PROOF AND GENERALIZATION OF THE CASSINI-CATALAN-TAGIURI-GOULD IDENTITIES

RUSSELL JAY HENDEL

ABSTRACT. The only published proof of the Gould identity which generalizes the Tagiuri, Catalan, and Cassini identities is based on exploration of general properties of a functional operator. In this paper we present a simply described method, the Tagiuri Generation Method (TGM), which can both generate and prove an infinite number of identities in an arbitrary number of parameters. In particular TGM generates and proves the Gould identity. This paper explores TGM and looks at one infinite family of identities generated by TGM. The identities that result from TGM are different from traditional Fibonacci identities in that indices of Fibonacci numbers occurring in these identities seem uniformly distributed. The paper makes this heuristic precise. Two open problems connected with TGM are also presented.

1. History and Overview

The identities

$$F_{n+1}F_{n-1} = F_n^2 + (-1)^n, \tag{1.1}$$

$$F_{n+a}F_{n-a} = F_n^2 + (-1)^{n+a+1}F_a^2, \tag{1.2}$$

$$F_{n+a}F_{n+b} = F_nF_{n+a+b} + (-1)^nF_aF_b, \tag{1.3}$$

and

$$F_{n+a}F_{n+b}F_{n+c} = F_nF_{n+a}F_{n+b+c} - F_nF_{n+b}F_{n+c+a} + F_nF_{n+c}F_{n+a+b} + (-1)^n\left(F_aF_bF_{n+c} + F_bF_cF_{n+a} - F_cF_aF_{n+b}\right) \tag{1.4}$$

are due to Cassini [3, p. 74], Catalan [3, p. 83], Tagiuri [3, p. 114], and Gould [2] respectively.

The proof of (1.4) presented in [2] is complicated, based on the general properties of the functional operator $Tf(x) = f(x+a)f(x+b) - f(x)f(x+a+b)$. In seeking a simpler proof of (1.4), the author discovered a general method, the Tagiuri Generation Method (TGM), for generating identities. The generated identities can have an arbitrary number of parameters. One does not need to separately prove each generated identity since one can prove a general result that TGM only produces true identities. The identities generated by TGM have some unusual properties. TGM allows the generation of infinite families of identities in several parameters. The goal of this paper is to present TGM, study the properties of one infinite family of identities generated by TGM, and to pose some open problems.

An outline of this paper is as follows. Section 2 illustrates a special case of TGM used to prove (1.4). Section 3 defines TGM in full generality. Section 4 shows that TGM only produces true identities, one main result of this paper. Section 5 presents examples of identities derived from TGM. Section 6 defines and studies index histograms of TGM identities. Index

Acknowledgment is given to the editor and several anonymous referees for spending extra time significantly improving the appearance and form of the paper.

histograms are very useful in studying properties of infinite families of identities generated by TGM. Section 7 notes some properties of TGM identities and how they differ from traditional identities. Section 8 studies one infinite family of identities generated by TGM. Section 9 concludes the paper and presents two open problems.

Before proceeding we clarify our use of the term parameters. Every identity must have at least one variable occurring in it, say n, since the identity asserts that a given equation is true for all n. By convention we say that such an identity has zero parameters. For example, (1.1) has zero parameters. Any additional variables are classified as parameters. Thus, (1.2)-(1.4) are identities with one, two, and three parameters respectively.

2. Proof of (1.4)

In this section we prove (1.4). The proof shows several main attributes of the definition of TGM which is presented in the following section.

TGM always starts with specification of p, the number of parameters in the desired identity and s, the coefficient of the negative summand. To prove (1.4), we let

$$p = 3, \qquad s = 1.$$

Using these values of p and s, we form the *start identity*

$$F_{n+a}F_{n+b}F_{n+c} = (1+s)F_{n+a}F_{n+b}F_{n+c} - sF_{n+a}F_{n+b}F_{n+c} = \\ F_{n+a}F_{n+b}F_{n+c} + 1 \times F_{n+a}F_{n+b}F_{n+c} - 1 \times F_{n+a}F_{n+b}F_{n+c}. \quad (2.1)$$

In (2.1) $p = 3$ is the number of parameters and $s = 1$ is the numerical coefficient of the negative summand.

Throughout this paper when dealing with a list of parameters, we may use two notations. We may refer to the parameters as $a_1, a_2, a_3, \dots$ or as $a, b, c, \dots$. Using this convention the second parameter can either be described as b or a_2.

The basic idea of TGM is to repeatedly use (1.3) to replace factors of the form $F_{n+x}F_{n+y}$, with $\{x, y\} \subset \{a, b, c\}$, in the three summands on the right-hand side of (2.1).

First, we use (1.3) to replace the factor containing the first and second parameters, $F_{n+a}F_{n+b}$, in the first summand on the right-hand side of (2.1). This yields

$$F_{n+a}F_{n+b}F_{n+c} = F_nF_{n+a+b}F_{n+c} + (-1)^nF_aF_bF_{n+c} + F_{n+a}F_{n+b}F_{n+c} - F_{n+a}F_{n+b}F_{n+c}. \quad (2.2)$$

Next, we apply (1.3) to the factor containing the second and third parameters, $F_{n+b}F_{n+c}$, in the second summand on the right-hand side of (2.1) (which is the third summand on the right-hand side of (2.2)). This yields

$$F_{n+a}F_{n+b}F_{n+c} = F_nF_{n+a+b}F_{n+c} + (-1)^nF_aF_bF_{n+c} + \\ F_{n+a}F_nF_{n+b+c} + (-1)^nF_{n+a}F_bF_c - F_{n+a}F_{n+b}F_{n+c}. \quad (2.3)$$

Finally, we apply (1.3) to the factor containing the first and third parameters, $F_{n+a}F_{n+c}$, in the third summand on the right-hand side of (2.1) (which is the fifth summand on the right-hand side of (2.3)). This yields

$$F_{n+a}F_{n+b}F_{n+c} = F_nF_{n+a+b}F_{n+c} + (-1)^nF_aF_bF_{n+c} + F_{n+a}F_nF_{n+b+c} + \\ (-1)^nF_{n+a}F_bF_c - F_nF_{n+b}F_{n+c+a} - (-1)^nF_aF_{n+b}F_c. \quad (2.4)$$

After an appropriate rearrangement, (2.4) is identical with (1.4). It follows that the above method has generated (1.4). It also follows that (1.4) is true since (2.1) is trivially true and all substitutions made are justified by (1.3).

The above derivation of (2.4) can be summarized with the notation

$$\langle p=3, s=1, (1,2), (2,3), (1,3)\rangle,$$

or more compactly,

$$\langle 3, 1, (1,2), (2,3), (1,3)\rangle. \tag{2.5}$$

We refer to (2.5) and (2.4) as the Tagiuri Generation Method, TGM, and the Tagiuri Generated Identity, TGI, respectively.

3. Definition of TGM

We need one more piece of notation and one more convention to completely define TGM. To motivate the ideas, consider the case $p=4, s=0$, yielding the start identity

$$F_{n+a}F_{n+b}F_{n+c}F_{n+d} = F_{n+a}F_{n+b}F_{n+c}F_{n+d}. \tag{3.1}$$

We would like to make two simultaneous substitutions of (1.3) to the right-hand side of (3.1), with one substitution applied to the factor $F_{n+a}F_{n+b}$ and the other substitution applied to the factor $F_{n+c}F_{n+d}$.

We use the notation (1,2;3,4) to indicate two simultaneous substitutions using (1.3). The parentheses indicate that the substitutions are made to one summand, and the semicolon separates pairs of indices i, j corresponding to the factor $F_{n+a_i}F_{n+a_j}$ to which (1.3) is applied to replace $F_{n+a_i}F_{n+a_j}$ with $F_nF_{n+a_i+a_j} + (-1)^nF_{a_i}F_{a_j}$.

For the rest of the paper, if $p = 2q$ is even, we make the following substitutions into any TGI studied:

$$a_1 = -q, a_2 = -(q-1), \dots, a_q = -1, a_{q+1} = 1, a_{q+2} = 2, \dots, a_{2q} = q. \tag{3.2}$$

We refer to the resulting identity as a Tagiuri Generated Identity With Substitution, TGIWS.

To clarify the concepts just introduced, we complete our analysis of (3.1).

Example 3.1. *We use the TGM*

$$\langle 4, 0, (1,2;3,4)\rangle.$$

The resulting TGI is

$$F_{n+a}F_{n+b}F_{n+c}F_{n+d} = F_n^2F_{n+a+b}F_{n+c+d} + F_aF_bF_cF_d + (-1)^nF_nF_{n+a+b}F_cF_d + (-1)^nF_nF_aF_bF_{n+c+d},$$

and the resulting TGIWS is

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} = F_{n-3}F_n^2F_{n+3} - 1 + (-1)^nF_n(F_{n-3} - F_{n+3}). \tag{3.3}$$

We summarize with the following definition.

Definition 3.2. *The Tagiuri Generation Method, TGM, indicated by*

$$\langle p, s, Q_1, Q_2, \dots, Q_{2s+1}\rangle,$$

where for each $k, 1 \le k \le 2s+1$, Q_k is of the form $(i_{k,1}, j_{k,1}; i_{k,2}, j_{k,2}; \dots; i_{k,m_k}, j_{k,m_k})$, for some m_k, with $1 \le m_k \le p/2$, and where for each $k, 1 \le k \le 2s+1$, the sets $(i_{k,q}, j_{k,q}), 1 \le q \le m_k$, are pairwise disjoint, refers to the process of using the start identity

$$\prod_{i=1}^{p} F_{n+a_i} = (1+s)\prod_{i=1}^{p} F_{n+a_i} - s\prod_{i=1}^{p} F_{n+a_i} \tag{3.4}$$

and then in the k-th summand of (3.4), $1 \le k \le 2s+1$, substituting (1.3) with $a = a_{i_{k,q}}, b = a_{j_{k,q}}$, for $1 \le q \le m_k$, with the convention that the k-th summand of (3.4) is $\prod_{i=1}^{p} F_{n+a_i}$ for $1 \le k \le s+1$, and $-\prod_{i=1}^{p} F_{n+a_i}$ for $s+2 \le k \le 2s+1$.

The resulting identity is called the Tagiuri Generated Identity, TGI.

If a substitution (such as (3.2)*) is made into the TGI, we call the resulting identity the Tagiuri Generated Identity With Substitution, TGIWS.*

Example 3.3. *Equations* (2.5) *and* (2.4) *illustrate a TGM and TGI respectively. Example 3.1 illustrates a TGM, TGI, and TGIWS.*

In this paper we use the acronym TGM in two senses. TGM can refer to the general Tagiuri Generation Method. When preceded by an article (e.g. a TGM), it refers to an application of Definition 3.2 with specific parameters. Similar comments apply to the acronyms TGI and TGIWS. Also, throughout the paper these acronyms will be used to refer both to single identities and families of identities.

4. Main Theorems

Theorem 4.1. *Every TGI and every TGIWS is true.*

Proof. Equation (3.4) is trivially true. Each substitution made is an application of (1.3) which is true. Further substitutions, such as (3.2), result in further true identities. □

It might seem natural to generalize TGM with use of other substitutional identities. For example, can there be a Gould Generating Method in which (1.4) is applied to triples of indices occurring in summands of the start identity (3.4)?

However, this would not increase the number of identities generated. Indeed, (1.4) itself is derived from a TGM as shown in Section 2. So, using (1.4) for substitution would not increase the number of derived identities. We summarize with the following result.

Theorem 4.2. *The set of identities generated by* (1.3) *from a start identity is not increased if substitutions from other TGI are used.*

5. Examples

This section presents further examples of the concepts introduced in Definition 3.2.

Example 5.1. *We consider the TGM*

$$\langle 4, 1, (1, 2), (2, 3), (3, 4)\rangle.$$

The resulting TGIWS is

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} = F_{n-3}F_nF_{n+1}F_{n+2} + F_{n-2}F_n^2F_{n+2} - F_{n-2}F_{n-1}F_nF_{n+3} - (-1)^n\left(F_{n+1}F_{n+2} + F_{n-2}F_{n-1} - F_{n-2}F_{n+2}\right). \quad (5.1)$$

Example 5.2. *We consider the TGM*

$$\langle 6, 0, (1, 2; 3, 4; 5, 6)\rangle.$$

The resulting TGIWS is

$$F_{n-3}F_{n-2}F_{n-1}F_{n+1}F_{n+2}F_{n+3} = F_{n-5}F_n^4F_{n+5} + 2F_{n-5}F_n - 4F_n^2 - 2F_nF_{n+5} + (-1)^n\left(F_{n-5}F_n^2F_{n+5} + 2F_{n-5}F_n^3 - 2F_n^3F_{n+5} - 4\right). \quad (5.2)$$

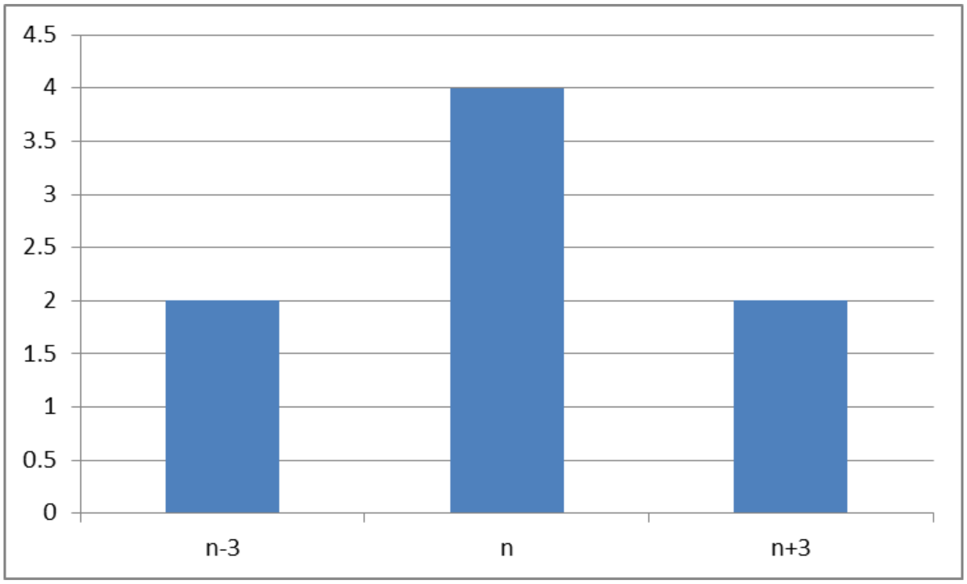

FIGURE 1. Index histogram for (3.3).

6. INDEX HISTOGRAMS

Identities (3.3) and (5.1) - (5.2) seem different from many traditional Fibonacci identities. To formalize this difference and to more generally study the properties of these identities, we introduce in this section the concept of an index histogram (or more loosely, an index distribution).

Definition 6.1. *The index histogram of a TGIWS is a histogram of the indices occurring in the right-hand side of the identity. In counting indices, we ignore numerical coefficients of summands and factors of* $(-1)^n$ *but we count powers with multiplicity, assume parenthetical expressions expanded, and assume all* F_x *numerically evaluated if* x *is independent of* n*.*

Using this definition Figures 1-3 present the index histograms of (3.3) and (5.1) - (5.2) respectively. Figure 4 presents the index histogram of $\langle 12, 0, (1,2;3,4;5,6;7,8;9,10;11,12)\rangle$; however, for reasons of space, we omit the actual TGIWS.

7. CHARACTERISTICS OF TGIWS

If we contrast (3.3) and (5.1) - (5.2) with traditional Fibonacci identities, noticeable differences emerge.

A traditional Fibonacci identity, for example, $F_{2n} = F_n L_n$, is short, compact, punchy, and unexpected. The elegance lies in the identity content.

Contrastively, (3.3) and (5.1) - (5.2) have a cumbersome appearance. Figures 1-4 show that the distribution of indices in these and similar identities seems almost uniform. This is precisely formulated in Section 8. The elegance of these identities lies in their generation and proof. Each of (3.3) and (5.1) - (5.2) follows from a one-line TGM followed by substitution of (3.2).

Thus, the Fibonacci identities presented in this paper are in a certain sense new and have different characteristics.

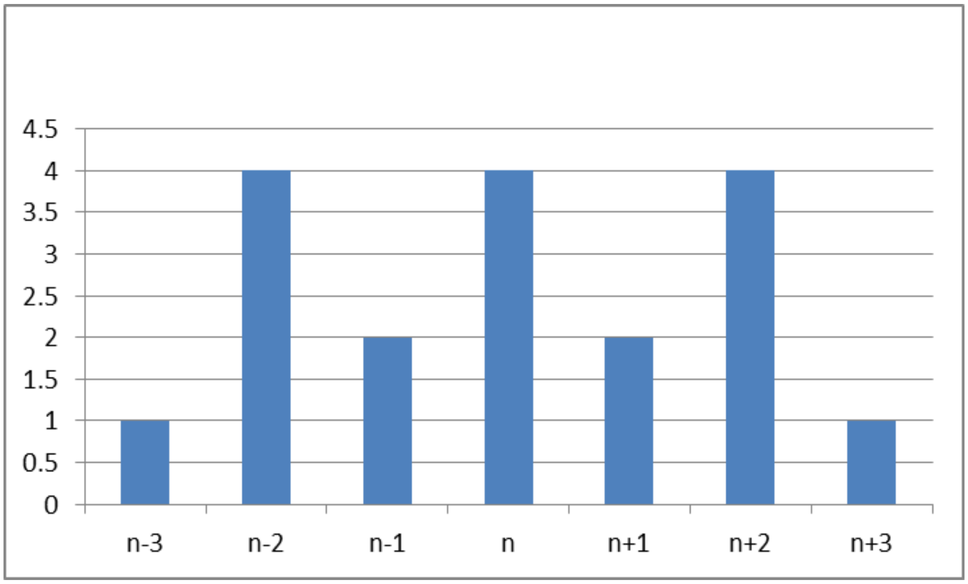

Figure 2. Index histogram for (5.1).

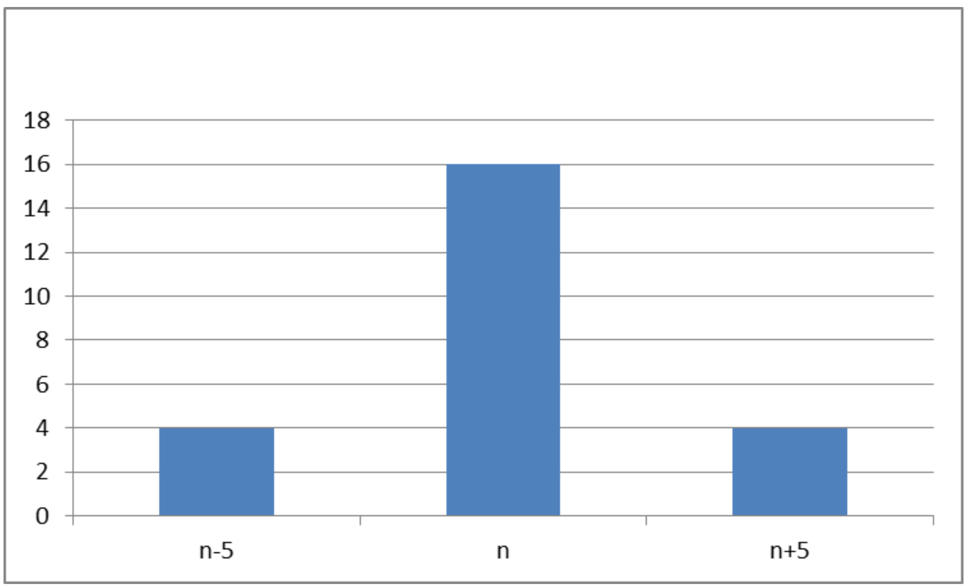

Figure 3. Index histogram for (5.2).

8. An Infinite Family of TGIWS

Definition 3.2 can be used to generate an infinite family of TGIWS.

One such infinite family consists of the TGIWS arising from

$$\langle 2q, 0, (1, 2; 3, 4; \ldots ; 2q-1, 2q)\rangle, \tag{8.1}$$

where q ranges over the positive integers.

Characteristics of this family are presented in Table 1.

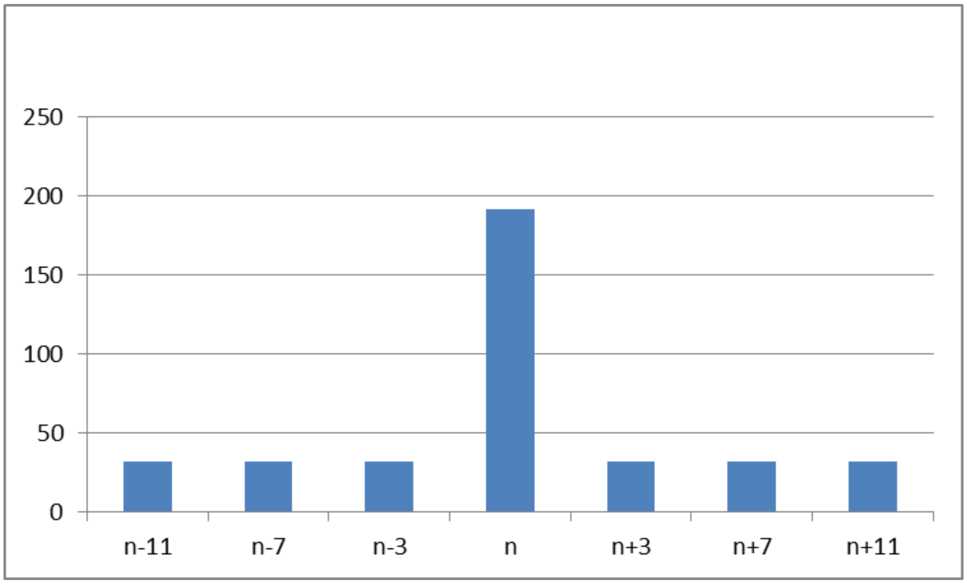

FIGURE 4. Index histogram for $\langle 12, 0, (1, 2; 3, 4; 5, 6; 7, 8; 9, 10; 11, 12)\rangle$.

$p = 2q$	q	Indices	# Indices	Weight n	Weight non-n	Total
2	1	n	1	2		2
4	2	$n, n \pm 3$	3	4	2	8
6	3	$n, n \pm 5$	3	16	4	24
8	4	$n, n \pm 7, n \pm 3$	5	32	8	64
10	5	$n, n \pm 9, n \pm 5$	5	96	16	160
12	6	$n, n \pm 11, n \pm 7, n \pm 3$	7	192	32	384
$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$
2q	q even	$n, n \pm (2q-1), n \pm (2q-5), \dots, n \pm 3$	$q+1$	$q2^{q-1}$	2^{q-1}	$q2^q$
2q	$q > 1$, odd	$n, n \pm (2q-1), n \pm (2q-5), \dots, n \pm 5$	q	$(q+1)2^{q-1}$	2^{q-1}	$q2^q$

TABLE 1. Statistics on the index histograms of the TGIWS derived from (8.1) for several values of q. The column header, '# Indices' refers to the number of distinct indices appearing in the identity. The column header, 'Weight n' refers to the number of occurrences of F_n in the identity where the count of occurrences follows Definition 6.1. Similarly, 'Weight non-n' refers to the number of occurrences of F_x for any $x \neq n$. 'Total' refers to the total number of F_x (for all x) occurring in the identity.

To illustrate the table interpretation, we review the entries in the row $p = 6$. Substituting (3.2) in the TGI arising from (8.1) yields the TGIWS (5.2) whose index histogram is presented in Figure 3. The indices occurring in (5.2) are $\{n, n-5, n+5\}$. Equation (5.2) has 16 occurrences of F_n and four occurrences each of F_{n-5} and F_{n+5}. Consequently, there is a total of 24 indices occurring in (5.2).

The empirically observed formulas in Table 1, for $q = 1$ to 6, are the basis for the next theorem.

Theorem 8.1. *Fix positive integer $q > 1$. Consider the TGIWS arising from (8.1). If q is even, then*
(a) The following $q+1$ indices are present in the TGIWS: $\{n-(2q-1), n-(2q-5), \dots, n-3, n, n+3, n+7, \dots, n+(2q-1)\}$;
(b) There are $q2^{q-1}$ occurrences of F_n;
(c) For each $F_x, x \neq n$, occurring in the identity, there are 2^{q-1} occurrences of F_x.
If q is odd, then
(a') The following q indices are present in the TGIWS: $\{n-(2q-1), n-(2q-5), \dots, n-5, n, n+5, n+9, \dots, n+(2q-1)\}$;
(b') There are $(q+1)2^{q-1}$ occurrences of F_n;
(c') For each $F_x, x \neq n$, occurring in the identity, there are 2^{q-1} occurrences of F_x.

Proof. Substituting(3.2) into the TGI obtained from (8.1) results in

$$\begin{aligned} F_{n-q}F_{n-(q-1)}\dots F_{n-1}F_{n+1}\dots F_{n+q} = &\left(F_nF_{n-(2q-1)} + (-1)^nF_{-q}F_{-(q-1)}\right)\times \\ &\left(F_nF_{n-(2q-5)} + (-1)^nF_{-(q-2)}F_{-(q-3)}\right)\times \\ &\ddots \\ &\left(F_nF_{n+(2q-1)} + (-1)^nF_{q-1}F_q\right). \end{aligned} \tag{8.2}$$

There are two cases to consider according to the parity of q. We first assume q is even.

Proof of (a). Using the counting conventions of Definition 6.1, we see that the set of indices occurring on the right-hand side of (8.2) is $\{n-(2q-1), n-(2q-5), \dots, n-3, n, n+3, n+7, \dots, n+(2q-1)\}$. This completes the proof of (a).

Proof of (b). The expansion of (8.2) requires adding products over all paths through the binomials on the right-hand side of (8.2). By Definition 6.1, in determining the weights (i.e. number) of indices in the resulting expansion of the right-hand side of (8.2), only the left-hand summand of each binomial matters.

A straightforward counting argument then shows that the total number of occurrences of F_n in the expansion of the right-hand side of (8.2) is

$$\binom{q}{q}\times q + \binom{q}{q-1}\times(q-1) + \binom{q}{q-2}\times(q-2)\dots = \sum_{i=0}^{q}\binom{q}{i}i = q2^{q-1}.$$

This can be justified with the following combinatorial argument: In expanding (8.2), for each $i, 0 \leq i \leq q$, we obtain a factor of F_n^i if we choose exactly i of the left-hand summands in the q binomials and $q-i$ of the right-hand summands in the remaining $q-i$ binomials. This completes the proof of (b).

Proof of (c). If $x \neq n$ and F_x occurs in the expansion of the right-hand side of (8.2), then there are 2^{q-1} occurrences of F_x in the expansion of the right-hand side of (8.2). This can be justified with the following combinatorial argument: For F_x to occur as a factor in the expansion of (8.2), we must select the left-hand summand of the (unique) binomial with F_x. We are then free to choose the left or right-hand summands in the remaining $q-1$ binomials. Clearly, there are 2^{q-1} ways to choose arbitrarily from the left and right-hand summands in

$q-1$ binomials. This completes the proof of (c) and completes the proof for the case when q is even.

If q is odd, then the proofs for the cases (a') and (c') are similar to the proofs of (a) and (c) and are omitted.

Proof of (b'). The arguments used in the q is even case for contributions of occurrences of F_n also apply to the odd case. However, when q is odd, there are additional contributions to the occurrences of F_n. To see this note that since q is odd, the $\frac{q+1}{2}$-th binomial in (8.2) is $F_n^2 + (-1)^n$, which makes a contribution of two F_n, while the other binomials only make a contribution of one F_n.

Thus, we obtain one extra factor of F_n that does not occur in the q is even case. These extra factors arise from selecting the left-hand summand of the (unique) binomial with F_n^2 and freely choosing the left or right-hand side of the remaining binomials. Clearly, there are 2^{q-1} ways of doing this. If we add this 2^{q-1} to the $q2^{q-1}$ occurrences contributed by the arguments for the q is even case, we obtain $(q+1)2^{q-1}$ occurrences as required.

This completes the proof of (b') and the theorem. □

As can be seen from Table 1 or Theorem 8.1, the numbers of occurrences of $F_x, x \neq n$, is either 0 or a constant independent of n.

Proposition 8.2. *Fix positive integer $q > 1$. Let S be the support of the index histogram of the TGIWS derived from* (8.1)*. Then the distribution of indices restricted to $S - \{n\}$ is discrete uniform.*

This proposition concretizes our heuristic observation that indices in a family of TGIWS tend to be almost uniformly distributed. Figure 4 illustrates this nicely.

We mention briefly an alternative method of generating an infinite family of identities. However, this method will not be further explored in this paper.

Fairgrieve and Gould [1] introduced the idea of creating symmetric forms of identities by adding all possible 'translates' of them. More precisely, if we add the $n \geq 3$ TGI obtained from $\langle n, 0, (1,2)\rangle, \langle n, 0, (2,3)\rangle, \langle n, 0, (3,4)\rangle, \ldots, \langle n, 0, (n-1, n)\rangle$, and $\langle n, 0, (n,1)\rangle$, we obtain the following infinite family of identities, where indices are taken modulo n so that $F_{x+j} = F_y$, $L_{x+j} = L_y$, and $a_{x+j} = a_y$, where $x + j \equiv y \pmod n$ and $y \in \{1, 2, \ldots, n\}$.

$$n \prod_{i=1}^{n} F_{n+a_i} = \sum_{j=0}^{n-1} \prod_{i=1}^{n-2} F_{n+a_{i+j}} F_n F_{n+a_{n-1+j}+a_{n+j}} + (-1)^n \left(\sum_{j=0}^{n-1} \prod_{i=1}^{n-2} F_{n+a_{i+j}} F_{a_{n-1+j}} F_{a_{n+j}} \right).$$

Historically, this identity was presented at the biennial, international, Fibonacci conference held in Caen, France in July 2016. The author initially attempted to prove this by a general Binet formula method. However, upon examination, a simpler proof using Definition 3.2 was found.

9. Conclusion

This paper has explored a specific method of generating identities, the Tagiuri Generation Method. The paper introduced definitions, notations, and index histograms of these identities. The paper also studied an infinite family of identities.

We close the paper with two open problems.

(i) Given an arbitrary identity, is there a way to recognize this identity as arising from a TGM with some type of substitution such as (3.2)?

Such a method, if found, would allow quick one or two-line proofs of some cumbersome identities.

(ii) Proposition 8.2 concretizes the heuristic that the distribution of indices in the TGIWS arising from (8.1) using (3.2) is almost uniform. Can this be generalized to other TGM and other infinite families of TGIWS?

References

[1] S. Fairgrieve and H. W. Gould, *Product Difference Fibonacci Identities of Simson, Gelin-Cesàro, Tagiuri and Generalizations,* The Fibonacci Quarterly, **43** (2005), 137–141.

[2] H. W. Gould, *The Functional Operator* $Tf(x) = f(x+a)f(x+b) - f(x)f(x+a+b)$, Mathematics Magazine, **37** (1964), 38–46.

[3] T. Koshy, *Fibonacci and Lucas Numbers with Applications,* John Wiley and Sons, New York, 2001.

MSC2010: 11B39

Department of Mathematics, Towson University, Baltimore, Md. 21252, USA
E-mail address: rhendel@towson.edu