WHAT I TELL YOU K TIMES IS TRUE ...

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Abstract. Some years ago while investigating generalized Zeckendorf representations, i.e. representations of integers as binary sums of k^{th} order Fibonacci numbers, we found that the fraction of 0's in the representation was a monotone increasing function of the number of bits used. It was relatively easy to show that this behavior was to be expected asymptotically, but was there an easy way to show that this fraction was always increasing? Specifically, could one find a K (presumably bigger than the order k) so that if the fraction was increasing for K consecutive steps, then it would always be increasing? Here, we introduce the **SP** (sorta positive) polynomials. We show that if the characteristic polynomial for a difference equation is a factor of an **SP** polynomial of degree K, then if the ratios of any two solutions to the equation are K in row increasing, then the ratios are always increasing. For the k^{th} order Fibonacci sequences $K = k+1$. For the fraction of 0's in the Zeckendorf representation, the characteristic polynomial is the square of a Fibonacci characteristic polynomial (i.e. degree $= 2k$) and we show that $K = 2k+2$.

1. Introduction

According to an old joke, the "rule of three" means that "What I tell you 3 times is true", with the example that "all odd numbers are prime" because "3 is prime", "5 is prime", and "7 is prime". (The inconvenient fact that "9 is not prime" can be ascribed to *experimental error*.)

On the other hand, as Bill Webb pointed out at this conference, many properties of Fibonacci numbers and related sequences can be proved by obvious inductions because the sequences obey simple difference equations. To prove that a sequence has a property, we can check that the property holds for the first K elements, where K is the order of the difference equation, and show that the difference equation implies that the property holds for the next element of the sequence.

Of course, the difficulty is finding the "right" difference equation which, in contrast to the joke, may have order K much greater than 3.

Our original motivating example was the fraction of 0's in the n-bit (generalized) Zeckendorf representation of integers. The question was: How many of these fractions need to be increasing to show that these fractions are **always** increasing?

Before giving conditions for **K-in-a-row** implies **always**, let's see an example in which this rule does **not** hold. Consider the difference equation

$$x_n = 2x_{n-1} - x_{n-2}$$

and the property $x_n > 0$. If the rule held for this example, there would be a K so that if

$$x_0 > 0,\ x_1 > 0,\ \ldots,\ x_{K-1} > 0$$

then $x_n > 0$ for **all** $n \geq 0$. But, the sequence $x_n = (K - n)$ satisfies this difference equation and

$$x_0 = K > 0,\ x_1 = K-1 > 0,\ \ldots,\ x_{K-1} = 1 > 0$$

but $x_K = 0 \not> 0$ and in fact $x_n < 0$ for all $n > K$. Hence, there is **NO** value of K so that **K-in-a-row-positive** implies **always positive**.

1.1. **Some Things We All Know.** In contrast to the above, we all know how to show familiar facts about the Fibonacci numbers, like Fibonacci numbers are MONOTONE INCREASING (for $n > 2$). The proof follows from the difference equation

$$f_n = f_{n-1} + f_{n-2}.$$

So if $f_{n-1} > 0$ and $f_{n-2} > 0$ then $f_n > f_{n-1}$. Or said another way, $0 < f_1 \le f_2$ implies $f_n > f_{n-1}$ for all $n > 2$. So, in this example, if I tell you that the sequence is increasing TWICE then it is ALWAYS increasing.

Unfortunately, things may not always be quite this easy. We all know that

$$\frac{f_n}{f_{n-1}} \longrightarrow \frac{1+\sqrt{5}}{2},$$

but these ratios are **NOT MONOTONE**, they oscillate above and below their limit.

On the other hand

$$\frac{f_n}{f_{n-1}+1} = \frac{1}{1}, \frac{1}{2}, \frac{2}{2}, \frac{3}{4}, \frac{5}{4}, \frac{8}{6}, \frac{13}{9}, \ldots \longrightarrow \frac{1+\sqrt{5}}{2},$$

and is **MONOTONE** increasing for $n \ge 4$. This result could be proved using the identity $f_{n+1} f_{n-1} - f_n^2 = \pm 1$, but we desire a more general approach. The Fibonacci polynomial (i.e. the characteristic polynomial of the Fibonacci difference equation) is $x^2 - x - 1$, a NonNegative polynomial. If we multiply this polynomial by $x - 1$ we get $x^3 - 2x^2 + 1$ which is **NOT** NonNegative, but is the characteristic polynomial of $Z_n = 2Z_{n-1} - Z_{n-3}$. Notice that $1 + f_n$ does satisfy this equation, but does not satisfy the Fibonacci DE (we will use DE as an abbreviation for difference equation), $f_n = f_{n-1} + f_{n-2}$, (the extra $+1$ gets in the way). Also f_n does satisfy this DE. So the monotone ratios that we've found are the ratios of two different solutions to the difference equation $Z_n = 2Z_{n-1} - Z_{n-3}$. We will consider a class of DE's for which this DE may serve as a prototype.

2. What's the Difference?

The behavior of solutions of difference equations can often be explained in terms of the properties of their associated characteristic polynomials. To explain our examples, we will contrast the well-known NonNegative polynomials [6] with the **SP** polynomials [7].

NonNegative Difference Equations

Difference Equation (k^{th} order):	$Z_n = c_1 Z_{n-1} + \ldots + c_k Z_{n-k}$.
Characteristic Polynomial:	$ch(x) = x^k - c_1 x^{k-1} - \ldots - c_k$.
NonNegative:	each $c_i \ge 0$ and $c_k > 0$.
Aperiodic (primitive):	$\gcd\{ i \mid c_i > 0 \} = 1$.

Some Facts about NonNegative Difference Equations and Polynomials [6]

- Unique positive real root λ_0.
- $\lambda_0 \ge |\lambda_i|$ for any other root λ_i.
- If aperiodic, $\lambda_0 > |\lambda_i|$ for any other root λ_i.
- If k Z_n's in a row are positive, then $Z_n > 0$ for all larger n.

- If $k + 1$ in a row are increasing, then Z_n is always increasing.
- If $\lambda_0 = 1$, then there is NO always increasing solution (there are never $k + 1$ in a row increasing).

SP (Sorta Positive) Difference Equations

Difference Equation (k^{th} order): $Z_n = b_{k-1} Z_{n-1} - b_{k-2} Z_{n-2} - \ldots - b_0 Z_{n-k}$.
Characteristic Polynomial: $S(x) = x^k - b_{k-1} x^{k-1} + b_{k-2} x^{k-2} + \ldots + b_1 x + b_0$.
Sorta Positive: $b_i \geq 0$ and $b_0 > 0$.
(We call these Sorta Positive since all the coefficients except one must be nonnegative, i.e. sort of positive.)

Some Facts about **SP** Polynomials and Difference Equations [1]

- $S(x)$ has 0 or 2 positive real roots λ_0 and λ_1.
- $\lambda_0 \geq \lambda_1$.
- All other roots are dominated by λ_1:
  $\lambda_0 \geq \lambda_1 > |\lambda_i|$.
- Z_n is $O(\lambda_0^n)$ if $\lambda_0 > \lambda_1$
  (for "reasonable" initial conditions, Z_n is $\Theta(\lambda_0^n)$).
- Z_n is $O(n \lambda_0^n)$ if $\lambda_0 = \lambda_1$
  (for "reasonable" initial conditions, Z_n is $\Theta(n \lambda_0^n)$).

In our example, $f_n = f_{n-1} + f_{n-2}$ is a NonNegative difference equation, $g_n = 2 g_{n-1} - g_{n-3}$ is an **SP** difference equation. The f_n equation has a single positive eigenvalue $\lambda_0 = (1 + \sqrt{5})/2$, while the g_n equation has two positive eigenvalues λ_0 and 1. As noted above f_n / λ_0^n cannot be monotone, but it's perfectly possible that g_n / λ_0^n *may* be monotone, and we'll see below that initial monotonicity implies always monotone.

3. SP (Sorta Positive Polynomials)

We defined **SP** polynomials so that we could show that certain ratios were monotone increasing. The following theorem gives the conditions.

Theorem 3.1. *(**SP** Theorem) If N_n and W_n are any two POSITIVE sequences which are solutions to a difference equation with a k^{th} order **SP** polynomial $S(x)$ as its characteristic polynomial, then the condition*

$$\frac{N_0}{W_0} \leq \frac{N_1}{W_1} \leq \ldots \leq \frac{N_{k-1}}{W_{k-1}}$$

with at least one of these inequalities strict, ($<$) implies

$$\frac{N_n}{W_n} \text{ is monotone increasing for all } n \geq k.$$

Proof. Consider

$$D_{i,j} = \begin{vmatrix} N_i & W_i \\ N_j & W_j \end{vmatrix} = N_i W_j - N_j W_i = W_i W_j \left(\frac{N_i}{W_i} - \frac{N_j}{W_j} \right).$$

[1] $T(n) = \Theta(g(n))$ means there exist positive constants c_1 and c_2 so that $c_1 |g(n)| \leq |T(n)| \leq c_2 |g(n)|$ for all sufficiently large n. $T(n) = O(g(n))$ means there exists a positive constant c so that $|T(n)| \leq c |g(n)|$ for all sufficiently large n.[6]

Clearly, $D_{i,j}$ is anti-symmetric, i.e. $D_{i,j} = -D_{j,i}$ and $D_{i,i} = 0$. Now consider $D_{n+1,n} = W_{n+1}\, W_n \left(\frac{N_{n+1}}{W_{n+1}} - \frac{N_n}{W_n} \right)$. If $\frac{N_n}{W_n}$ is increasing ($\geq$) or strictly increasing ($>$) then $D_{n+1,n} \geq 0$ or $D_{n+1,n} > 0$. Then assuming that N_n and W_n satisfy an **SP** difference equation,

$$\begin{aligned} D_{n+1,n} &= b_{k-1}\, D_{n,n} - b_{k-2}\, D_{n-1,n} - \ldots - b_0\, D_{n+1-k,n} \\ &= b_{k-2}\, D_{n,n-1} + \ldots + b_0\, D_{n,n+1-k}. \end{aligned}$$

If

$$\frac{N_{k-1}}{W_{k-1}} \geq \frac{N_{k-2}}{W_{k-2}} \geq \ldots \geq \frac{N_0}{W_0},$$

then $D_{k-1,k-i} \geq 0$ for $i \in [2, k]$, and if least one of these inequalities is strict, $D_{k-1,0} > 0$. So $D_{k,k-1} > 0$ and by induction $D_{n,n-1} > 0$ for all $n \geq k$, and the corresponding ratios are strictly increasing. □

As an immediate application of this theorem, consider the generalized Fibonacci polynomial $x^k - x^{k-1} - \ldots - 1$ which is not **SP** but when multiplied by $x - 1$ the product polynomial is $x^{k+1} - 2x^k + 1$ which is **SP**. Let $\langle f_n \rangle = \langle 1, 1, 2, 4, 7, 13, 24, \ldots \rangle$ be the 3rd order Fibonacci numbers. Clearly the ratio

$$\frac{f_{n+1}}{f_n} = \frac{1}{1}, \frac{2}{1}, \frac{4}{2}, \frac{7}{4}, \frac{13}{7}, \ldots$$

is converging to λ_0, the positive root of $x^3 - x^2 - x - 1$, but this convergence is not monotone. On the other hand, $3 + f_n$ is a solution to the difference equation associated with $x^4 - 2\,x^3 + 1$, and, of course, so is f_{n+1}. Now consider

$$\frac{f_{n+1}}{3 + f_n} = \frac{1}{4}, \frac{2}{4}, \frac{4}{5}, \frac{7}{7}, \frac{13}{10}, \ldots .$$

This sequence is also converging to λ_0, but now because the characteristic polynomial $x^4 - 2\,x^3 + 1$ is **SP**, the convergence is monotone.

3.1. Multiply SP.

As we saw above, not all polynomials are **SP**, but sometimes we may be able to "change" a polynomial into **SP** form.
If $p(x)$ is a polynomial, is there another polynomial $r(x)$, so that $p(x)\, r(x)$ is **SP**?
If so we say that $p(x)$ is **multiply SP**.

Necessary Conditions for Multiply **SP**:

- $p(x)$ has at most TWO positive real roots.
- IF $p(x)$ has one positive real root, then this root dominates all other roots.
- IF $p(x)$ has two positive real roots λ_1 and λ_2,
  $\lambda_1 \geq \lambda_2$ and λ_2 dominates all other roots.
- IF $p(x)$ is a multiple of a PERIODIC NonNegative polynomial,
  then $p(x)$ is NOT multiply **SP**
  (because the positive real root is NOT dominant).

Theorem 3.2. *If $p(x)$ is a polynomial with a unique positive dominant root λ_0 and $\frac{p(x)}{x - \lambda_0}$ has only positive coefficients, then $p(x)$ is multiply* **SP**, *and the resulting* **SP** *polynomial $S(x)$ can be taken so that $\lambda_0 > |\,\lambda_i\,|$ for every λ_i which is a root of $S(x)$.*

3.2. **Multiply SP in MOD Form.**

Theorem 3.3. *IF* $x^n \equiv q_n(x) \quad \bmod p(x)$
AND $x^{n-1} \equiv q_{n-1}(x) \quad \bmod p(x)$
AND $\exists \beta$ *so that* $q_n(x) - \beta\, q_{n-1}(x)$ *has only negative coefficients*
THEN $x^n - \beta\, x^{n-1} - q_n(x) + \beta\, q_{n-1}(x)$ *is* **SP**
(and is a multiple of $p(x)$*).*

Computing $x^n \mod p(x)$

If $p(x) = x^k + p_1\, x^{k-1} + p_2\, x^{k-2} + \ldots + p_{k-1}\, x + p_k$
$x^k \equiv -p_1\, x^{k-1} - p_2\, x^{k-2} - \ldots - p_{k-1}\, x - p_k \;\; \mathbf{mod}\, p(x)$

$$\begin{bmatrix} -p_1 & 1 & 0 & \ldots & 0 \\ -p_2 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & & \\ -p_k & 0 & 0 & \ldots & 0 \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{pmatrix} = \begin{pmatrix} \hat{q}_1 \\ \hat{q}_1 \\ \vdots \\ \hat{q}_k \end{pmatrix}$$

$$\begin{aligned} x^r &\equiv q_1\, x^{k-1} + \ldots + q_k \\ x^{r+1} &\equiv q_1\, x^k + \ldots + q_k\, x \\ &\equiv q_1\,(-p_1 x^{k-1} - \ldots - p_k) + q_2\, x^{k-1} + \ldots + q_k\, x \\ &\equiv \hat{q}_1\, x^{k-1} + \ldots + \hat{q}_k \quad \bmod p(x). \end{aligned}$$

Our procedure is to repeatedly compute $x^r \mod p(x)$, until there is a β so that $x^r - \beta x^{r-1}$ is an **SP** polynomial. The matrix times vector calculates $x^r \mod p(x)$ from $x^{r-1} \mod p(x)$.

3.3. **Examples.** Let us demonstrate this method for two Fibonacci examples.

Example 1: $p(x) = x^2 - x - 1$. In this example, the matrix has 1's in its first column because $p(x)$ has -1's for its two coefficients.

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad M \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x^3 \equiv 2\,x + 1 \quad \bmod p(x)$$
$$x^2 \equiv x + 1 \quad \bmod p(x)$$

Taking $\beta = 2$,

$$x^3 - 2\,x^2 \equiv -1 \quad \bmod p(x).$$

So $S(x) = x^3 - 2\,x^2 + 1$. Notice that this is just the polynomial we found in Section 1 which allowed us to show that $f_n\,/\,(\,f_{n-1} + c\,)$ is monotone increasing.

From the formulation above, we could choose other values for β. For example, if we took $\beta = 4$, we'd get the polynomial $x^3 - 4\,x^2 + 2\,x + 3$, which would allow us to show that $(\,3^n + f_n\,)\,/\,(\,3^{n-1} + f_{n-1}\,)$ is monotone increasing. But this will be of little help if we're interested in Fibonacci numbers because all it says is that 3^n grows much faster than the Fibonacci numbers. What we're doing is introducing an extra root. To get formulas that help us understand Fibonacci numbers we'd need this root to be $\leq \lambda_0$ (and even $= \lambda_0$ may be too big).

Example 2: $p(x) = (x^2 - x - 1)^2 = x^4 - 2x^3 - x^2 + 2x + 1.$

The corresponding matrix and the first few iterates are:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \\ -1 \end{pmatrix} \to \begin{pmatrix} 5 \\ 0 \\ -5 \\ -2 \end{pmatrix} \to \begin{pmatrix} 10 \\ 0 \\ -12 \\ -5 \end{pmatrix}$$

$$\begin{aligned} x^6 &\equiv 10x^3 + 0x^2 - 12x - 5 \mod p(x) \\ 2x^5 &\equiv 10x^3 + 0x^2 - 10x - 4 \mod p(x). \end{aligned}$$

So when we choose $\beta = 2$,

$$S(x) = x^6 - 2x^5 + 2x + 1 \equiv 0 \mod p(x).$$

It's easy to check that $S'(\lambda_0) = 0$ and so the roots of $S(x)$ are bounded by λ_0. Of course, this is obvious because $p(x)$ has a double root at λ_0. This result will allow us to show that for the usual ($k = 2$) Fibonacci numbers, ratios like $(n A_n + B_n)/(n C_n + D_n)$ are monotone increasing (or decreasing). In the next section, we will see that similar results hold for the k^{th} order Fibonacci numbers regardless of k.

4. Fibonacci and **SP**

The Fibonacci Polynomials are Multiply **SP**

- $\mathrm{ch}(x) = x^k - x^{k-1} - \ldots - 1$
- $r(x) = x - 1$
- $r(x)\,\mathrm{ch}(x)$ is **SP**
- $r(x)\,\mathrm{ch}(x) = x^{k+1} - 2x^k + 1$

So $\dfrac{F_n + c}{F_{n-1} + c}$ is monotone increasing,

if

- F_n is any solution to the k^{th} order Fibonacci DE, and
- $F_n + c$ is positive, and
- this ratio is increasing for $k + 1$ consecutive values of n.

Squared Fibonacci Polynomials are Multiply **SP**

- $\mathrm{ch}(x) = x^k - x^{k-1} - \ldots - 1$
- $r(x) = 5x^2 - 2x + 1$
- $r(x)\,[\mathrm{ch}(x)]^2$ is **SP**
- Coefficients
  $5, -12, 0, 0, 4, \ldots, 4(k-3), 4k+2, 4(k-2), 4(k-2), \ldots, 8, 4, 0, 1$
- $k = 2$ 5, -12, 0, 10, 0, 0, 1
- $k = 3$ 5, -12, 0, 0, 14, 4, 0, 1.

Here we're not using the **MOD** calculation from the previous section. Instead we are "guessing" a multiplier and showing that it works. It is somewhat surprising that the single polynomial $r(x) = 5x^2 - 2x + 1$ serves as the multiplier for all squared Fibonacci polynomials regardless of the degree k. It is also pleasant that this multiplier is of low degree and so checking for monotone ratios only involves two more points after the initial conditions have been specified.

5. Zeckendorf Representation

It is fairly well known that natural numbers can be represented in a Fibonacci base, that is, if x is any natural number,

$$x = \sum b_i f_i$$

where the f_i are the Fibonacci numbers, 1, 2, 3, 5, 8, 13, ... and while the b_i's depend on x, each b_i is in $\{0, 1\}$. Further the n bit representation of x is unique when no two consecutive b_i's can both be 1 (see [11]). If instead of the usual Fibonacci numbers which are based on the recurrence $f_i = f_{i-1} + f_{i-2}$, one uses the k^{th} order Fibonacci numbers based on the recurrence $f_i = f_{i-1} + f_{i-2} + \ldots + f_{i-k}$, a representation still exists and is unique if no k consecutive b_i's are allowed to be 1. These representations are called the Zeckendorf representations [9, 4, 3]. One can think of the usual binary (base 2) representation as the limit of the Zeckendorf representations when $k = \infty$. Zeckendorf representations have many applications, for example in data structures [4], reliable data transmission [1][2], and cryptography [10].

Theorem 5.1. *(Zeckendorf's Theorem) Every positive integer has a unique representation as a sum of k^{th}-order Fibonacci numbers which uses no k consecutive Fibonacci numbers.*

$$n = \sum b_i f_i$$

where each $b_i \in \{0, 1\}$.

If we define the set of Zeckendorf, $\mathcal{Z}$, strings of length n as the binary strings of length n with no k consecutive 1's, then the number of such strings $\#_n$ is f_{n+2}, where f_{n+2} stands for the $(n+2)^{\text{nd}}$ Fibonacci number of order k. This fact follows from a simple argument. A $\mathcal{Z}$-string of length n starts with either 0 or 10 or 100, ..., or 11...10 ($k-1$ 1's), and for each of these prefixes the string after the prefix is a $\mathcal{Z}$ string of shorter length, which gives the recurrence

$$\#_n = \#_{n-1} + \ldots + \#_{n-k}$$

with the initial conditions $\#_1 = 2$, $\#_2 = 4$, ..., $\#_{k-1} = 2^{k-1}$, $\#_k = 2^k - 1$. (For consistency, one could put $\#_0 = 1$.) But this recurrence is just the recurrence for the Fibonacci numbers of order k with the initial conditions off-set by 2.

We can count W_n, the number of bits in $\mathcal{Z}$ strings of length n, and obviously $W_n = n\, f_{n+2}$. Since the bits are only 0's and 1's, we could count N_n, the number of 0 bits in $\mathcal{Z}$ strings of length n. Obviously the number of 1's is $W_n - N_n$. A formula for N_n is a little harder to come by, but a difference equation for N_n is easy to state:

$$N_n = N_{n-1} + N_{n-2} + \ldots + N_{n-k} + f_{n+2}.$$

This equation follows from the above observation about the possible prefixes, and the f_{n+2} term comes from the fact that each prefix has exactly one 0. Since half of the bits in each binary string are 0's, the initial conditions for N_n are

$$N_i = i\, 2^{i-1} \qquad \text{for} \quad i \in [0, k].$$

For $k = 2$, it's easy to obtain a formula for N_n:

$$N_n = \frac{1}{5}\, \{ n\, (f_{n+4} + f_{n+2}) - 2\, f_n \},$$

and this gives

$$\frac{N_n}{W_n} = \frac{1}{5}\left\{\frac{f_{n+4} + f_{n+2}}{f_{n+2}} - \frac{2\,f_n}{n\,f_{n+2}}\right\}.$$

From this formula,

$$\lim_{n\longrightarrow\infty}\frac{N_n}{W_n} = \frac{\lambda_0^2 + 1}{5} \approx .7236,$$

and further, since the first term converges exponentially while the second (negative) term converges like $1/n$, the ratio N_n/W_n will be approaching its asymptotic value from below and in an asymptotically monotone fashion.

n	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8
1	.5000	.5000	.5000	.5000	.5000	.5000	.5000
2	.6666	.5000	.5000	.5000	.5000	.5000	.5000
3	.6666	.5713	.5000	.5000	.5000	.5000	.5000
4	.6875	.5769	.5333	.5000	.5000	.5000	.5000
5	.6923	.5833	.5378	.5161	.5000	.5000	.5000
6	.6984	.5909	.5417	.5191	.5079	.5000	.5000
7	.7017	.5944	.5450	.5214	.5097	.5039	.5000
8	.7045	.5973	.5481	.5233	.5111	.5049	.5020
9	.7066	.5998	.5500	.5249	.5122	.5057	.5025
10	.7083	.6016	.5516	.5263	.5131	.5064	.5030
∞	.7236	.6184	.5663	.5379	.5218	.5125	.5071

TABLE 1. The proportion of 0's in the n bit Zeckendorf representation based on the k^{th} order Fibonacci numbers.

When we calculated a few values (see Table 1) [3], it appeared that convergence was *always* monotone and not just asymptotically monotone. We also calculated some of these ratios for other values of k, and still found that the convergence seemed to be always monotone.

Armed with the tools we've developed above, we can now prove this convergence. First, we use the fact that squared Fibonacci polynomials are multiply **SP** and the **SP** Theorem to obtain:

Theorem 5.2. *If N_n and W_n are two positive solutions to the k^{th} order double Fibonacci difference equation, and N_n/W_n is increasing for the first $2\,k + 2$ values of n, then N_n/W_n is always increasing.*

For example, if A_n, B_n, C_n, and D_n are sums of Fibonacci numbers, then

$$\frac{n\,A_n + B_n}{n\,C_n + D_n}$$

will be *always* increasing if this ratio is *initially* increasing.

Next, we recall that for the Zeckendorf Ratios $N_n\,/\,W_n$, N_n and W_n satisfy the double Fibonacci difference equation, i.e.

W_n = number of bits in n-bit Z-representation: $\quad W_n = n\,f_{n+2}$,

N_n = number of 0 bits in n-bit Z-representation: $\quad N_n = N_n + \ldots + N_{n-k} + f_{n+2}$.

Let

$$L\,[X_n] = X_n - X_{n-1} - \ldots - X_{n-k}$$

then

$$L^2\,[W_n] \;=\; L\left[\sum i\, f_{n+2-i}\right] \;=\; 0$$
$$L^2\,[N_n] \;=\; L\,[f_{n+2}] \;=\; 0.$$

Theorem 5.3. *The proportion of 0's in the n bit k^{th} order Zeckendorf representation is a strictly increasing function of n for*

$$\begin{cases} n \;\geq\; k-1, & \text{if } k \;\geq\; 3 \\ n \;\geq\; 3, & \text{if } \;k \;=\; 2\,. \end{cases}$$

We just need to show these ratios are *initially* increasing. The following relations make this calculation easy:

$$N_{-j} \;=\; W_{-j} \;=\; 0 \quad \text{for } j \in [\,2\,,\,k\,]\,, \text{ and}$$
$$2\,N_i \;=\; W_i \quad \text{for } i \in [\,0\,,\,k-1\,]\,.$$

6. Conclusion

Many results about Fibonacci numbers may be proved by checking that the result holds for a *few* initial values, because an associated difference equation leads to a simple inductive proof. Here we showed that this same strategy can be used for other properties by increasing the order of the associated difference equation. So, while a number of Fibonacci properties obey the Bellman's dictum: [5]

What I tell you 3 times is true,

other properties may require:

What I tell you K times is true

(with $K > 3$).

In particular, we introduced the idea of **SP** polynomials, and showed how to convert Fibonacci polynomials into **SP** polynomials. These **SP** polynomials enable us to prove monotone increasing ratios from initially increasing ratios. Specifically, we showed if F_n is any solution to a k^{th} order Fibonacci DE and c is a constant, then, if $F_n + c$ is positive, $(\,F_n \,+\, c\,)/(\,F_{n-1} \,+\, c\,)$ is *always* increasing if this ratio is *initially* increasing. Also we showed that

$$(\,n\,A_n \,+\, B_n\,)/(\,n\,C_n \,+\, D_n\,)$$

will be *always* increasing if this ratio is *initially* increasing, where $A_n,\; B_n,\; C_n,\; D_n$ are any solutions to a Fibonacci DE and the numerator and denominator are positive. We used these results to show that the fraction of 0's in the n bit k^{th} order Zeckendorf representation is always increasing (except for one situation in which the ratios for n bits and $n+1$ bits is the same).

For our Fibonacci example, we were able to convert the non-negative polynomials into **SP** polynomials by increasing the degree by 1 or 2, but for general non-negative polynomials the needed degree increase may be substantial, e.g. from degree d to degree d^2 (or even larger).

In summary, mis-quoting the Bellman (see the illustration) [5]:

What I tell you K times is true,

but K may be much larger than the defining k for a sequence. Luckily, for Fibonacci sequences K is only 1 or 2 more than the obvious k.

We agree with Richard Guy's [8] observation that we can be misled about inferring long term behavior of sequences from their initial behavior because *there are too few small numbers to go around*, but if we allow K to be unbounded then we **may** be able to predict eventual behavior based on the first K observations. But, we also showed, by example, that there are properties which do not obey such a K-in-a-row rule for any K.

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