FIBONACCI WORDS AND THE CONSTRUCTION OF A "QUASICRYSTALLINE" FIVEFOLD STRUCTURE

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ABSTRACT. Since their discovery in 1982 by Dan Shechtman, quasicrystals are still somewhat mysterious. The atoms they are made of arrange themselves on long distances in a fivefold symmetry that is forbidden by the rules of crystallography. The Penrose tilings are one of the rare examples presenting such a symmetry. It is therefore natural to assume that atoms in quasicrystals should be organized in a similar way. However, the rules of placement of the basic bricks composing these tilings are not sufficient to solve the mystery since after a certain extension, they offer several placement possibilities of these bricks thus provoking a decoherence of the system. As a consequence, there is a need for a global, fundamental, geometric logic to explain the formation of quasicrystals; this is precisely what this article proposes through a method of construction in several steps based on the intrinsic particularities of the infinite Fibonacci word.

INTRODUCTION

The importance of symmetries in all fields of science no longer needs to be demonstrated. Since antiquity, they have been studied, classified, sorted out in categories (examples in [13]) and have allowed the understanding of many observable phenomena, in astrophysics as well as biology, crystallography, and particle physics. The formation of the whole universe seems to be orchestrated by symmetry requirements. Periodicity is a quality that allows some symmetries to repeat themselves infinitely often in space. When atomic arrangements organize themselves in a periodical way in materials, they generally produce crystals with particular physical, chemical, and electronic characteristics. The only groups of symmetries having this quality in a two-dimensional plane space are 2-, 3-, 4- or 6-fold symmetries and they produce seventeen different types of paving [8]. We know no periodical fivefold paving [1] and therefore no crystal shape presenting such a symmetry. However, in 1982, Daniel Shechtman, who was working on new aluminum alloys, observed with astonishment a diffraction diagram revealing a fivefold atomic structure in one of the materials he had just created [5, 7]. Soon the relation was made with a discovery of the physicist Roger Penrose who, ten years earlier, had described a method to tile the plane allowing a general fivefold symmetry that was regular but non-periodic.

Penrose tilings [10] and Shechtman's crystals [2] were then called "quasiperiodic" [11] to mean "almost periodic." Fivefold symmetries and geometric structures based on these symmetries are always connected with the golden ratio [12] as well as the Fibonacci sequence and the infinite Fibonacci word [3]. However, one enigma remains since the rules for placing the Penrose tiles only permit the paving of the entire plane in a fivefold symmetry, but are not sufficient to explain the formation of quasicrystals. After a certain development, these rules allow several possible placements for these bricks, always provoking a decoherence of the system and leading to an impasse. The placement rules only have a local scope and do not respond to a global geometric logic [4, 6], hence their inability to solve this puzzle. The geometric logic capable of explaining the organization of atoms in quasicrystals and the way they grow from a basic germ is still lacking.

My research on the Fibonacci sequence and the infinite Fibonacci word allowed me to discover a repetitive mirror effect composing another, superior, infinite word, composed of three letters b, c and d; see [9]. (In fact, the reader who wants to understand my point of view on the construction presented herein will be greatly helped by consulting the webpage [9].) This infinite word superposes itself perfectly on the infinite Fibonacci word with a recurrence Φ^3 (where $\Phi = (1+\sqrt{5})/2$) when S and L, which compose the Fibonacci word, respectively have the values Φ and Φ^2. This recurrence of the mirror effect allowed me, by taking the successive scales down, to determine an origin O as the center of symmetry, situated at infinity but absolutely localizable. This centering allows the fivefold symmetry to spread and to find equilibrium on the whole plane.

Since a good drawing is worth a thousand words, I invite you to follow this method and build a logical quasicrystalline structure in perfect adequacy with the infinite Fibonacci word that is nothing else but the result of a development process: the Fibonacci sequence. In theory, this method requires no complex mathematical calculations, only the use of a ruler and a pair of compasses, thus respecting the tradition of ancient Greek geometry. However, computer software offers a precision and makes drawing so much more comfortable that we are not going to deprive ourselves of them.

Step 1: Figure 1

From a basic circle, we trace two perpendicular axes (x; y) passing through the circle's center Z. We determine the point B, midpoint of the radius [ZA]. From the point C we draw a circular arc of center B which meets the x axis in D. We obtain with this simple figure, three segment lengths ED, DZ and ZA. These three segments are in the proportion of $\Phi = (1+\sqrt{5})/2$, that is, of the golden ratio:

$$ED \cdot \Phi = DZ; \; DZ \cdot \Phi = ZA.$$

This implies that if $ED = 1$, then $DZ = \Phi$ and $ZA = \Phi^2$. Or if $ED = \Phi$, then $DZ = \Phi^2$ and $ZA = \Phi^3$.

It should be recalled [9] that when the two letters S and L that compose the infinite Fibonacci word represent two different spacings of parallel straight lines having the respective values Φ and Φ^2, there is a superior infinite word composed of the letters b, d and c covering the infinite Fibonacci word with recurring scales of order Φ^3 and revealing a mirror effect in every module [b], [d] and [c].

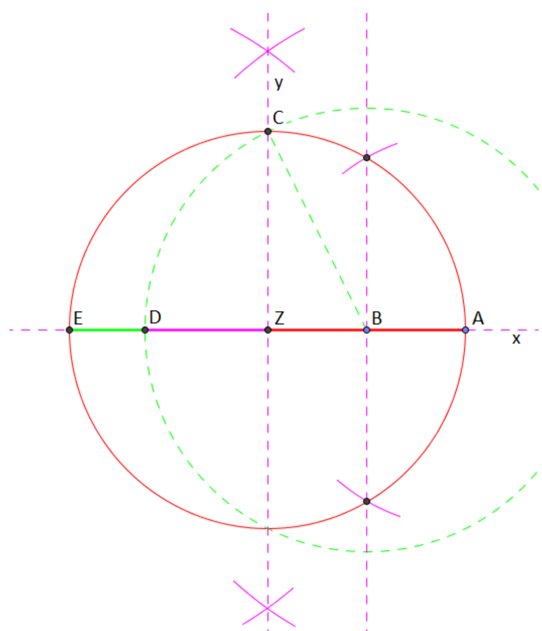

Figure 1. Golden ratio on the x axis.

Step 2: Figures 2, 3, and 4

Now, if we trace, from the original circle, a new circle (green) of center D and radius DZ, we obtain a new point F on the x axis. Then we draw a circle with center E (yellow) and radius ED to obtain the point G on the x axis. We trace a circle with center F (red) and radius FE and obtain H on the x axis. This circle will be tangent to the original circle and of proportion $1/\Phi^3$; we can therefore repeat the operation infinitely often. In the other direction, we trace a new circle with center A (yellow) and radius AD, then repeat the same operation with the new circle.

We notice that the intersections of the successive circles determine points that, once linked together with straight lines, materialize two new axes forming a 72° angle and intersecting at the origin O. We therefore obtain a homothetic transformation with center O and factor $K = \Phi$. This 72° angle leads us to consider a fivefold symmetry.

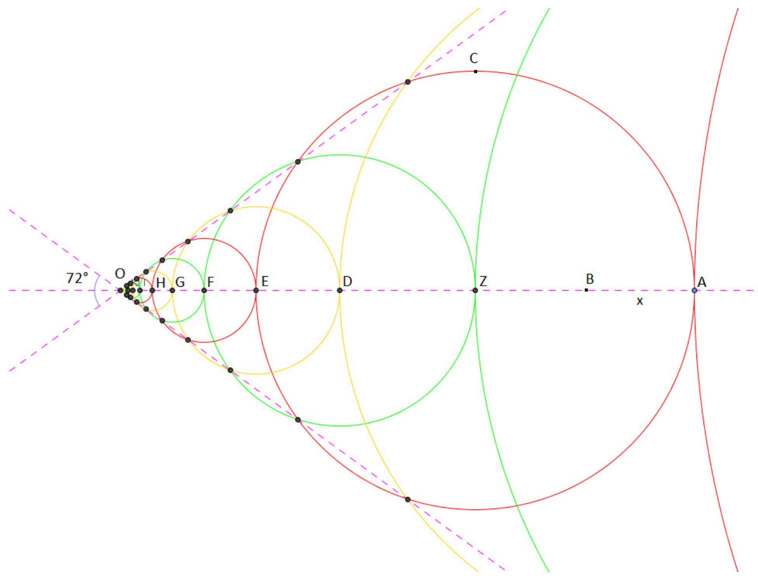

Figure 2. The golden ratio can be spread to infinity on the x axis.

In such a system, the origin O can be found at $-\infty$ whereas the successive circles grow towards $+\infty$; the interesting thing here is that we can find the exact location of the origin O and its distance from any circle C_n.

If we consider a circle C_n as having diameter 1, the following circle C_{n+1} will have a diameter Φ. If now we draw the circle centered at O tangent to C_{n+1}, it will be identical to a circle of diameter 1. If we draw the straight line (D) tangent to both the circle centered at O and C_n this line will be parallel to the x axis (Figure 4). This implies that the distance d between any circle C_n and the origin O can be calculated as follows: $d = C_{n-1}/2 = (C_n/\Phi)/2$. (See also Figure 3.)

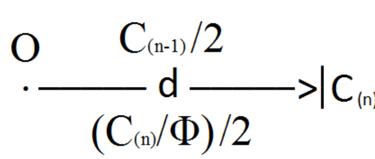

Figure 3. Computing the distance d between the circle C_n and the origin O

For example, the ‘unit’ circle of diameter 1 will be at a distance $(1/\Phi)/2$ from the origin O, or approximately at 0.30901699. The circle C_1 of diameter Φ will be at a distance $\frac{1}{2}$ from origin O.

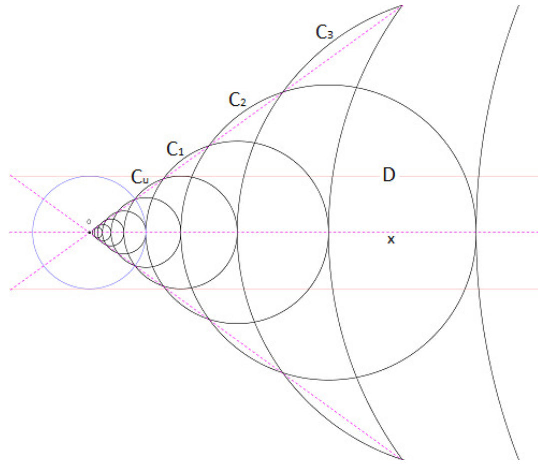

Figure 4. A circle centered at the origin O and tangential to any circle C_n of the system will be in the golden ratio therewith.

Step 3: Figures 5, 6, 7, 8

Now all we need to do is use the points determined by the intersections of the consecutive circles as centers and draw new series of circles in the same way we did on the x axis. We obtain two new symmetry axes and we understand from the 36° and 72° angles that we will have ten series of circles on 360°. We can also draw a series of circles centered at O and draw tangents to the previous circles.

We observe on Figure 6 that all the circles of the system (not centered at the origin O) contain the same pattern formed by the surrounding circles, and that all the circles centered at O and tangent to the other circles will contain the same system of circles continuing infinitely many times. The blue straight lines on Figures 6 and 10 are the "hesitant straight lines" described further.

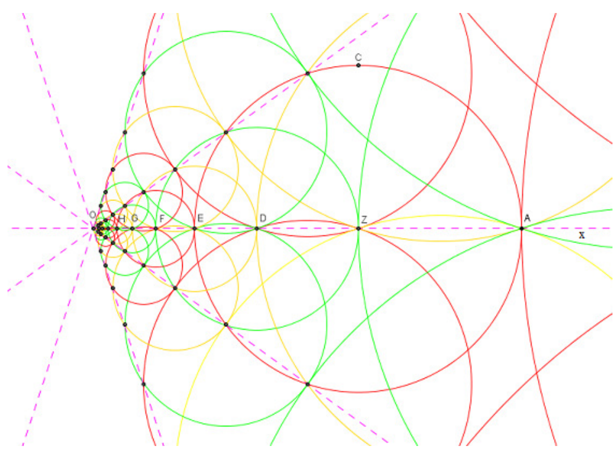

Figure 5. The intersections of successive circles form two new axes at 36° with respect to x and are the centers of circles which propagate along these axes.

The monochrome Figure 7 reveals a five or tenfold central symmetry with a recurrence Φ, narrowing towards $-\infty$ and expanding towards $+\infty$. If we want to superpose a quantic system on it, like the Fibonacci sequence with its quanta S and L, logically it will not originate at O since it is located at infinity and is, by definition, unattainable. As a consequence, S, the first quantum and

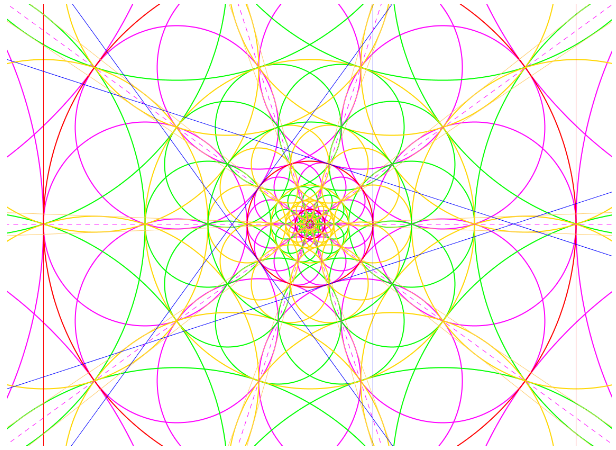

Figure 6. The ten circles systems over 360°.

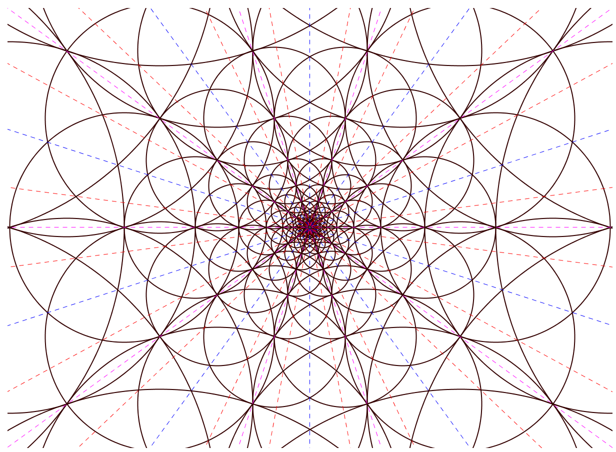

Figure 7. The ten circles monochrome systems and their symmetry axes.

the first Fibonacci word, with which we associate a gap-value of Φ, will occupy any circle C_1 of this system and will be located at a distance $\frac{1}{2}$ from the origin O, as shown in Figure 2.

The first quantum S ($=\Phi = C_1$) (the first Fibonacci word) is at a distance $\frac{1}{2}$ of O; it then disappears from the infinite Fibonacci word. The circle tangent to C_1 in ascending order will be C_4, Φ^3 bigger than C_1; its diameter will then have a value Φ^4 and will contain the first three quanta of the infinite Fibonacci word, that is to say LSL. The next tangent circles, C_7, C_{10}, C_{13}, C_∞ will contain the rest of the infinite Fibonacci word.

In Figure 8, we can notice that a double wave is propagating along each symmetry axis and that half of its length increases by a factor of Φ each time it crosses the axis. This double wave, formed by the arcs of circles that go through the symmetry axes, is propagating in a beam angle of about 4.345415°.

At this point one could wonder about the link with quasicrystals and the infinite Fibonacci word.

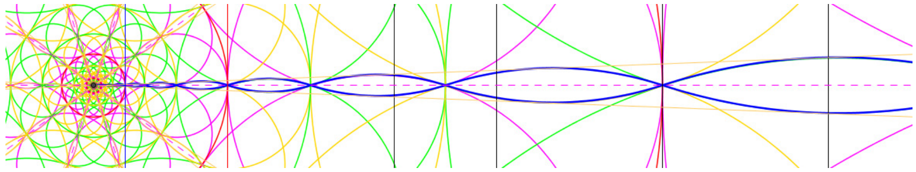

Figure 8. Double wave propagating along the ten axes of symmetry.

Step 4: Figures 9, 10, 11

Now if we draw a network of parallel lines, perpendicular to the symmetry axis x, with successive spacings that obey the substitution rules of the Fibonacci sequence (the infinite Fibonacci word); the two spacings S and L having respectively the values Φ and Φ^2; taking the tangent as the starting point of any circle of the structure we have constructed; this tangent (red line) being perpendicular to the symmetry axis passing through the center of this circle; and making the circle contain the first three spacings LSL of the infinite Fibonacci word (Figure 9); then the spacings distribution will obey the infinite Fibonacci word.

When repeating the operation symmetrically on the other side of the symmetry center O, we notice that each circle centered on the x axis contains some of the S and L spacings of the "mirror" Fibonacci word on both sides of its own center. We can verify this property. The limit is reached when the circles become too small to contain the quanta of S and L spacings of the system, hence a central singularity that I call "black hole" in my research.

Of course, we could have attributed the value 1 to the diameter of any circle (not centered at the origin O); the following circles would have had the respective diameters: Φ; Φ^2; Φ^3; Φ^4; Φ^5... We would then have used the diameters Φ and Φ^2 to determine the spacings of our network of straight lines. We would have obtained the same result since the following circle, tangent to the circle whose diameter is Φ, i.e., S, would have had diameter Φ^4, and would therefore have contained LSL.

In these conditions, the central spacing containing the center of symmetry O between the first two red lines has value Φ^3. The center of origin O is therefore at $\Phi^3/2$, which corroborates the results of my research on supersymmetry. This central spacing has to contain only one straight line dividing it into two spacings L and S, or S and L, so as to not generate gaps in the continuity of the plane. This straight line passing optimally at 0.5 from O can be found on either side of it, we therefore observe a spontaneous symmetry-breaking phenomenon; this straight line is called a "hesitant straight line."

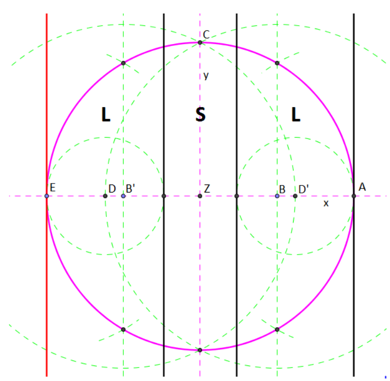

Figure 9. Insert gaps [L, S, L] in a circle.

Thanks to the three different systems of successive tangent circles (red, yellow, and green) centered on the x axis, we will be able to check the recurrence of the three mirror effects in all the circles of

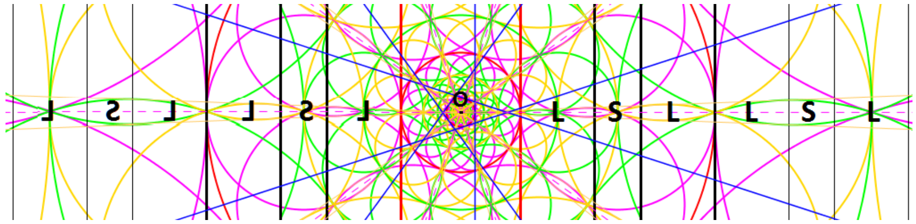

Figure 10. Distribution spacings L and S of the infinite Fibonacci word in the two opposite directions from the origin O.

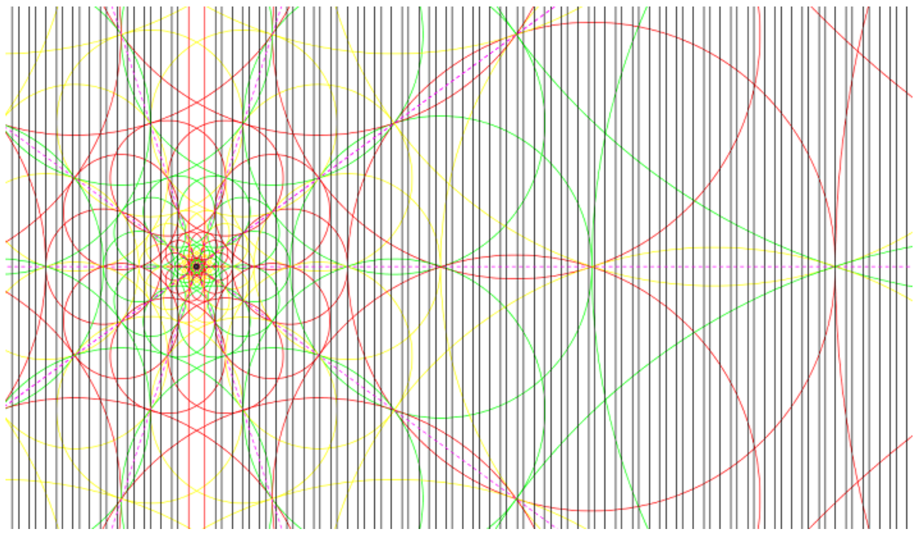

Figure 11. Distribution gaps L and S Fibonacci infinite word on the x axis.

the same color. The red circles will always be centered at a spacing S and their circumferences will always pass between two spacings L of the infinite Fibonacci word. The yellow circles will always be centered at a spacing L and their circumferences will always pass in the middle of a spacing L of the infinite Fibonacci word. The green circles will always be centered at a spacing L and their circumferences will always pass in the middle of a spacing S of the infinite Fibonacci word. The distribution of the spacings S and L inside these circles will always be the mirror image of their centers.

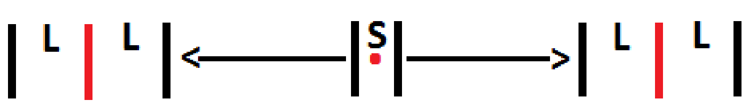

Figure 12. Distribution of the spacings (odd Fibonacci numbers) in the red circles

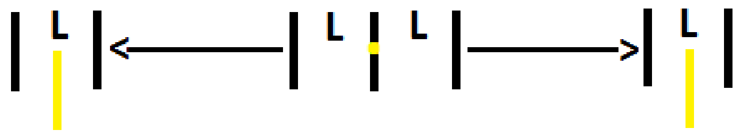

Figure 13. Distribution of the spacings (odd Fibonacci numbers) in the yellow circles

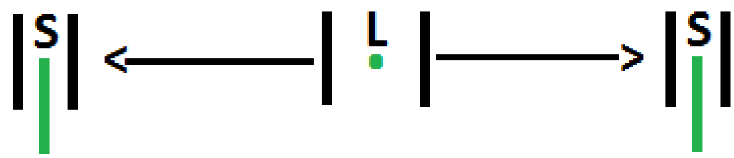

Figure 14. Distribution of the spacings (odd Fibonacci numbers) in the green circles

Diameter of the circle	Fibonacci numbers Number of spacings in the circle
C_1 Φ	r 1
C_2 Φ^2	y 1
C_3 Φ^3	g 2
C_4 Φ^4	r 3
C_5 Φ^5	y 5
C_6 Φ^6	g 8
C_7 Φ^7	r 13
C_8 Φ^8	y 21
C_9 Φ^9	g 34
C_{10} Φ^{10}	r 55
C_{11} Φ^{11}	y 89
C_{12} Φ^{12}	g 144
C_{13} Φ^{13}	r 233
C_{14} Φ^{14}	y 377
C_{15} Φ^{15}	g 610
C_{16} Φ^{16}	r 987
C_{17} Φ^{17}	y 1597
C_{18} Φ^{18}	g 2584
Etc.	Etc.

Logic of the mirror distribution of the spacings S and L in circles of the same color:

When S gives LSL, and L gives LSLSL, given an amplification factor of order Φ^3:

	Mirror distribution of the spacings in the red circles	Diameter of the red circles
1	S	Φ
3	LSL	Φ^4
13	LSLSL LSL LSLSL	Φ^7
55	LSLSL LSL LSLSL LSL LSLSL LSLSL LSL LSLSL LSLSL LSL LSLSL LSL LSLSL	Φ^{10}
	Etc...	Etc...

	Mirror distribution of the spacings in the yellow circles	Diameter of the yellow circles
1	$\frac{1}{2}$L $\frac{1}{2}$L	Φ^2
5	$\frac{1}{2}$L SL LS $\frac{1}{2}$L	Φ^5
21	$\frac{1}{2}$L SL LSL LSLSL LSLSL LSL LS $\frac{1}{2}$L	Φ^8
	Etc...	Etc...

	Mirror distribution of the spacings in the green circles	Diameter of the green circles
2	$\frac{1}{2}$S L $\frac{1}{2}$S	Φ^3
8	$\frac{1}{2}$S L LSLSL L $\frac{1}{2}$S	Φ^6
34	$\frac{1}{2}$S L LSLSL LSLSL LSL LSLSL LSL LSLSL LSLSL $\frac{1}{2}$S	Φ^9
	Etc...	Etc...

Thanks to this logic of propagation and distribution of the spacings in each of the successive tangent circles, and the entanglement of the three systems (red, yellow and green), we understand that the mirror effect will be preserved at infinity in all circles centered on the x axis. The spacings (S and L) contained in all the circles of the same color amplified by a factor Φ^3 and put end to end recreate the infinite Fibonacci word. This logic works in the division of any circle in order to push back the quanta S and L towards $-\infty$ as well as in their multiplication towards $+\infty$.

Step 5: Figures 15 and 16

Eventually, what we need to do is to reproduce this network of straight lines in the same way on each of the symmetry axes, which amounts to making it revolve four times around the origin O with a 36° angle. We finally obtain a “pentagrid” that is perfectly balanced and regular over the whole plane, and five hesitant straight lines. The network of circles, the mirror effects intrinsic to the infinite Fibonacci word and the network of hesitant straight lines are, symmetrically speaking, in perfect harmony. This “pentagrid” possesses only four different segment lengths delimited by the intersections of its straight lines.

When S=Φ and L=Φ^2, these four lengths are (with $\lambda = \sqrt{\Phi^2+1}$) :

- i ˜0.324919696... $= (1/\lambda)\ /\Phi = \tan 18°$
- h ˜0.5257311118... $= 1/\lambda$
- f ˜0.8506508079... $= \Phi/\lambda$
- d ˜1.37638192... $= \Phi^2/\lambda = \tan 54°$

We are now done with drawings, even though the bravest of my readers may still be expanding this network of straight lines that I named “Alpha Network” in my research.

However, the consequences of the symmetry-breaking phenomenon near the origin and on the rest of the plane still has to be explained. But I have no other choice than to invite you to consult my website [9], since it is not possible to explain everything in this article. What matters here is that we managed, thanks to this construction and development logic, to maintain a fivefold symmetry and the coherence of four different lengths over the whole plane, therefore obtaining a structure similar to the atomic arrangement in quasicrystals. Figure 15 shows the final result. You can verify on your construction the stability of the four lengths (i, h, f, d) over a long distance.

After rotating the parallel straight lines network, each circle of the system illustrated in figure 4 will contain a mirror image of the spacings (S and L) from its center and the axis of symmetry passing through it. The perfect pentagonal and decagonal symmetry, as well as the stability of the eight surfaces and four different segment lengths that characterize the “Alpha Network,” are maintained on the whole plane.

The “Alpha Network” is a perfectly regular pentagrid, formed by eight different surfaces and four irrational segment lengths defined by the intersection of the straight lines, except in the central

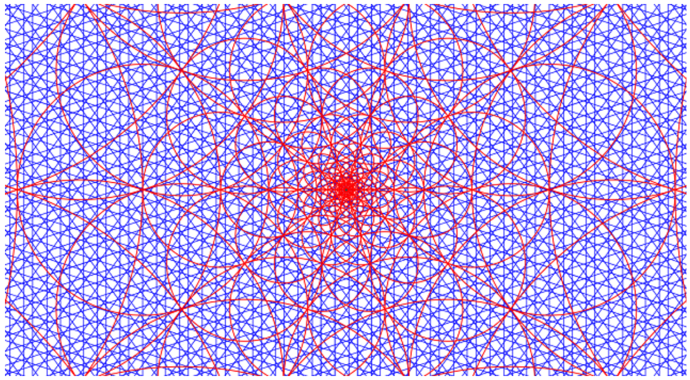

Figure 15. Final result of the five stages of construction.

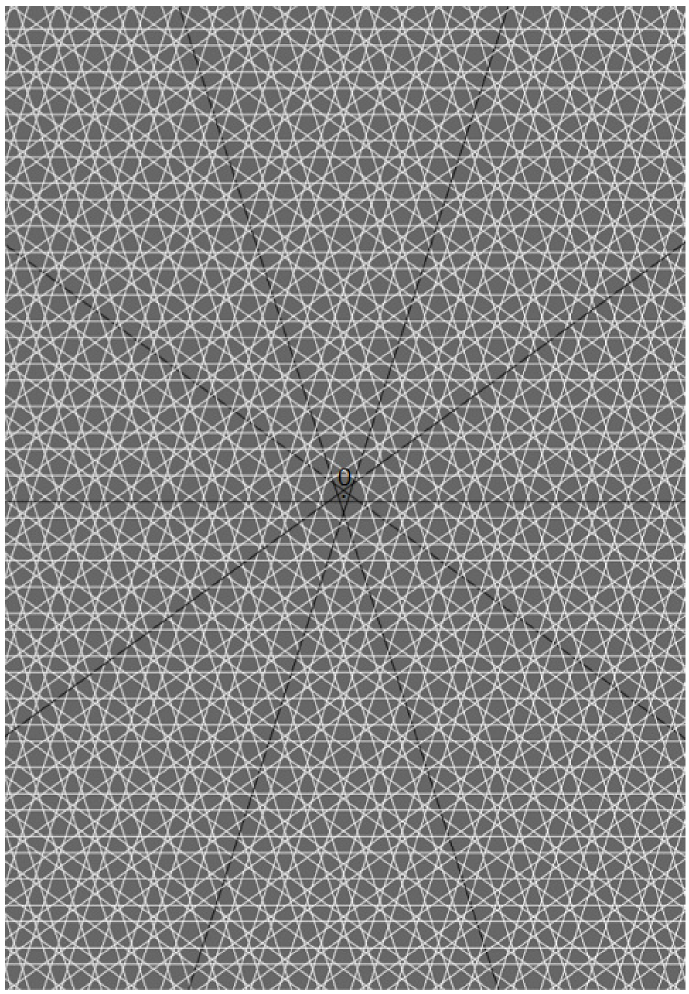

Figure 16. Alpha Network.

singularity, that I named "black hole", and that contains the five "hesitant" straight lines and the first Fibonacci word. This grid balances itself on the whole plane by propagating a fivefold symmetry and keeps its characteristics as long as its expansion in all its directions follows exactly the infinite Fibonacci word.

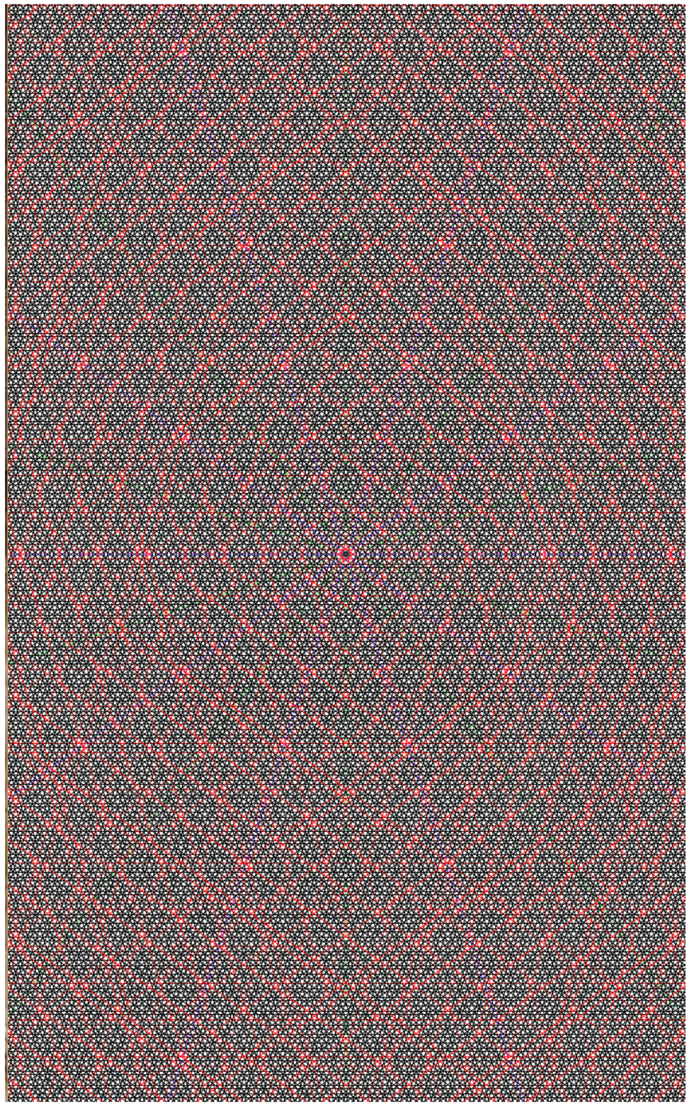

FIGURE 17. Extended "Alpha Network"

Without going into too much detail, I would like to point out that, depending on the respective places of the five hesitant straight lines in their original Φ^3 spacing on 180°, there are four possible states for the Alpha Network (Figure 18):

- State 1: [L,S] at 0°; [S,L] at 36°; [L,S] at 72°; [S,L] at 108°; [L,S] at 144°.
- State 2: [S,L] at 0°; [L,S] at 36°; [L,S] at 72°; [L,S] at 108°; [S,L] at 144°.
- State 3: [L,S] at 0°; [L,S] at 36°; [L,S] at 72°; [L,S] at 108°; [L,S] at 144°.
- State 4: [L,S] at 0°; [L,S] at 36°; [S,L] at 72°; [L,S] at 108°; [L,S] at 144°.

Any other possibility would amount to one of those above. The first three of these states require a central singularity. This decagonal singularity, that I call "black hole" in my research, is different because it can contain other lengths than those already cited (states 1, 2, and 3 have six more lengths in the black hole). State 4 restores stability in the central zone that contains only the four lengths (i, h, f, d); the black hole then disappears since it cannot be seen anymore and merges into the "Alpha Network." The four different states generated by the potential movement of the hesitant straight lines do not change anything to the visual aspect of the whole network; only the central singularity undergoes a major upheaval.

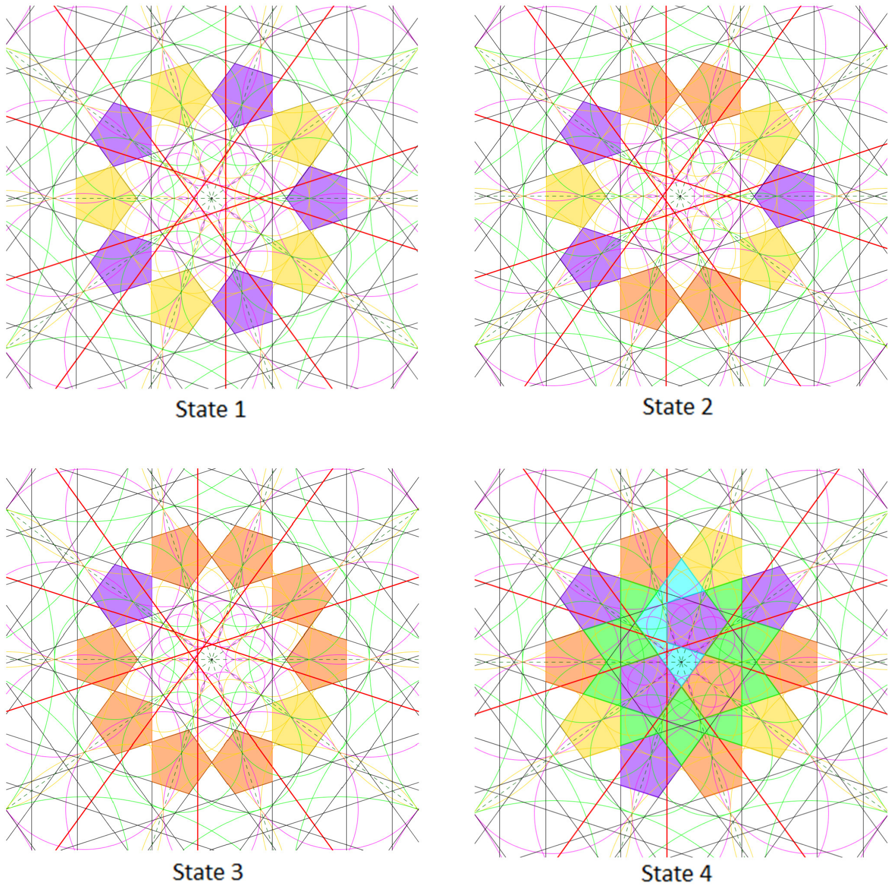

FIGURE 18. The four different states of the Alpha network according to the respective placement of five straight hesitant.

Figures 19 and 20 display a picture of the Alpha Network altered by the filters of a graphics software. Readers who are familiar with the subject will immediately recognize in these patterns a quasiperiodic order and a quasicrystalline logic.

Figure 21 is a picture, touched up out of concern for aesthetics, of the Alpha Network and the network of circles that superposes itself on it.

The Alpha Network can be paved or reconstituted by juxtaposing three basic bricks, including a regular pentagon possessing three different states depending on whether it is crossed by no straight line, one straight line or two straight lines of the subjacent network. I call these bricks the "bricks of creation." They are represented and described in Figure 22. All sides of the polygons that form these bricks are equal; their value is the length d of the Alpha Network, except for the two longer sides of the cup, which are twice longer.

We can also tile the "Alpha Network" by linking together the centers of all the regular pentagons with polygons; we will obtain a second set of bricks (Figure 24) and a "secondary" tiling (Figures 25 and 26).

The secondary bricks can be further simplified in a set of two bricks (Figures 27 and 28), but they will no longer contain individually the information of the subjacent Alpha network (regarding

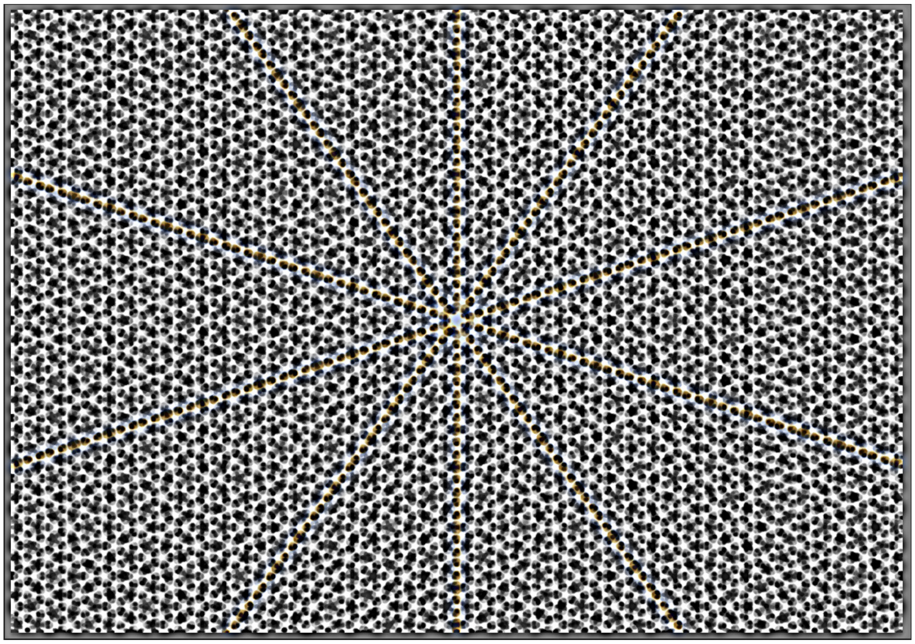

FIGURE 19. (State 1, with central singularity)

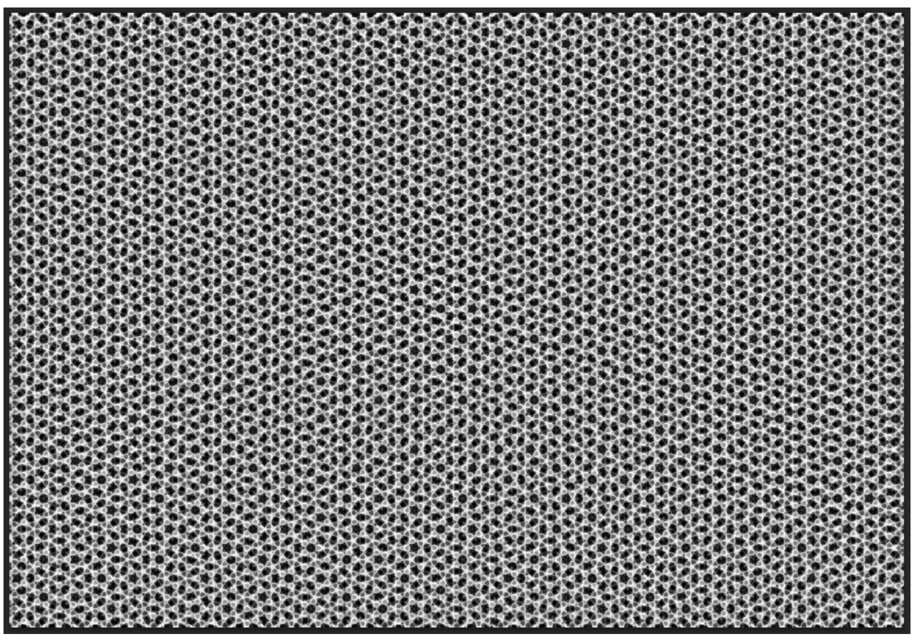

FIGURE 20. (State 4, without singularity)

the segments passing through them) and, with these two bricks only, it will be as difficult to tile the plane in a fivefold symmetry as with the Penrose tiles, of which they are components (Figures 29 and 30).

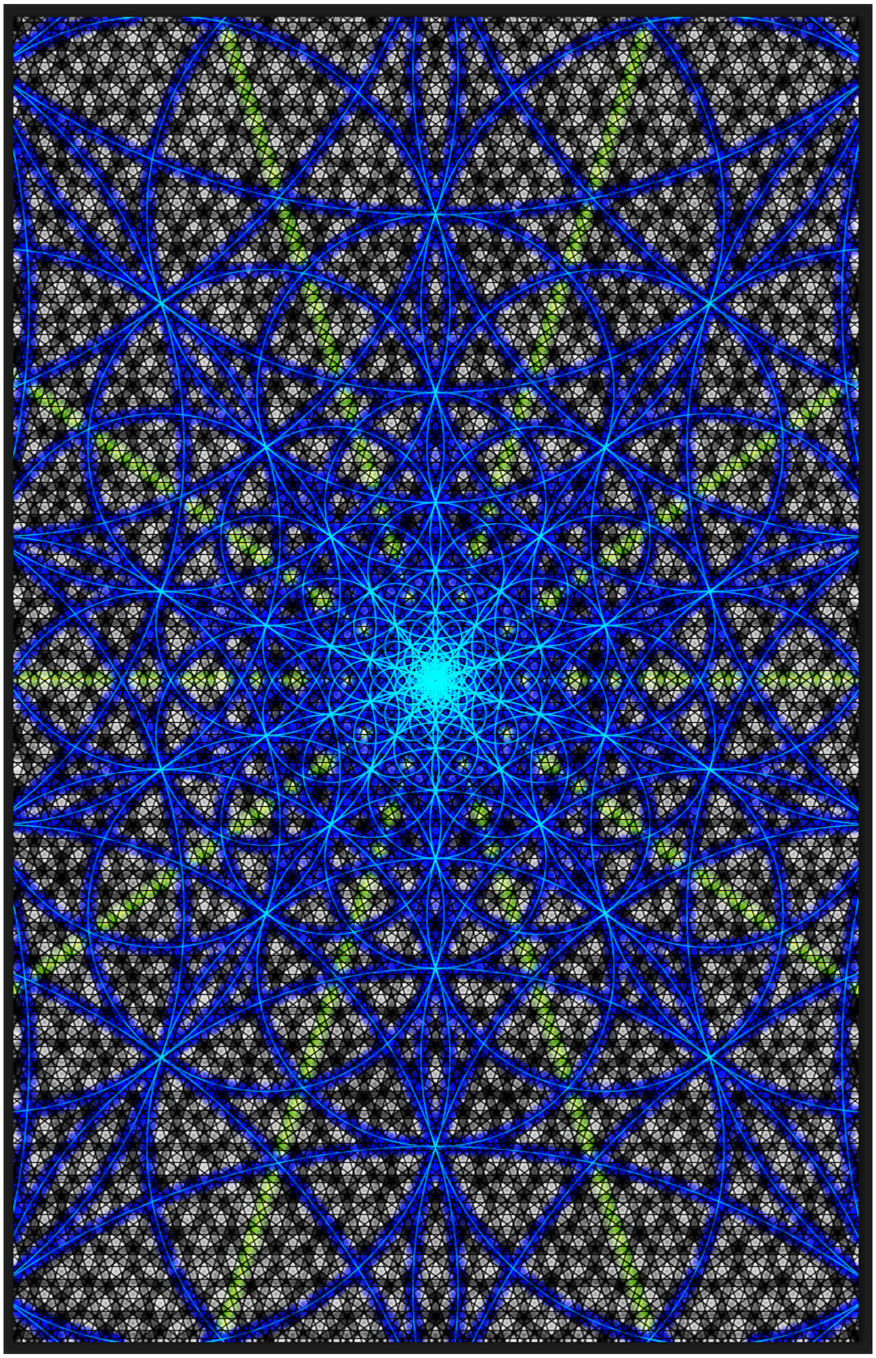

FIGURE 21. Alpha network and the network of circles are harmoniously linked.

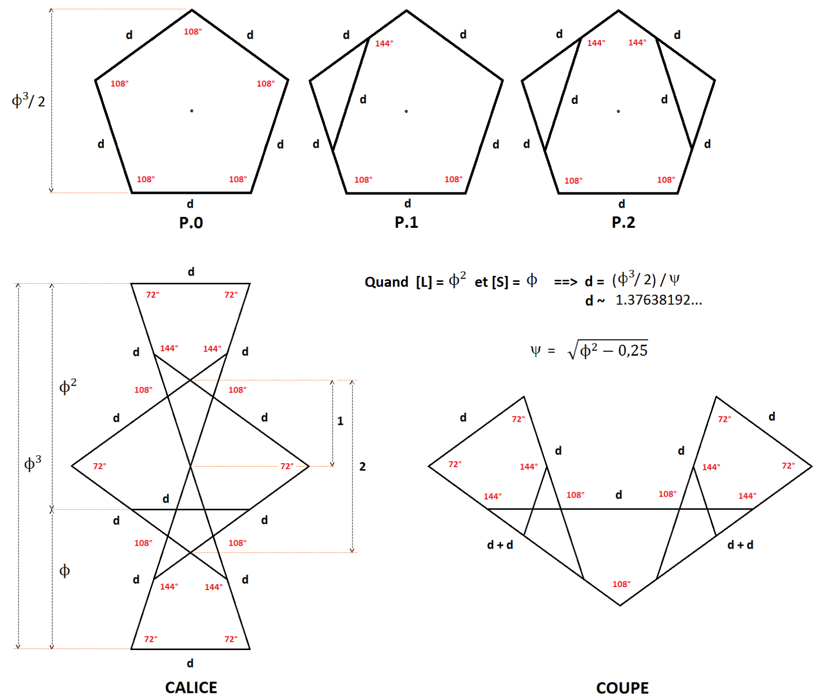

FIGURE 22

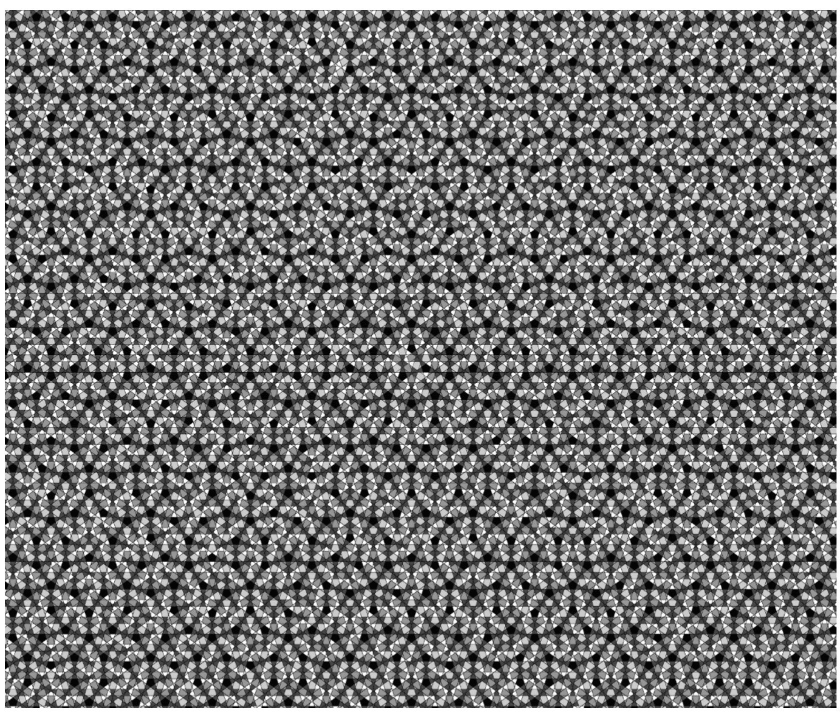

FIGURE 23. Tiling of the plane with the bricks of creation

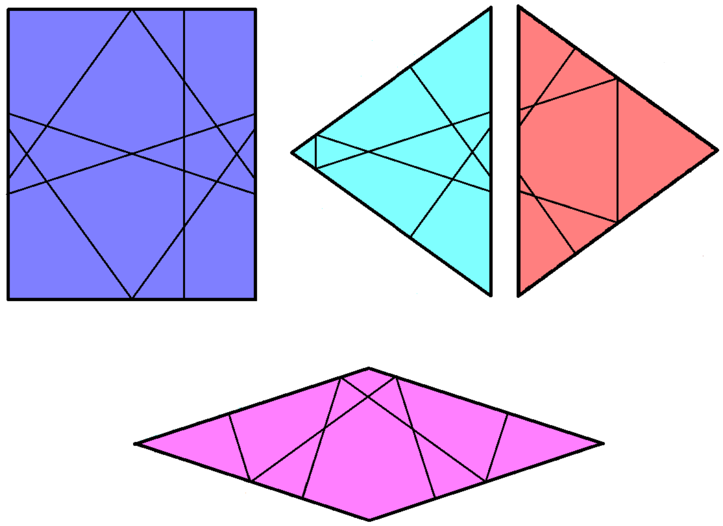

FIGURE 24. Bricks of the secondary tiling

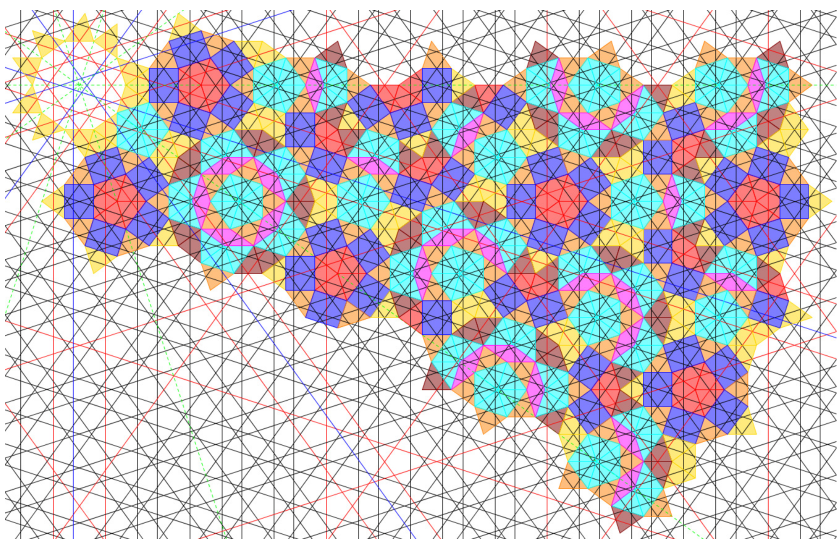

FIGURE 25. Arrangement of secondary bricks on the Alpha network

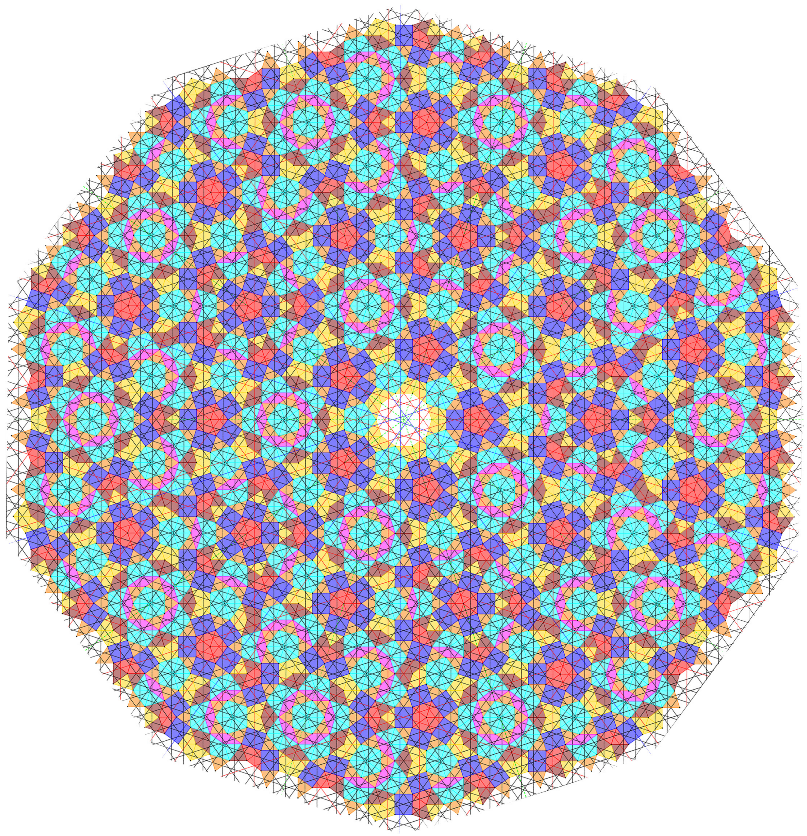

Figure 26. Secondary tiling

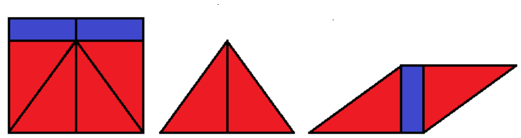

Figure 27. The bricks of the secondary tiling can be decomposed into two basic bricks

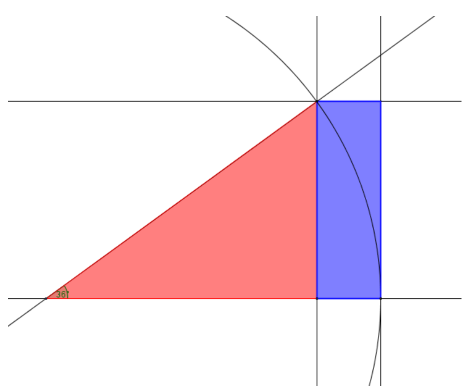

Figure 28. Construction of the bricks of the simplified secondary tiling

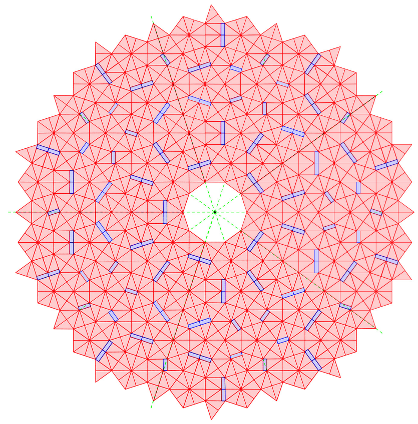

Figure 29. Simplified secondary tiling

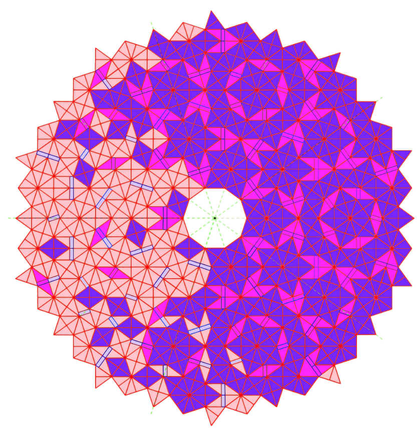

Figure 30. Simplified secondary tiling and Penrose bricks

Conclusion

I have demonstrated, with the construction of this structure, that the infinite Fibonacci word is in fact a partition of mirror effects and symmetries; that the homothetic recurrence of this partition based on the golden ratio and its infinite character determines the place of an origin O and reveals a spontaneous symmetry-breaking phenomenon; that the combination of these elements on 360° implies a fivefold symmetry that expands over the whole plane. As a consequence, the Alpha Network explains and solves, thanks to its geometric and symmetrical particularities, the problem of atomic arrangement in quasicrystals and how they grow from one germ.

I also showed that the origin O and its precise location, inferred from this geometric demonstration, is the true starting point of the Fibonacci sequence and the infinite Fibonacci word, what leads one to think that the spontaneous symmetry-breaking phenomenon could be the source of energy and the dynamic cause of this propagation and development system. The fact that this result can be obtained without much mathematics gives it a primitive and instinctive character. Many models reveal the Fibonacci sequence in processes used by Mother Nature (phyllotaxy). We therefore realize that nature is able to adapt itself to the Fibonacci recurrence, which is a great source of fascination.

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