LINEAR RECURRENCES ORIGINATING FROM POLYNOMIAL TREES

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ABSTRACT. Let T^* be the set of polynomials in x generated by these rules: $0 \in T^*$, and if $p \in T^*$, then $p+1 \in T^*$ and $xp \in T^*$. Let $g(0) = \{0\}$, $g(1) = \{1\}$, $g(2) = \{2, x\}$, and so on, so that the cardinality of $g(n)$ is given by $G_n = 2^{n-1}$ for $n \geq 1$, and T^* can be regarded as a tree whose nth generation consists of nodes labeled by the polynomials in $g(n)$. Let $T(r)$ be the subtree of T^* obtained by substituting r for x and deleting duplicates. For various choices of r, the cardinality sequence G_n satisfies a linear recurrence relation.

1. INTRODUCTION

Let T^* be the set of polynomials in x generated by these rules: $0 \in T^*$, and if $p \in T^*$, then $p+1 \in T^*$ and $xp \in T^*$. We regard T^* as a tree that grows in successive generations: $g(0) = \{0\}$, $g(1) = \{1\}$, $g(2) = \{2, x\}$,

$$\begin{aligned} g(3) &= \{3, 2x, x+1, x^2\} \\ g(4) &= \{4, 3x, 2x+1, 2x^2, x+2, x^2+x, x^2+1, x^3\}, \end{aligned}$$

and so on, as in Figure 1.

The purpose of this article is to describe T^* and some of its subtrees. To see that the polynomials in T^* accrue without duplication, suppose instead that there is a duplicate, q, and assume that it is the first duplicate to occur. Write $q = p$, where p occurs before q. If $p = xp_1$ and $q = xq_1$ then $p_1 = q_1$, contrary to the firstness of q, and likewise if $p = p_1 + 1$ and $q = q_1 + 1$. The remaining case is that $p(x) = xp_1(x)$ and $q(x) = q_1(x) + 1$ (or vice versa), but then $p(0) = 0$ and $q(0) > 0$; this contradiction implies that that there are no duplicates, so that $|g(n)| = 2^{n-1}$ for $n \geq 1$.

Next, we shall show for $n > 0$ that the number of polynomials of degree k in $g(n)$ is the binomial coefficient $C(n-1, k)$ for $k \in [0, n-1]$. As a first inductive step, $C(1-1, 0)$ counts the polynomials of degree 0 in $g(1) = \{1\}$. Assume for arbitrary $m \geq 1$ that the number of polynomials of degree k in $g(m)$ is $C(m-1, k)$ for $k \in [0, m-1]$, and suppose that $h \in [0, m]$. Every $p(x)$ in $g(m+1)$ that has degree h is of one of two kinds: $p(0) > 0$ or $p(0) = 0$. There are $C(m-1, h)$ polynomials $p(x) - 1$ in $g(m)$ for which $p(0) > 0$, together with $C(m-1, h-1)$ polynomials $p(x)/x$ in $g(m)$ for which $p(0) = 0$. Therefore, the number of polynomials in $g(m+1)$ of degree h is

$$C(m-1, h) + C(m-1, h-1) = C(m, h),$$

which finishes an inductive proof.

A third easily proved property of T^* is that it consists precisely of the polynomials all of whose coefficients are in the set $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$. This follows from the obvious fact that if $p(x) \in T^*$ and $c \in \mathbb{Z}_{\geq 0}$, then $xp(x) + c \in T^*$.

For arbitrary $p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$ in T^*, it is easy to determine the generation $g(n)$ that contains $p(x)$: count a_0 steps from $p(x)$ back to $p(x) - a_0$; then 1 step back to $(p(x) - a_0)/x = a_m x^{m-1} + a_{m-1} x^{m-2} + \cdots + a_1$; then a_1 steps back to $a_m x^{m-2} + a_{m-1} x^{m-3} + \cdots + a_2$; and so on, until reaching 0, for a total of $a_0 + 1 + a_1 + 1 + \cdots + a_{m-1} + 1 + a_m$ steps. That is,

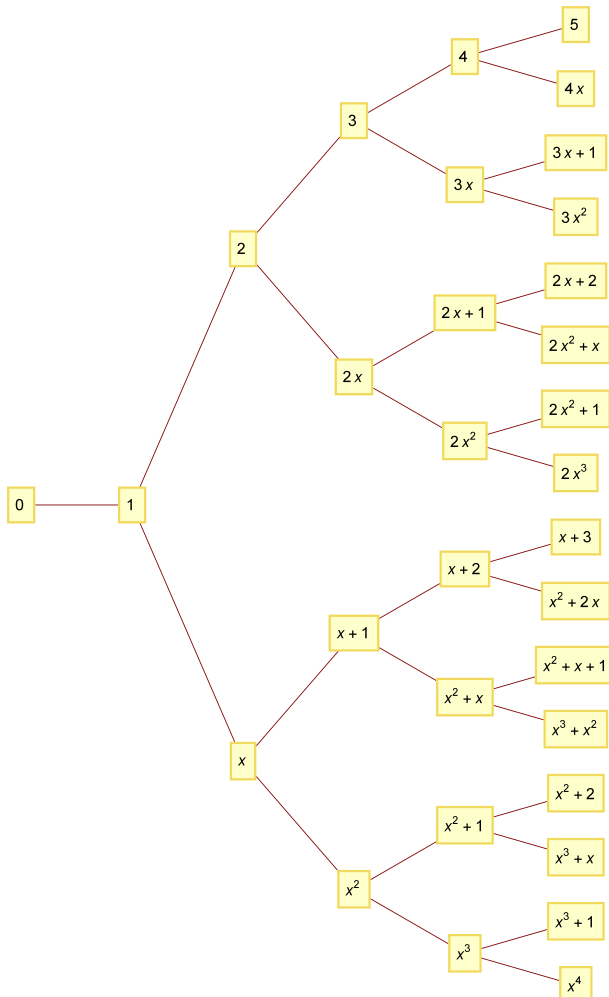

FIGURE 1. The tree T^*, generations $g(0)$ to $g(5)$

$p(x) \in g(p(1)+m)$. For example, the number of steps from 0 to the mth Fibonacci polynomial, $F_m(x)$, is $m + F_m$, since $F_m(1) = F_m$. (Here, $F_m(x)$ is defined by the recurrence $F_m(x) = xF_{m-1}(x) + F_{m-2}(x)$ with $F_0(x) = 0$, $F_1(x) = 1$, and F_m is defined by $F_m = F_{m-1} + F_{m-2}$ with $F_2 = F_1 = 1$.)

Another easily proved property of T^* that involves Fibonacci numbers arises when we ask how many even polynomials are in $g(n)$. Recall that $p(x)$ is *even* if $p(-x) = p(x)$ and *odd* if $p(-x) = -p(x)$. Let u_n be the number of odd polynomials in $g(n)$, and v_n the number of even. Starting with $u_1 = 0$ and $v_1 = 1$, we have $v_n = v_{n-1} + u_{n-1}$ and $u_n = v_{n-1}$. Consequently, $u_n = F_{n-1}$ and $v_n = F_n$. As a corollary, suppose that $\sqrt{2}$ is substituted for x in T^*; then the number of rationals (integers, actually) in $g(n)$ is F_n.

The rest of this article concerns subtrees of T^*, which appear in various guises, which we shall call (1) polynomial form, (2) numeric form; and (3) tuple form. For the first of these, suppose that

$$q(x) = x^m - q_{m-1}x^{m-1} - \cdots - q_1 x - q_0$$

is a fixed polynomial of degree $m \geq 1$. Then the substitution of

$$q_{m-1}x^{m-1} + \cdots + q_1 x + q_0$$

for x^m throughout T^* produces a labeling of T^* that includes duplicates. We consider two methods for their removal. In the first, the order in which the substitutions in each generation $g(n)$ are made is "top-to-bottom"; i.e., with reference to Figure 1, starting at n and proceeding down to x^{n-1}. The other order is "bottom-to-top". The two trees that result need not be isomorphic, as we shall see in Section 2. For the top-to-bottom order, we denote the resulting tree by $T(q(x))$. The bottom-to-top tree we denote by $\widehat{T}(q(x))$. Note that $T(q(x))$ can be regarded as T^* mod $q(x)$.

Next, suppose that r is a nonzero complex number. Substituting r for x in T^* gives a tree whose nodes are numbers, and, after top-to-bottom deletion of all duplicates, we are left with a tree which we denote by $T(r)$. Likewise, bottom-to-top deletion of duplicates yields a tree $\widehat{T}(r)$. If r is a zero of an irreducible polynomial $q(x)$, then $T(r)$ is isomorphic to $T(q(x))$, and $\widehat{T}(r)$ is isomorphic to $\widehat{T}(q(x))$. If r is a zero of a reducible polynomial, then $T(r)$ is isomorphic to a subtree of $T(q(x))$, and $\widehat{T}(r)$ is isomorphic to a subtree of $\widehat{T}(q(x))$. We call $T(r)$ and $\widehat{T}(r)$ *numeric forms* of a subtree of T^*.

In Sections 2 and 3, we discuss two particular examples of trees and see, for instance, that although $T(r)$ and $\widehat{T}(r)$ have much in common, they need not be isomorphic. Much of the paper henceforth studies the sequence $(G_n)_{n\geq 0}$, where $G_n = |g(n)|$ and $g(n)$ is the nth generation of the tree $T(r)$, for various r. We either prove or conjecture that the sequences (G_n) are linear recurrences of special types that depend on r. In Section 4, we prove that (G_n) is (eventually) a linear recurrence for all $r = \sqrt{d}$, $d \geq 2$ an integer. Section 5 contains numeric tables for G_n corresponding to various specific trees, and a list of recurrence conjectures for (G_n) in six cases of families of trees. Section 6 deals with the trees $T(1/d)$ and $T(-d)$ for which we also prove or state theorems. Mathematica programs are given in a seventh section.

Before continuing, we summarize the main results of this introduction as a theorem.

Theorem 1.1. *Let $g(n)$ be the nth generation of the tree T^*, and G_n the cardinality of $g(n)$. Then*

(1) $G_n = 2^{n-1}$ *for* $n \geq 1$.

(2) *The number of polynomials of degree k in $g(n)$ is $C(n-1,k)$.*

(3) *T^* consists of the polynomials that have nonnegative integer coefficients.*

(4) *If $p(x)$ has degree m, then $p(x) \in g\big(p(1)+m\big)$.*

(5) *The number of odd polynomials in $g(n)$ is F_{n-1}, and of even, F_n.*

2. Examples: T(golden ratio)

Example 2.1. Let $q(x) = x^2 - x - 1$ and $r = (1+\sqrt{5})/2$. The tree $T(r)$ grows as in Figure 2.

Example 2.2. Let $q(x) = x^2 - x - 1$ and $r = (1+\sqrt{5})/2$. The tree $\widehat{T}(r)$ grows as in Figure 3.

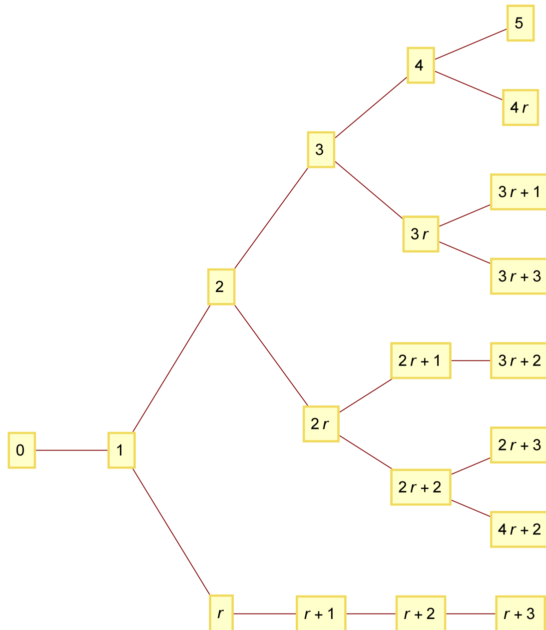

FIGURE 2. The tree $T((1+\sqrt{5})/2)$, generations $g(0)$ to $g(5)$

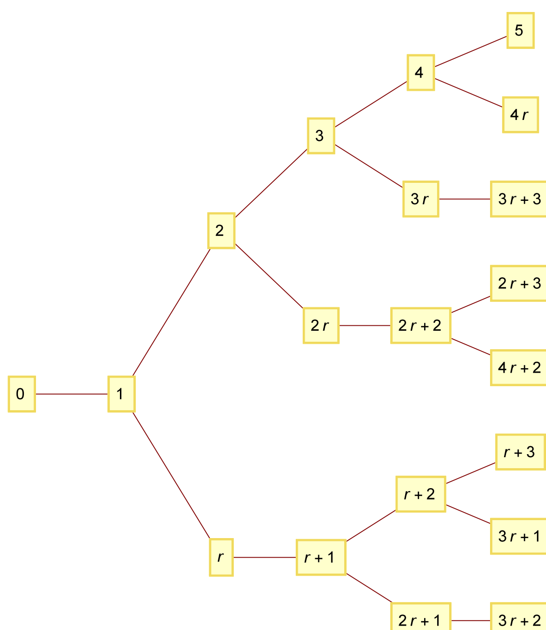

FIGURE 3. The tree $\widehat{T}((1+\sqrt{5})/2)$, generations $g(0)$ to $g(5)$

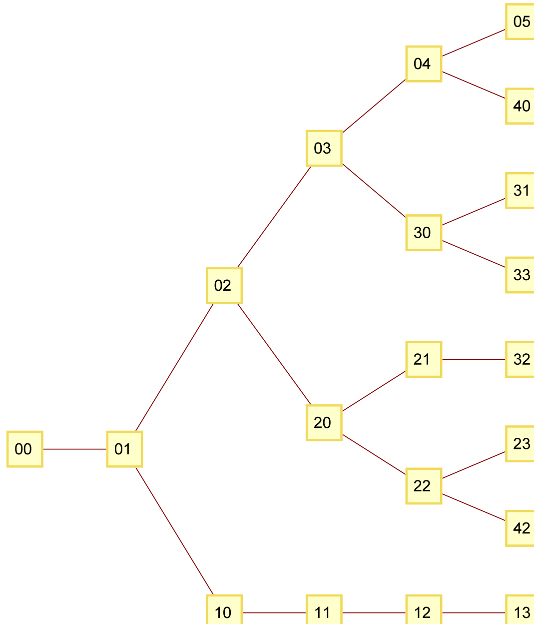

FIGURE 4. The tree $T(1,1)$, generations $g(0)$ to $g(5)$

Continuing with $r = (1+\sqrt{5})/2$, let $g(n)$ be the nth generation of $T(r)$ and let $\widehat{g}(n)$ be the nth generation of $\widehat{T}(r)$. Figures 2 and 3 show that $g(n) = \widehat{g}(n)$ for $n = 0, 1, \ldots, 5$, and it is clear by induction that $g(n) = \widehat{g}(n)$ for all $n \geq 0$. Consider next the sizes of these generations: for those shown in the two figures, we have $G_n = 1, 1, 2, 3, 5, 8, \cdots$. In a bad moment, one might expect the sequence to continue with $13, 21, 34$, but actually the next three terms are $12, 18, 25$. The sequence is A252864 in the Online Encyclopedia of Integer Sequences [5]. Stoll [4] conjectured that

$$G_n = G_{n-1} + G_{n-3} \text{ for } n \geq 12.$$

Although $T(r) = \widehat{T}(r)$ as sets, it is easy to see that as trees, $T(r)$ and $\widehat{T}(r)$ are not isomorphic. For example, $T(r)$ has the path from 3 to $r+3$ in which three consecutive nodes each have outdegree 1, but $\widehat{T}(r)$ has no such path.

We turn now to a third way to represent a subtree of T^*, mentioned in Section 1 as tuple form. Write

$$q(x) = x^2 - q_1 x - q_0.$$

The tree $T = T(q_1, q_0)$ is defined as follows: $(0,0) \in T$, and if $(j,k) \in T$, then $(j, k+1) \in T$ and $(jq_1 + k, jq_0) \in T$, with duplicates removed as they occur (here, using the top-to-bottom method). As an example, take $q_1 = q_0 = 1$. It is easy to see that every ordered pair of nonnegative integers occurs exactly once in $T(1,1)$. We write (j,k) as jk and note that $T(1,1)$ grows as in Figure 4. Of course, $T(1,1)$ is isomorphic to the tree in Figure 2. Clearly, $T(1,1)$ results from $T(r)$, for $r = (1+\sqrt{5})/2$, by replacing each $jr + k$ by (j,k).

3. Example: $T(i)$

Here, we take $(q_1, q_0) = (0, -1)$, corresponding to $q(x) = x^2 + 1$ and $r = i$. Every Gaussian integer occurs exactly once in this tree, $T(i)$. We wish to determine which numbers are in each generation $g(n)$. It is convenient to work with nodes represented as vectors (j, k), with these rules of generation: $(j, k) \to (j, k+1)$ and $(j, k) \to (k, -j)$. There are nine easily verifiable types of containment:

C0: $(0, 0) \in g(0)$
C1: $(n-1, 0) \in g(n)$ for $n \geq 2$
C2: $(0, n) \in g(n)$ for $n \geq 1$
C3: $(3-n, 0) \in g(n)$ for $n \geq 4$
C4: $(0, 2-n) \in g(n)$ for $n \geq 3$
C5: $j \geq 1,\ k \geq 1,\ j+k+1 \leq n \Longrightarrow (j, k) \in g(n)$ for $n \geq 3$
C6: $j \geq 1,\ k \geq 1,\ j+k+3 \leq n \Longrightarrow (-j, k) \in g(n)$ for $n \geq 5$
C7: $j \geq 1,\ k \geq 1,\ j+k+3 \leq n \Longrightarrow (-j, -k) \in g(n)$ for $n \geq 5$
C8: $j \geq 1,\ k \geq 1,\ j+k+2 \leq n \Longrightarrow (j, -k) \in g(n)$ for $n \geq 4$

Taking all the containments together gives the results in Table 1.

Table 1. Counting nodes in $g(n)$ of $T(i)$

n	C0	C1	C2	C3	C4	C5	C6	C7	C8	G_n
0	1	0	0	0	0	0	0	0	0	1
1	0	1	0	0	0	0	0	0	0	1
2	0	1	1	0	0	0	0	0	0	2
3	0	1	1	1	0	1	0	0	0	4
4	0	1	1	1	1	2	0	0	1	7
5	0	1	1	1	1	3	1	1	2	11
6	0	1	1	1	1	4	2	2	3	15
7	0	1	1	1	1	5	3	3	4	19
8	0	1	1	1	1	6	4	4	5	23

The nine contaiment types enable an easy proof that for $n \geq 4$, the number of nodes in $g(n)$ is $4n - 9$, as stated without proof in [2].

4. The tree $T(\sqrt{d})$, d not a square, yields a recurrence of order d

Here we prove that, for the tree $T(\sqrt{d})$, d not a square, the nth generation cardinality, G_n, is the sum of the previous d generation cardinalities G_{n-i}, $1 \leq i \leq d$, for n large enough. The proof is a rewriting and a generalization of the elegant method of proof used by Michael Stoll in the case $d = 2$ in [4].

Let $d \geq 2$ be an integer, but not a square integer. Denote the set of numbers $a + b\sqrt{d}$, where a and b are nonnegative integers, by $\mathbb{Z}_{\geq 0}[\sqrt{d}\,]$.

For a number x in $\mathbb{Z}_{\geq 0}[\sqrt{d}\,]$ define the length of x, $\ell(x)$, as the *minimal* number of steps needed to obtain x from 0, where two steps are allowed, namely $y \mapsto y + 1$ and $y \mapsto y\sqrt{d}$. Clearly $g(n) = \{x;\ \ell(x) = n\}$. Note that every number x of the prescribed form $a + b\sqrt{d}$ has a unique expansion of the form

$$c_0 + c_1\sqrt{d} + c_2\sqrt{d}^{\,2} + \cdots + c_k\sqrt{d}^{\,k}, \tag{4.1}$$

where the c_i's are in $\{0, 1, \ldots, d-1\}$ and $c_k > 0$. When convenient, we write k_x for k and, if $k \geq 1$, c_x^- for c_{k-1}.

To see that (4.1) holds, write both a and b in base d and use $d = \sqrt{d}^2$. The uniqueness of the writing comes from the uniqueness of the d-ary expansion of any nonnegative integer and the fact that $a + b\sqrt{d} = a' + b'\sqrt{d}$, a, b, a' and b' integers, implies $a = a'$ and $b = b'$.

Lemma 4.1. *Suppose $d \geq 2$. Let $x = c_0 + c_1\sqrt{d} + c_2\sqrt{d}^2 + \cdots + c_k\sqrt{d}^k$, where $c_i \in \{0, 1, \ldots, d-1\}$, $0 \leq i \leq k$, and $c_k \neq 0$. Then $\ell(x) = \ell'(x)$ unless $d = 2$ and $x = 1$ or $\sqrt{2}$, where*

$$\ell'(x) := \begin{cases} k + s(x), & \text{if } d \geq 3, \\ k + s(x) + c_{k-1} - 1, & \text{if } d = 2, \end{cases}$$

with the convention that $c_{-1} = 0$ if $k = 0$, and $s(x) = \sum_{i=0}^{k} c_i$.

Proof. Since the set of x's of the form $a + b\sqrt{d}$, where a and b are nonnegative integers, is countable and these numbers can be listed in increasing order, we can use a proof by induction. Note that, for $d = 2$, $\ell(2) = 2 = \ell'(2)$ and $\ell(1 + \sqrt{2}) = 3 = \ell'(1 + \sqrt{2})$. So, when $d = 2$, assume $x > 1 + \sqrt{2}$ with consequently $k_x \geq 2$, and assume $x > 1$ when $d \geq 3$. By the inductive hypothesis we have, in case $x - 1$ and $x/\sqrt{d}$ are of the prescribed form $a + b\sqrt{d}$, with a and b nonnegative integers, that $\ell(x - 1) = \ell'(x - 1)$ and $\ell(x/\sqrt{d}) = \ell'(x/\sqrt{d})$. If $c_0 \geq 1$ in x, then clearly $\ell(x) = \ell(x - 1) + 1 = \ell'(x - 1) + 1 = \ell'(x)$, where the last equality holds because $k_{x-1} = k_x$ and $s(x - 1) = s(x) - 1$, and $c^-_{x-1} = c^-_x$ when $d = 2$ because $k_x \geq 2$. If $c_0 = 0$, then either $a = 0$ or $a = c_{2m}d^m +$ (possibly) higher terms $c_{2i}d^i$, ($i > m \geq 1$ and $c_{2m} \neq 0$), where $x = a + b\sqrt{d}$. Then

$$\ell(x) \leq \ell(x/\sqrt{d}) + 1 = \ell'(x/\sqrt{d}) + 1 = \ell'(x). \tag{4.2}$$

If $a = 0$, then (4.2) is an equality because $x - 1 \notin \mathbb{Z}_{\geq 0}[\sqrt{d}]$. Assume $a > 0$. Since $\ell(x) = 1 + \min\{\ell(x/\sqrt{d}), \ell(x - 1)\}$, we have, from (4.2), $\ell'(x) = \ell(x)$ if $\ell(x/\sqrt{d}) \leq \ell(x - 1)$, that is, if $\ell'(x) - 1 \leq \ell'(x - 1)$. Now

$$a - 1 = (d - 1)\sum_{i=0}^{m-1} d^i + (c_{2m} - 1)d^m + \cdots .$$

Suppose first $d \geq 3$. Then, $\ell'(x-1) = k_{x-1} + \big(s(x) + m(d-1) - 1\big)$. But $k_{x-1} \geq k_x - 2$ with equality iff $k_x = 2m$, $c_{2m} = 1$ and $c^-_x = 0$. Hence, $\ell'(x-1) \geq (k_x + s(x)) + m(d-1) - 3 \geq \ell'(x) - 1$ with equality in the latter inequality iff $d = 3$ and $m = 1$.

Suppose now $d = 2$. If $k_{x-1} = k_x$, then $\ell'(x-1) \geq k_x + \big(s(x) + m - 1\big) - 1 \geq k_x + s(x) - 1 \geq \ell'(x) - 1$. Suppose $k_{x-1} < k_x$. If $m = 1$, then, since $x > 1 + \sqrt{2}$ and $c_0 = 0$, we find that $x = \sqrt{2} + 2$. Thus, $\ell'(x - 1) = 3 \geq 4 - 1 = \ell'(x) - 1$. Hence, we may now assume $m \geq 2$. If $k_{x-1} = k_x - 2$, then $c^-_x = 0$ and $\ell'(x-1) \geq 2m - 2 + \big(s(x) + m - 1\big) - 1 \geq 2m + s(x) - 2 = \ell'(x) - 1$. If $k_{x-1} = k_x - 1$, then $c^-_x = 1$ and $\ell'(x - 1) = 2m - 1 + \big(s(x) + m - 1\big) + 0 \geq 2m + s(x) = \ell'(x) > \ell'(x) - 1$. □

Definition. For $i = 0, \ldots, d - 1$, write S^i for the set of x in $\mathbb{Z}_{\geq 0}[\sqrt{d}]$ with $c_0 \equiv i \pmod{d}$, and S^i_n for $g(n) \cap S^i$.

Consider the following $2d - 1$ conditional steps

$$x \in S^{d-1} \mapsto x\sqrt{d} \in S^0, \quad \text{and} \quad x \in S^i \begin{array}{l} \nearrow \; 1 + x \in S^{i+1}, \\ \searrow \; x\sqrt{d} \in S^0, \end{array} \qquad (0 \leq i \leq d-2). \tag{4.3}$$

Lemma 4.2. *If $d \geq 3$, then there is exactly one path from 0 to x that uses the steps in (4.3) for all x in $\mathbb{Z}_{\geq 0} + \mathbb{Z}_{\geq 0}\sqrt{d}$, and this path is minimal, i.e., it contains $\ell(x)$ steps. If $d = 2$, then for all x satisfying $\ell(x) \geq 3$, there is a unique minimal path from one of the three nodes 3, $2\sqrt{2}$ and $1 + \sqrt{2}$ in $g(3)$ to x using the steps (4.3).*

Proof. The existence of a path is easily seen to be true as the steps in (4.3) allow moving from any $c_1 + c_2\sqrt{d} + \cdots + c_k\sqrt{d}^{k-1}$ to $c_0 + c_1\sqrt{d} + c_2\sqrt{d}^2 + \cdots + c_k\sqrt{d}^k$ for any c_0, $0 \leq c_0 < d$. The uniqueness can be seen by observing that all nonzero $x \in \mathbb{Z}_{\geq 0}[\sqrt{d}]$ have at most one predecessor using any of the $2d - 1$ functions described in (4.3), namely $x/\sqrt{d}$ if $x \in S^0$, and $x - 1$ otherwise. Suppose $d \geq 3$. Each of the $2d - 1$ steps in (4.3) produces an increment of 1 in the ℓ' function. Since $\ell'(0) = 0$, this path reaches x in $\ell'(x)$ steps. By Lemma 4.1, $\ell'(x) = \ell(x)$ so that this path to x is minimal. If $d = 2$, then applying one the three steps in (4.3) to some x with $\ell(x) \geq 3$ also increases ℓ' by 1. (This is not true before the third generation as $\sqrt{2} \mapsto 1 + \sqrt{2}$ is a step in (4.3), but $\ell'(1 + \sqrt{2}) = 3$ and $\ell'(\sqrt{2}) = 1$.) □

Theorem 4.3. *Let $d \geq 2$ be a non-square integer. Consider the tree generated by the two steps $x \mapsto 1 + x$ and $x \mapsto x\sqrt{d}$ starting with $x = 0$. Then, with G_k denoting the cardinality of the generation $g(k)$, we find that*

$$G_{n+d} = \sum_{i=0}^{d-1} G_{n+i},$$

for all $n \geq 3$, if $d = 2$, and for all $n \geq 1$, if $d \geq 3$.

Proof. Suppose either $d = 2$ and $n \geq 5$, or, $d \geq 3$ and $n \geq d + 1$, then, by (4.3) and Lemma 4.2, we see that

$$G_n = S_{n-1}^{d-1} + 2\sum_{i=0}^{d-2} S_{n-1}^i = G_{n-1} + \sum_{i=0}^{d-2} S_{n-1}^i.$$

But $S_{n-1}^0 = G_{n-2}$, $S_{n-1}^1 = S_{n-2}^0 = G_{n-3}$, $S_{n-1}^2 = S_{n-2}^1 = S_{n-3}^0 = G_{n-4}$, $\cdots$, $S_{n-1}^{d-2} = S_{n-2}^{d-3} = \cdots = S_{n+1-d}^0 = G_{n-d}$. Thus, $G_n = \sum_{i=1}^d G_{n-i}$. □

Remark. When $d = 2$, we find that $G_n = L_{n-1}$ for all $n \geq 3$, where L_k is the kth Lucas number. When $d \geq 3$ and not a square, then $G_i = 2^{i-1}$ for $1 \leq i \leq d - 1$ and $G_d = 2^d - 1$. For $d = 3$, (G_n) was seen as a companion to the tribonacci sequence and G_n proved to count compositions of n into parts 1 or 2 $\pmod 3$ in [3]. More generally, for all $d \geq 3$, G_n was proved to count compositions of n into parts $1, 2, \cdots, d - 1 \pmod d$ in [1, Theorem 8].

Remark. If $d = m^2$ for some integer $m \geq 2$, then it was shown in [2, Theorem 2.1] that $G_{n+m} = \sum_{i=0}^{m-1} G_{n+i}$, for all $n \geq 2$, with $G_i = 2^{i-1}$, $1 \leq i \leq m - 1$, and $G_m = 2^{m-1} - 1$. However, the method used for proving Theorem 4.3 also works for the case $d = m^2$ provided we use the m-ary representation of positive integers. We briefly illustrate this for $m = 3$. For x a positive integer, write

$$x = c_0 + c_1 3 + c_2 3^2 + \cdots + c_k 3^k,$$

where the c_i's are 0, 1 or 2, and $c_k \neq 0$. Define $\ell'(x) := k + s(x)$, where $s(x) = \sum_{i=0}^k c_i$. Then we can show inductively that $\ell(x) = \ell'(x)$. Denote the positive integers congruent to $i \pmod 3$ by S^i, and the intersection of S^i with $g(n)$ by S_n^i. Then we may observe that with the five steps

$$x \in S^2 \mapsto 3x \in S^0, \text{ and } x \in S^i \begin{array}{l} \nearrow \; 1 + x \in S^{i+1}, \\ \searrow \; 3x \in S^0, \end{array} \qquad (i = 0 \text{ or } 1),$$

there is a unique minimal path from 0 to every positive integer x (each $x \in \mathbb{Z}_{\geq 1}$ has a unique predecessor, namely $x/3$ if $x \in S^0$ and $x - 1$ if $x \in S^1 \cup S^2$). Then, for $n \geq 4$, $G_n = S_n^0 + S_n^1 + S_n^2 = G_{n-1} + S_{n-1}^0 + S_{n-1}^1 = G_{n-1} + G_{n-2} + S_{n-2}^0 = G_{n-1} + G_{n-2} + G_{n-3}$.

5. More examples

Recalling that $q(x)$ in Example 4 is simply $x^2 - 2$, it is natural to discover that for other choices of $q(x) = x^2 - q_1 x - q_0$, the resulting generations $g(n)$ appear to have linearly recurrent cardinality sequences (G_n). Some proofs of the computer-detected recurrences reported in Table 5 and 6 might be challenging. Most of the entries in the column headed "linear recurrence" are conjectured as "eventual"; i.e., the recurrence applies after some unspecified number of initial terms. An alternative way to represent these recurrences, including initial terms, is by Mathematica, as in Program 5 in Section 7. The number of initial terms before linear recurrence applies is quite remarkable in some cases. For example, for $q(x) = x^2 + x - 1$, linear recurrence applies after 24 initial terms, as indicated by the following Mathematica code:

$$\text{Join}[\{1, 1, 2, 4, 7, 11, 16, 23, 31, 43, 62, 90, 131, 191, 279, 408, 597, 873, 1279, 1874, 2746\}, \text{LinearRecurrence}[\{1, 0, 1\}, \{4023, 5896, 8641\}, 30]]$$

Table 5. Linear recurrences for (G_n)

$q(x)$	G_n	linear recurrence
$x^2 - 1$	$1, 1, 2, 4, 4, 5, 6, 7, 8, 9, 10,$	$2, -1$
$x^2 - 2$	$1, 1, 2, 3, 4, 7, 11, 18, 29,$	$1, 1$
$x^2 - 3$	$1, 1, 2, 3, 6, 11, 20, 37, 68,$	$1, 1, 1$
$x^2 - 4$	$1, 1, 2, 4, 7, 14, 27, 52, 100,$	$1, 1, 1, 1$
$x^2 - x + 1$	$1, 1, 2, 4, 7, 11, 16, 22, 28, 34, 40,$	$2, -1$
$x^2 - 2x + 1$	$1, 1, 2, 4, 7, 11, 16, 22, 29, 37, 46,$	$3, -3, 1$
$x^2 - 3x + 1$	$1, 1, 2, 4, 7, 13, 23, 42, 75, 136,$	$1, 2, -1$
$x^2 - 4x + 1$	$1, 1, 2, 4, 8, 15, 29, 56, 107, 206,$	$1, 1, 2, -1$
$x^2 - 5x + 1$	$1, 1, 2, 4, 8, 16, 31, 61, 120, 236,$	$1, 1, 1, 2, -1$
$x^2 - x - 1$	$1, 1, 2, 3, 5, 8, 12, 18, 25,$	$1, 0, 1$
$x^2 - 3x - 1$	$1, 1, 2, 4, 8, 15, 29, 55, 104,$	$1, 1, 1, 0, 1$
$x^2 - 5x - 1$	$1, 1, 2, 4, 8, 16, 32, 63, 125, 247,$	$1, 1, 1, 1, 1, 0, 1$
$x^2 - 2x - 1$	$1, 1, 2, 4, 7, 13, 23, 40, 70, 123,$	$2, -1, 1$
$x^2 - 4x - 1$	$1, 1, 2, 4, 8, 16, 31, 61, 119, 232,$	$2, -1, 2, -1, 1$
$x^2 - 6x - 1$	$1, 1, 2, 4, 8, 16, 32, 64, 127, 253,$	$2, -1, 2, -1, 2, -1, -1, 1$

For each $q(x)$ in Table 5, an unspecified but compelling number of terms satisfying the conjectured recurrence were checked. For example, for $q(x) = x^2 - x + 1$, the number of terms checked for the recurrence was 142, beginning with $G_8 = 28$. Also, regarding Table 5, note, for example, that the tree $T(x^2 - 4)$ contains isomorphic images of $T(2)$ and $T(-2)$ as proper subtrees.

Table 6. Linear recurrences for (G_n)

$q(x)$	G_n	(conjectured) linear recurrence
x^2-3x+2	$1,1,2,4,8,15,27,47,80,134,222,$	$3,-2,-1,1$
x^2-4x+2	$1,1,2,4,8,15,29,55,105,200,381,$	$1,2,0,-1$
x^2-5x+2	$1,1,2,4,8,16,31,61,120,235,461,$	$1,1,2,0,-1$
x^2-6x+2	$1,1,2,4,8,16,32,63,125,248,492,$	$1,1,1,2,0,-1$
x^2-4x+3	$1,1,2,4,8,16,31,59,111,207,384,$	$3,-2,0,-1,1$
x^2-5x+3	$1,1,2,4,8,16,31,61,119,233,455,$	$1,2,0,0,-1$
x^2-6x+3	$1,1,2,4,8,16,32,63,125,248,491,$	$1,1,2,0,0,-1$
x^2-7x+3	$1,1,2,4,8,16,32,64,127,253,504,$	$1,1,1,2,0,0,-1$

For each $q(x)$ in Table 6, a number of terms satisfying the conjectured recurrence were checked. For example, for $q(x)=x^2-3x+2$, the number of terms checked for the recurrence was 37, beginning with $G_4=8$. Tables 5 and 6 suggest that patterns associated with each form of equation depend on two cases: (1) two real zeros, or (2) two nonreal zeros, and the examples in the two tables are arranged in groups that suggest the following conjectures:

1. If $q(x)=x^2-kx+1$, where $k>2$, then (G_n) satisfies the linear recurrence with coefficients

$$(\overbrace{1,1,\dots,1}^{k-2 \text{ terms}},2,-1), \text{ and initial terms } (1,1,2,\dots,2^{k-1},2^k-1).$$

2. If $q(x)=x^2-(2k+1)x-1$, where $k>0$, then (G_n) satisfies the linear recurrence with coefficients

$$(\overbrace{1,1,\dots,1}^{2k+1 \text{ terms}},0,1), \text{ and initial terms } (1,1,2,\dots,2^{2k+1},2^{2k+2}-1).$$

3. If $q(x)=x^2-2kx-1$, where $k>1$, then (G_n) satisfies the linear recurrence with coefficients

$$(\overbrace{2,-1,2,-1,\dots,2,-1}^{2k \text{ terms}},1), \text{ and initial terms } (1,1,2,\dots,2^{2k},2^{2k+1}-1).$$

4. If $q(x)=x^2-kx+2$, where $k>3$, then (G_n) satisfies the linear recurrence with coefficients

$$(\overbrace{1,1,\dots,1}^{k-3 \text{ terms}},2,0,-1), \text{ and initial terms } (1,1,2,\dots,2^{k-1},2^k-1).$$

5. If $q(x)=x^2-kx+3$, where $k>4$, then (G_n) satisfies the linear recurrence with coefficients

$$(\overbrace{1,1,\dots,1}^{k-4 \text{ terms}},2,0,0,-1), \text{ and initial terms } (1,1,2,\dots,2^{k-1},2^k-1).$$

6. If $q(x) = x^2 - kx + (k-1)$, where $k > 2$, then (G_n) satisfies the linear recurrence with coefficients

$$(3, -2, \overbrace{0, 0, \ldots, 0}^{k-3 \text{ terms}}, -1, 1), \text{ and initial terms}$$
$$(1, 1, 2, \ldots, 2^k, 2^{k+1} - 1, 2^{k+2} - 5, 2^{k+3} - 17).$$

6. The trees $T(1/d)$ and $T(-d)$

Suppose that $d \geq 2$. Recall that $T(1/d)$ is obtained by substituting $1/d$ for x in the polynomial tree T^* and using top-to-bottom deletion of duplicates. Equivalently, starting with 0, we successively apply the following rules for growing a tree T': if $x \in T'$, then $x + 1 \in T'$ and $x/d \in T'$, with duplicates removed as they occur. The resulting tree T' is $T(1/d)$. It is inductively clear that $T(1/d)$ consists of 0 and all rational numbers a/d^i such that a and d are relatively prime positive integers. We note without proof another way to generate successive generations $g(n)$, without duplicates during the process, as follows: $g(0) = \{0\}$, and for $n \geq 1$,

$$g(n) = \{x + 1 : x \in g(n-1)\} \cup \{x/d : x \in g(n-1) \text{ and } x < d\}.$$

Theorem 6.1. *Suppose that $d \geq 2$. For the tree $T(1/d)$, the cardinality sequence (G_n) satisfies the linear recurrence equation*

$$G_n = G_{n-1} + G_{n-2} + \cdots + G_{n-d}, \tag{6.1}$$

with initial values

$$(G_0, G_1, \ldots, G_d) = (1, 1, 2, 2^2, \ldots, 2^{d-1}).$$

Proof. The initial values are inherited from T^*. We have

$$G_{d+1} = 2^d - 1 = 2^{d-1} + 2^{d-2} + \cdots + 2^0 = G_d + G_{d-1} + \cdots + G_1,$$

which serves as a first inductive step. Suppose for arbitrary $n \geq d + 1$ that (6.1) holds. It is easy to see that if, for some w in $g(n)$, one of the numbers $w + 1$ or w/d is not in $g(n + 1)$ because it is already in an earlier $g(m)$, then that number is $x/d + 1$ for some x in $g(n - d)$. Specifically, the number arises from x as follows:

$$x \to x + 1 \to x + 2 \to \cdots \to x + d \to (x + d)/d,$$

which is already in $g(n - d + 2)$, as indicated by

$$x \to x/d \to x/d + 1.$$

Therefore,

$$\begin{aligned}
G_{n+1} &= 2G_n - \#\{w \in g(n) : w \geq d\} \\
&= 2G_n - \#\{w \in g(n) : w = x + d \text{ for some } x \text{ in } g(n-d)\} \\
&= 2G_n - G_{n-d} \\
&= G_n + (G_{n-1} + G_{n-2} + \cdots + G_{n-d}) - G_{n-d} \\
&= G_n + G_{n-1} + \cdots + G_{n+1-d},
\end{aligned}$$

so that (6.1) holds for all $n \geq d + 1$. □

Note that G_n in Theorem 6.1 can be counted by the number of fractions having denominators $1, d, d^2, \ldots, d^{n-1}$. For example, for $d = 3$, the fractions in $g(7)$ are counted as $1 + 2 + 6 + 14 + 14 + 6 + 1$; i.e., 1 fraction with denominator 1, and 2 with denominator 3, and 6 with denominator 3^2, etc. The seven summands are identical to row 7 of a tribonacci triangle (A224598 in [5]).

As a corollary to Theorem 6.1, the generation sizes for the tree $T(1/2)$ comprise the classical Fibonacci sequence: $G_n = F_{n+1}$ for $n \geq 0$.

Theorem 6.2. *Suppose that $d \geq 2$. For the tree $T(-d)$, the sequence (G_n) satisfies the linear recurrence (6.1) beginning at G_{2d+2}.*

A proof of Theorem 6.2 is omitted. Note, in particular, that the tree $T(-2)$ has $(G_n) = (1, 1, 2, 4, 5, 8, 13, \ldots)$, obtained by substituting 4 for 3 in the Fibonacci sequence. Initial values of (G_n), for selected values of d, are shown in Table 7.

Table 7. Initial values of (G_n)

d	$(G_0, G_1, \ldots, G_{2d+1})$
2	$(1, 1, 2, 4, 5, 8)$
3	$(1, 1, 2, 4, 8, 14, 25, 46)$
4	$(1, 1, 2, 4, 8, 16, 30, 58, 111, 214)$
5	$(1, 1, 2, 4, 8, 16, 32, 62, 122, 240, 471, 926)$
6	$(1, 1, 2, 4, 8, 16, 32, 64, 126, 250, 496, 984, 1951, 3870)$
7	$(1, 1, 2, 4, 8, 16, 32, 64, 128, 254, 506, 1008, 2008, 4000, 7967, 15870)$

7. MATHEMATICA PROGRAMS

This section shows Mathematica (version ≥ 7) code used to generate trees and sequences found elsewhere in the article. These may prove useful for further research.

Program 1 generates the polynomial tree T*

```
Expand[NestList[DeleteDuplicates[Flatten[Map[{#+1,x*#}&,#],1]]&,{1},7]]
```

Program 2 draws Figure 1

```
f:={#+1,x #}&;
graph=Most[Flatten[Map[Thread[{#,#}->f[#]]&,Flatten[Nest[f,0,4]]]]]
t=TreePlot[Expand[graph],Left,0,VertexLabeling->True,ImageSize->400]
```

Program 3 generates the tree $T(r)$

```
r=Sqrt[2];z=10;
t=Expand[NestList[DeleteDuplicates[Flatten[Map[{#+1,r*#}&,#],1]]&,{0},z]];
s[0]=t[[1]];
s[n_]:=s[n]=Union[t[[n+1]],s[n-1]]
g[n_]:=Complement[s[n],s[n-1]];
Column[Table[g[n],{n,z}]]
Table[Length[g[n]],{n,z}]
```

Program 4 generates the tree $T(1/d)$ as in Theorem 6.1

```
d=3;g[0]={0};g[1]={1};
g[n_]:=g[n]=Union[1+g[n-1],(1/d) Select[g[n-1],#<d&]]
u=Table[g[n],{n,0,7}]
Map[Length,u]
```

Program 5 generates the sequence $(g(n))$ for $q(x) = x^2 - 3x + 2$, as in Table 6

```
LinearRecurrence[{3,-2,-1,1},{1,1,2,4},30]
```

References

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[5] Online Encyclopedia of Integer Sequences, **https://oeis.org/**

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