PROBLEM PROPOSALS

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These fourteen problems were posed by participants of the Seventeenth International Conference on Fibonacci Numbers and Their Applications, Université de Caen Normandie, Caen, France, Tuesday, June 28 and Friday, July 1, 2016. A few revisions, solutions and partial solutions, received during or after the sessions, are included.

Problem 1, posed by Curtis Cooper: Find a linear recurrence relation
Let a, b, c and $d \neq 0$ be integers, and let

$$M = \begin{pmatrix} a & b & c & d \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $\{y_n\}_{n=0}^{\infty}$ be the sequence of coefficients of $-x^2$ in the polynomial

$$\det(xI - M^n).$$

The first few terms of the sequence $\{y_n\}$ are

TABLE 1. Values of y_n

n	y_n
0	-6
1	b
2	$2ac + 2d - b^2$
3	$3a^2d + b^3 + 3bd - 3abc - 3c^2$
4	$-4a^2bd - 2a^2c^2 + 4ab^2c - 8acd - b^4 - 4b^2d + 4bc^2 - 6d^2$
5	$-5a^3cd + 5a^2b^2d + 5a^2bc^2 - 5a^2d^2 - 5ab^3c + 5abcd+$
	$5ac^3 + b^5 + 5b^3d - 5b^2c^2 + 5bd^2 + 5c^2d$

Find a linear recurrence relation for $\{y_n\}$, if one exists.

Solution by *Steven Miller, Williams College, Williamstown, MA; Peter J. C. Moses, Redditch, Worcestershire, England; Murat Sahin, Ankara University, Ankara, Turkey; Thotsaporn Thanatipanonda, Mahidol University International College, Nakornpathom, Thailand (independently).* By [1], there is a recurrence relation, of order $\binom{4}{2} = 6$. Using a computer algebra system, the recurrence is

$$y_n = -by_{n-1} - (d + ac)y_{n-2} + (c^2 - 2bd - a^2d)y_{n-3}$$
$$+ d(d + ac)y_{n-4} - bd^2y_{n-5} + d^3y_{n-6} \text{ for } n \geq 6.$$

Reference:
[1] P. T. Young, *On lacunary recurrences*, The Fibonacci Quarterly, 41.1 (2003), 41–47.

Problem 2, posed by Charlie Cook: 6×6 Fibonacci-like magic square

In [1], Brown showed that for $n \geq 2$ there do not exist any $n \times n$ magic squares with distinct entries chosen from a set of Fibonacci numbers. In [2], Freitag discovered a 4×4 magic square and an algorithm for constructing an infinite family of such magic squares, $[F_a]_4$, having magic constant F_{a+8} :

$$[F_a]_4 = \begin{pmatrix} F_{a+2} & F_{a+6} & F_{a+1} + F_{a+6} & F_{a+4} \\ F_{a+3} + F_{a+6} & F_{a+3} & F_{a+1} + F_{a+5} & F_a + F_{a+4} \\ F_{a+2} + F_{a+5} & F_a + F_{a+6} & F_{a+5} & 2F_{a+1} \\ F_{a+1} + F_{a+4} & F_{a+1} + F_{a+3} & F_a + F_{a+2} & F_{a+7} \end{pmatrix}.$$

In [3] we found

$$[F_a]_5 = \begin{pmatrix} F_a & F_{a+1} & F_{a+3} & F_{a+5} & F_{a+7} \\ F_{a+3} & F_{a+5} & F_{a+7} & F_a & F_{a+1} \\ F_{a+7} & F_a & F_{a+1} & F_{a+3} & F_{a+5} \\ F_{a+1} & F_{a+3} & F_{a+5} & F_{a+7} & F_a \\ F_{a+5} & F_{a+7} & F_a & F_{a+1} & F_{a+3} \end{pmatrix} +$$

$$\begin{pmatrix} F_{a+9} & F_{a+13} & F_{a+17} & F_{a+11} & F_{a+15} \\ F_{a+11} & F_{a+15} & F_{a+9} & F_{a+13} & F_{a+17} \\ F_{a+13} & F_{a+17} & F_{a+11} & F_{a+15} & F_{a+9} \\ F_{a+15} & F_{a+9} & F_{a+13} & F_{a+17} & F_{a+11} \\ F_{a+17} & F_{a+11} & F_{a+15} & F_{a+9} & F_{a+13} \end{pmatrix}$$

with magic number F_{a+18}. But we were unable to find a 6×6 Fibonacci-like magic square with just one or two Fibonacci numbers representing each matrix entry. We did find a 6×6 with one, two, three and even four Fibonacci numbers in the matrix entries but that was of no value since any numbers could be entered and a Zeckendorf representation could be found. Can such a matrix be found with a sum of at most two Fibonacci numbers representing each entry?

References:

[1] J. L. Brown, Jr., *Reply to Exploring Fibonacci magic squares*, The Fibonacci Quarterly, 3.2 (1965), 146.

[2] H. T. Freitag, *A magic square involving Fibonacci numbers*, The Fibonacci Quarterly, 6.1 (1968), 77-80.

[3] M. R. Bacon, C. K. Cook and R. J. Hendel, *Extending Freitag's Fibonacci-like magic square to other dimensions*, The Fibonacci Quarterly, 50.2 (2012), 119-128.

Problem 3, posed by Andreas M. Hinz: History of the Chinese rings

What are the origins of the Chinese rings? This historical problem proposal stems from the following original sources:

- Stewart Culin *Korean games*, 1895: Hung Ming (Zhuge Liang, 181–234)
- Luca Pacioli *De viribus quantitatis*, ∼1500
- Hans Ehemann *Nürnberger Zankeisen*, 1546?
- Gerolamo Cardano *De subtilitate*, 1550
- John Wallis *De Algebra Tractatus*, 1693
- Yoriyuki Arima *Shūki Sanpō*, 1769
- G. C. L. (= Georg Christoph Lichtenberg) *Ueber das Spiel mit den künstlich verflochtenen Ringen*, 1769
- Zhu Xiang Zhuren *Little Wisdom*, ∼1821

- Louis Gros *Théorie du Baguenodier*, 1872 (Édouard Lucas, 1880/2)

Other names for the Chinese rings include the following:
Korean: *yu gaek ju* (delay guest instrument);
English: *tiring irons, Cardano's rings*; Italian: *anelli di Cardano*;
French: *baguenaudier, vétille* (trifle), *jeu d'esclave* (slave's game);
Russian: *meleda*; Swedish: *Sinclairs bojor* (Sinclair's shackles);
Finnish: *Vangin lukko* (Prisoner's lock), *Siperian lukko* (Siberian lock).

Problem 4, posed by Clark Kimberling: The 3-gap theorem and Fibonacci numbers

Let r and h be real numbers and $n > 0$. Write the fractional parts

$$\{r+h\}, \{2r+h\}, \ldots, \{nr+h\}$$

in increasing order as

$$a_1 < a_2 < \cdots < a_n$$

and let

$$D_n = \{a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1}\}.$$

By the famous **3-gap theorem**, $|D_n| \leq 3$. For $r = (1+\sqrt{5})/2$, and arbitrary h, find the set $S(h)$ of n such that $|D_n| = 2$. Examples:

h	$S(h)$
$2/r$	$F = \{1, 2, 3, 5, 8, \ldots\}$
$3/r$	$F \cup \{4\}$
$5/r$	$F \cup \{4, 7\}$
$8/r$	$F \cup \{4, 7, 12\}$
0	$F \cup (F - 1)$
π	$F \cup \{6\}$

Problem 5, posed by Andreas M. Hinz: Jeux Scientifiques par Édouard Lucas (1842-1891)

Can you locate any of the missing items in the following list of scientific games of Édouard Lucas:

First series (1889) 1. La Fasioulette, 2. La Pipopipette, 3. La Tour d'Hanoï, 4. L'Icosagonal ou le Jeu des Vingt Forts, 5. L'Arithmétique diabolique ou le Calcul infernal, 6. Les Pavés Florentins du Père Sébastien. **Item 6 is missing!**

Second series (advertised) Les Tablettes Siamoises, Le Pauvre Ermite du Moulin de la Galette, Le Désespoire d'un Épicier, La Ballade des Pendus, La Colère du Charcutier, L'Arithmétique abracadabrante. **All are missing!**

Théorie des nombres, tome premier (1891). **Volume two is missing!**

Problem 6, posed by Steven J. Miller: Asymptotic behavior of variance of generalized Zeckendorf decompositions

We begin with the Fibonacci numbers indexed thus: $F_1 = 1, F_2 = 2$, and $F_{n+1} = F_n + F_{n-1}$. Then every positive integer can be written uniquely as a sum of non-adjacent terms. This is known as the Zeckendorff decomposition, and can be generalized to many other recurrence relations (with various restrictions). Interestingly, the Fibonacci sequence can be characterized as the unique sequence that has such a decomposition. Regarding decomposition of integers in $[F_n F_{n+1})$, let X, as a random variable, be the number of summands in a decompostion. Lekkerkerker showed mean(X) grows linearly with n. Others showed that variance(X) also grows linearly with n and that the limiting distribution of X is Gaussian. While the linear growth of the mean is easy to prove, the variance to date has defied elementary analysis. We can easily show, through generating functions and partial fractions, that variance(X) is $Cn + d + o(1)$ for some constants C and d, but currently the only proofs that $C > 0$ are somewhat involved. Can it be proved that variance(X) grows at least as fast as $\log\log\log n$ (or some other slow-growing function of n)?

Problem 7, posed by Dale Gerdemann: Sums of shallow diagonals of the fibonomial triangle

Prove for $a(n) = A181926(n)$ (sums of shallow diagonals of the fibonomial triangle),

$$\lim_{n\to\infty} \frac{\frac{a(n)}{a(n-1)}}{\frac{a(n-4)}{a(n-5)}} = \frac{1+\sqrt{5}}{2}.$$

Problem 8, posed by Dale Gerdemann: Palindromic sums

Let $a(n) = 3a(n-1) - a(n-2)$, as in A001906, bisection of Fibonacci sequence, and consider the following identities:

$$\begin{aligned}
5a(n) &= a(n+1) + 2a(n) + a(n-1),\ n \geq 1\\
6a(n) &= 2a(n+1) + 2a(n-1),\ n \geq 1\\
7a(n) &= a(n+2) + a(n-2),\ n \geq 2\\
8a(n) &= a(n+2) + f(n) + a(n-2),\ n \geq 2\\
9a(n) &= a(n+2) + 2f(n) + a(n-2),\ n \geq 2\\
10a(n) &= a(n+2) + a(n+1) + a(n-1) + a(n-2),\ n \geq 2\\
11a(n) &= a(n+2) + a(n+1) + a(n) + a(n-1) + a(n-2),\ n \geq 2\\
12a(n) &= a(n+2) + a(n+1) + 2a(n) + a(n-1) + a(n-2),\ n \geq 2
\end{aligned}$$

The sums here can be found using a greedy algorithm. Prove that these sums are always palindromes in the sense that every term $c \cdot a(n+k)$ is matched by a term $c \cdot a(n-k)$.

Problem 9, posed by Russell Jay Hendel: Herta's sequence

Herta Taussig Freitag, a famous Fibonaccianess, lived to the ripe age of

$$91 = 7 \times 13 = L_4 \times F_7 = L_4 \times F_{L_4}.$$

Thus she culminated her life by unifying the Fibonacci and Lucas numbers. In her memory, it seems natural to define

$$\text{Herta's sequence } = \{L_n \times F_{L_n}\}_{-\infty<n<\infty}.$$

Traditionally, when a recursive sequence is introduced, it is customary to seek generalizations of Fibonacci-Lucas identities and relations. Typical Fibonacci-Lucas identities, formulas and relations [1] are due to Binet, Catalan, Cassini-Simson, O'Cagne, and Gelin-Cesaro. Some additional identities and relations not named after people are the asymptotic, addition, and double angle formulae as well as the conjugation, recurrence and successor relations. Many more exist.

The open problem is to find a set of formulae and relations for Herta's sequence. To encourage readers to solve this problem, we note that a slightly similar problem—similar in that the index is increasing non-linearly—may be found in problem B-788, [2]. Define

$$G_n = F_{n^2}, \quad n \geq 1,$$

and prove the asymptotic formula

$$G_{n+1} \sim L_{2n+2} G_n,$$

and prove the following almost-exact recurrence (i.e., with error term asymptotically 0):

$$G_{n+1} \sim L_{2n+1} G_n + \beta^2 G_{n-1}.$$

References:

[1] P. Chandra and E. W. Weisstein, *Fibonacci Number* MathWorld: http://mathworld.wolfram.com/FibonacciNumber.html.

[2] R. J. Hendel (Proposer) and H.-J. Seiffert (Solver), *Problem B-788, Asymptotic Analysis*, The Fibonacci Quarterly, **33.2**, (1995), 182 and 34.4, (1996), 375-376.

Problem 10, posed by Peter G. Anderson: Prove that $2|F_{3n}$.

Discussion: The issue is not, of course, to resolve a long outstanding, difficult problem, but to show how the result can follow from a wide variety of well-used or obscure techniques, such as these: counting arguments, induction, King matrix, Binet formula, generating functions, geometry, etc.

Problem 11, posed by Clark Kimberling: Distinct products property

Suppose that S is a set of integers such that if U and V are distinct finite subsets of S, then

$$\prod_{x \in U} x \neq \prod_{y \in V} y.$$

What linear recurrence sequences, as sets S, have this property?

(It follows from Carmichael's Theorem that the set of Fibonacci numbers is an example. There appear to be many others.)

Problem 12, posed by Andreas M. Hinz: Hanoi sequences

This problem consist of three parts, based on [1] and [2].

12.1 Matchings in H_3^n.

The number of spanning trees for Hanoi graphs H_3^n (for their definition see [1, p. 94]), i.e. their complexity, is known to be [1, Theorem 2.24]:

$$\begin{aligned}\tau(H_3^n) &= 3^{\frac{1}{4}(3^n-1)+\frac{1}{2}n} \cdot 5^{\frac{1}{4}(3^n-1)-\frac{1}{2}n} \\ &= \left(\sqrt{\frac{3}{5}}\right)^n \cdot \left(\sqrt[4]{15}\right)^{3^n-1} \\ &= 1, 3, 135, 20\,503\,125, \ldots .\end{aligned}$$

For the number of matchings m_n (see [1, Exercise 2.19]) we have the recurrence

$$m_{n+1} = m_n^3 + 3m_n\ell_n^2 + 3\ell_n^2 k_n + k_n^3\,,$$

where ℓ_n is the number of matchings of H_3^n with one, k_n with two, and j_n with three perfect states deleted [1, p. 274f]:

$$\begin{aligned}k_{n+1} &= m_n\ell_n^2 + m_n k_n^2 + 2k_n\ell_n^2 + k_n^3 + 2j_n k_n \ell_n + j_n^2 k_n, \\ j_{n+1} &= j_n^3 + 3j_n k_n^2 + 3k_n^2\ell_n + \ell_n^3, \\ \ell_{n+1} &= j_n k_n^2 + j_n\ell_n^2 + 2\ell_n k_n^2 + \ell_n^3 + 2m_n\ell_n k_n + m_n^2\ell_n,\end{aligned}$$

with the seeds $m_0 = 1 = \ell_0$ and $k_0 = 0 = j_0$. The first few values are $m_n = 1, 4, 125, 4\,007\,754, \ldots$. Problem 12.1 is to find a closed formula for m_n.

12.2 Total distance of perfect states in $\overrightarrow{H}_3^n$

In Lucas's Second Problem (cf. [1, Chapter 3]) we have for the total distance of perfect state 0^n:

$$\mathrm{d}(0^n) = \sum_{\sigma\in\mathfrak{T}^n} \mathrm{d}(\sigma, 0^n) = 0, 2, 26, 281, 3287, 43543, \ldots$$

(Compare this with

$$\mathrm{d}(0^n) = \sum_{\sigma\in T^n} \mathrm{d}(s, 0^n) = 2\cdot 3^{n-1}(2^n - 1) = 0, 2, 18, 126, 810, 5022, \ldots$$

for H_3^n with $2^{-n}\cdot \overline{\mathrm{d}}(0^n) \longrightarrow \frac{2}{3}$, where $\overline{\mathrm{d}}(0^n)$ stands for the average distance from 0^n.) We know that $2^{-n}\cdot\overline{\mathrm{d}}(0^n) \longrightarrow a \in \left[\frac{1}{12}, \frac{5}{8}\right]$ as $n \to \infty$ (cf. [1, Exercise 3.5]). Problem 12.2 is to evaluate a.

12.3 The Star Tower of Hanoi

Paul K. Stockmeyer introduced in 1994 the question to find the optimal number of moves $\mathrm{d}(2^n, 3^n)$ if the perfect tower is to be transferred from one leave of the underlying star graph to another in this version of the Tower of Hanoi. Assuming a Frame-Stewart-like strategy, i.e. to cut the tower into m_n larger and $n - m_n$ smaller discs (see [1, p. 255-254] for a description of both, the problem and its solution), he obtained the value

$$\widetilde{\mathrm{d}}(2^n, 3^n) = 2\sum_{d=1}^{n} s_d,\ m_n = \lfloor \mathrm{lt}(s_n)\rfloor + 1,$$

where $s = 1, 2, 3, 4, 6, 8, 9, 12, 16, \ldots$ is the 3-smooth sequence of 3-smooth numbers $2^j\cdot 3^k$, $j, k \in \mathbb{N}_0$, in increasing order. The analogous problem to find $\mathrm{d}(0^n, 1^n)$, i.e. to get from the center to a leave, does not have an explicit solution though. The corresponding recurrence is $\widetilde{\mathrm{d}}(0^0, 1^0) = 0$ and for $n \in \mathbb{N}$:

$$\widetilde{\mathrm{d}}(0^n, 1^n) = \min\left\{\widetilde{\mathrm{d}}(0^m, 2^m) + \widetilde{\mathrm{d}}(2^m, 1^m) + \lambda \mid m \in [n]_0\right\}$$

where $\lambda = \frac{1}{2}(3^{n-m} - 1)$. This leads to

$$\widetilde{\mathrm{d}}(0^n, 1^n) = 0, 1, 4, 7, 14, 23, 32, 47, 68, 93, 120, 153, 198, 255, \dots$$

Problem 12.3 is to find a closed formula for this sequence and decide whether these solutions are indeed optimal; i.e. $\widetilde{\mathrm{d}}(0^n, 1^n) = \mathrm{d}(0^n, 1^n)$ and $\widetilde{\mathrm{d}}(2^n, 3^n) = \mathrm{d}(2^n, 3^n)$, a fact which for $n = 0, \dots, 15$ has been confirmed computationally by Stockmeyer.

References:

[1] Hinz, A. M., Klavžar, S., Milutinović, U., Petr, C., *The Tower of Hanoi—Myths and Maths*, Springer, Basel, 2013.

[2] Lucas, É, *Récréations Mathématiques*, 4 vols., Gauthier-Villars et fils, Paris, 1882–1894.

Problem 13, posed by Larry Ericksen: Jacobi polynomials

For Jacobi polynomials $P_n^{(a,b)}(x)$, obtain polynomials $f_n(z)$ in (13.1) and $g_n(z)$ in (13.2) of degree n in variable z. Establish triangles with integer coefficients $c_{n,i}$ and $d_{n,i}$ at each level n.

$$f_n(z) = \sum_{k=0}^{n} P_k^{(0,z-2k)}(3) \; = \; \frac{1}{n!} \sum_{i=0}^{n} c_{n,i}\, z^i, \tag{13.1}$$

$$g_n(z) = \sum_{k=0}^{n} P_k^{(-1,z+1-2k)}(3) \; = \; \frac{1}{n!} \sum_{i=0}^{n} d_{n,i}\, z^i. \tag{13.2}$$

Evaluate the Jacobi sums in (13.1) and (13.2) at $z = n$, and prove the following identities for Fibonacci numbers F_n :

$$f_n(n) = F_{2n+2} \quad \text{and} \quad g_n(n) = F_{2n+1}.$$

Examples: Triangles of coefficients for $0 \le n \le 3$.

	Coefficients $c_{n,i}$				
$n\backslash i$	0	1	2	3	$f_n(n)$
0	1				1
1	2	1			3
2	2	5	1		8
3	-6	8	9	1	21

	Coefficients $d_{n,i}$				
$n\backslash i$	0	1	2	3	$g_n(n)$
0	1				1
1	1	1			2
2	0	3	1		5
3	0	-1	6	1	13

Partial Solutions by Thotsaporn Thanatipanonda, Mahidol University International College, Nakornpathom, Thailand; Larry Ericksen, Millville, New Jersey, USA (independently). The identities for $f_n(n)$ and $g_n(n)$ can be proved by the Zeilberger algorithm or by using generating functions, which reduces to showing

$$\sum_{k=0}^{n}\sum_{j=0}^{k} \binom{k}{j} \binom{n-k+j}{j} = F_{2n+2},$$

$$\sum_{k=0}^{n}\sum_{j=0}^{k} \binom{k-1}{k-j} \binom{n-k+j}{j} = F_{2n+1}.$$

Problem 14, posed by Christian Ballot: Are these two sequences identical?

Let $\alpha \simeq 1.46557$ be the dominant zero of $x^3 - x^2 - 1$ and $(N_n)_{n\ge 0}$ be the fundamental recurrence associated with $x^3 - x^2 - 1$. That is, $N_{n+3} = N_{n+2} + N_n$ for all $n \ge 0$ with $N_0 = N_1 = 0$ and $N_2 = 1$. Define the function $f(x) := \lfloor \alpha^3 x \rfloor$.

Consider the sequences $(u_n)_{n\ge 0}$ and $(v_n)_{n\ge 0}$, where u_n and v_n are defined for all $n \ge 0$ by

$$u_n := f^n(1), \tag{14.1}$$

$$v_n := N_{3n+3} - \sum_{k=0}^{\lfloor (3n-14)/12 \rfloor} N_{3n-14-12k}, \tag{14.2}$$

where f^0 is the identity function and f^n stands for the n-fold composite function $f \circ \cdots \circ f$.

The problem posed during the conference was this: Do we have $u_n = v_n$ for all $n \geq 0$? About three weeks later, the proposer gave an affirmative answer, and a solution now appears in a submitted paper titled 'On functions expressible as words on a pair of Beatty sequences'. Following is an outline of the initial motivation and of the proof. We wanted to know whether the sequence $u_n = \lfloor \alpha^3 \lfloor \alpha^3 \lfloor \cdots \lfloor \alpha^3 \rfloor \cdots \rfloor \rfloor \rfloor$ (n nested pairs $\lfloor\ \rfloor$) is a linear recurrence. We had found experimentally that the first so-many u_n terms followed the formula given in (14.2), and, using PARI, that $u_n = v_n$ for $0 \leq n \leq 199$. We showed that (v_n) is a seventh-order recurrence with characteristic polynomial $(x^4 - 1)(x^3 - 4x^2 + 3x - 1)$. Then we got a closed form for v_n from which it is easy to verify that, for n large enough, $v_{n+1} = \lfloor \alpha^3 v_n \rfloor$ and thus conclude by induction.