} - W_{k(n+2j)} &= (-1)^{jk}W_{kn}, \\ F_{jk+2}H_{k(n+j)} - F_{jk}H_{k(n+j)+2} &= (-1)^{jk}H_{kn}, \\ L_{jk+1}H_{k(n+j)} - F_{jk}\overline{H}_{k(n+j)+1} &= (-1)^{jk}H_{kn}, \\ F_{jk-2}H_{k(n+j)} - F_{jk}H_{k(n+j)-2} &= (-1)^{jk+1}H_{kn}, \\ L_{jk-1}H_{k(n+j)} - F_{jk}\overline{H}_{k(n+j)-1} &= (-1)^{jk+1}H_{kn}. \end{aligned}$$

4. Special Cases of (3.1)–(3.7)

It is instructive to consider certain special cases of (3.1)–(3.7). Here, we need to keep in mind the last two paragraphs of Section 2.

In (3.2) and (3.3) take $\{W_n\}$ to be $\{F_n\}$, and in (3.5) take $\{H_n\}$ to be $\{F_n\}$. In each of these three cases, let $(j,k)=(2,1)$. Then (3.2), (3.3), and (3.5) become, respectively,

$$\sum_{i=1}^{n} 2^{i-1} F_i F_{i+2} = 2^n F_n F_{n+1},$$

$$\sum_{i=1}^{n} 3^{i-1} F_i F_{i+3} = 3^n F_n F_{n+1},$$

$$\sum_{i=1}^{n} 4^{i-1} F_i L_{i+2} = 4^n F_n F_{n+1}.$$

In (3.6), let $j \geq 3$ be odd, and take $k=1$. Then we obtain

$$\sum_{i=1}^{n} F_{j-2}^{i-1} F_i F_{i+1} \cdots F_{i+j-2} F_{i+j-3} = \frac{1}{F_j} \times F_{j-2}^{n} F_n \cdots F_{n+j-1}. \tag{4.1}$$

For $j=3$, (4.1) becomes

$$\sum_{i=1}^{n} F_i^2 F_{i+1} = \frac{1}{2} \times F_n F_{n+1} F_{n+2},$$

a result first found by Block [3].

We now consider two special cases of (3.7). When $(j,k)=(1,1)$ and $\{H_n\}$ is $\{F_n\}$, (3.7) becomes

$$\sum_{i=1}^{n} 2^{i-1} L_{i-1} = 2^n F_n,$$

which is equivalent to (1.5). Next let $(j,k)=(1,1)$, and take $H_n = \overline{F}_n = L_n$. Then by (1.2) we see that $\overline{H}_n = \overline{\overline{F}}_n = 5F_n$, and (3.7) becomes

$$\sum_{i=1}^{n} 2^{i-1} F_{i-1} = \frac{1}{5}\left[2^i L_i\right]_0^n = \frac{1}{5}\left(2^n L_n - 2\right). \tag{4.2}$$

We have not seen (4.2) in the literature, and so we offer (4.2) as a counterpart to (1.5).

Staying with (3.7), taking $(j,k)=(1,3)$, and specializing $\{H_n\}$ to $\{F_n\}$, we obtain

$$\sum_{i=1}^{n} 3^{i-1} L_{3i-1} = \frac{1}{2} \times 3^n F_{3n}.$$

Finally for (3.7), recall that $F_n L_n = F_{2n}$, and take $(j,k)=(3,1)$. Then specializing $\{H_n\}$ to $\{F_n\}$, we have

$$\sum_{i=1}^{n} 3^{i-1} F_i F_{2i+2} = \frac{1}{2} \times 3^n F_n F_{n+1} F_{n+2}.$$

We leave the task of writing down further special cases of (3.1)–(3.7) to the interested reader.

5. Concluding Comments

Results analogous to those presented here, where the right side contains squared terms, can be found in [6]. In [6], all the sums have weight terms of 1 or $(-1)^i$. Similar results with weight terms other than 1 or $(-1)^i$ seem to be rare. Indeed, we have discovered only two such results. These results are

$$\sum_{i=1}^{n} 2^{2i-2} H_{i+2}\overline{H}_{i-1} = \left[2^{2i} H_i^2\right]_0^n,$$

$$\sum_{i=1}^{n} 3^{2i-2} H_i^2 H_{i+1}^2 H_{i+3}\overline{H}_{i+1} = \frac{1}{4}\left[3^{2i} H_i^2 H_{i+1}^2 H_{i+2}^2\right]_0^n.$$

When we replace $\{H_n\}$ by $\{F_n\}$, the two sums above become, respectively,

$$\sum_{i=1}^{n} 2^{2i-2} F_{i+2} L_{i-1} = 2^{2n} F_n^2, \tag{5.1}$$

$$\sum_{i=1}^{n} 3^{2i-2} F_i^2 F_{i+1}^2 F_{i+3} L_{i+1} = \frac{1}{4} \times 3^{2n} F_n^2 F_{n+1}^2 F_{n+2}^2. \tag{5.2}$$

The sum (5.1) occurs in the recent paper of Treeby [8]. We have not managed to find results similar to (5.1) and (5.2) where the right side contains cubes or higher powers.

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