RECURRENCES FOR ENTRIES OF POWERS OF MATRICES

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Abstract. In this note, giving course to a challenge in a recent paper of Larcombe [2], we find the entries of any nth power of a 3×3 matrix, and as a byproduct, we recover Larcombe's result on 2×2 matrices. Further, we look at block matrices and show an invariance result for the powers of such matrices.

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1. Introduction

In [2], Larcombe considered the matrix $A = \begin{pmatrix} a_1 & b_1 \\ d_1 & e_1 \end{pmatrix}$ and showed that its powers $A^n = \begin{pmatrix} a_n & b_n \\ d_n & e_n \end{pmatrix}$ satisfy the invariance property $b_n/d_n = b_1/d_1$, for all $n \geq 1$ (or, $b_n d_1 = b_1 d_n$ without assuming that $d_1 \neq 0$). While the result could have been obtained (as the author of [2] also observed) from previous work by McLaughlin [3] or Williams [5] (among others), Larcombe provided several proofs for this result. He challenged the community to find extensions of this result to higher dimension matrices. It is the intent of this note to look at 3×3 matrices and find closed forms for the entries of their nth powers, as well as point toward the general case, as well, although, the results are not as simple as the one for the 2×2 matrices. Further, we consider block matrices and show an invariance result for their powers.

Throughout, for a matrix $A = (a_{ij})_{1\leq i,j\leq k}$, let $\mathrm{Tr}(A) = \sum_{i=1}^k a_{ii}$ denote the trace of A, and let $\det(A)$ denote the determinant of A. We also let I_k be the $k\times k$ identity matrix.

2. Powers of 3×3 matrices

We will start with an observation from which we can derive all the previous results, quite easily, in fact. It is known (this is customarily called Cayley-Hamilton theorem) that a matrix $A = (a_{ij})_{1\leq i,j\leq k}$ satisfies its own characteristic polynomial $p_A(\lambda) = \det(\lambda I_k - A) = \lambda^k - c_{k-1}\lambda^{k-1} - \cdots - c_1\lambda - (-1)^{k-1}\det(A) I_k$, with $c_{k-1} = \mathrm{Tr}(A)$, $c_{k-2} = \frac{\mathrm{Tr}(A^2)-\mathrm{Tr}(A)^2}{2}$, etc., with all the coefficients being described in terms of the eigenvalues of A. Thus,

$$A^k = c_{k-1}A^{k-1} + \cdots + c_1 A + (-1)^{k-1}\det(A) I_k.$$

Now, multiplying throughout by A^{n+1-k}, we easily obtain the next (obviously, known) result.

Lemma 2.1. *The entries $a_{ij}^{(n)}$ of A^n all satisfy the recurrence*

$$x_{n+1} = c_{k-1}x_n + \cdots + c_1 x_{n-k+2} + (-1)^{k-1}\det(A) x_{n-k+1}.$$

For easy writing, let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ and write its powers as $A^n = \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & h_n & j_n \end{pmatrix}$, $n \geq 1$.

Further, we set $T_1 := \mathrm{Tr}(A)$, $T_2 := \frac{\mathrm{Tr}(A^2)-\mathrm{Tr}(A)^2}{2}$, $D := \det(A)$.

Theorem 2.2. *The entries $a_n, b_n, \ldots,$ of A^n all satisfy the following recurrence*

$$x_{n+1} = T_1 x_n + T_2\, x_{n-1} + D\, x_{n-2}, \quad n \geq 3. \tag{2.1}$$

Further, let $\beta = T_1/3$, $\alpha = \frac{1}{3}\sqrt{T_1^2 + 3T_2} = \sqrt{\operatorname{Tr}((A - \beta I_3)^2/6)}$. The nth powers of A are

$$A^n = \begin{cases} U(-\alpha+\beta)^n + (V + Wn)\ \left(\frac{\alpha}{2}+\beta\right)^n & \text{if the eigenvalues of } A \text{ are } -\alpha, \frac{\alpha}{2}+\beta, \frac{\alpha}{2}+\beta \\ U + (V+Wn)\ \left(-\frac{1}{2}\right)^n & \text{if the eigenvalues of } A \text{ are } \alpha+\beta, -\frac{\alpha}{2}+\beta, -\frac{\alpha}{2}+\beta \\ (U + Vn + Wn^2)\sqrt[3]{D} & \text{if the eigenvalues of } A \text{ are all } \sqrt[3]{D} \\ U\lambda_0^n + V\lambda_1^n + W\lambda_2^n & \text{if the eigenvalues of } A \text{ are distinct,} \end{cases}$$

for some matrices U, V, W determined by the initial conditions A^n, for $n = 0, 1, 2$.

Proof. From Lemma 2.1 we obtain (2.1). Next, we need to solve the equation

$$x^3 - T_1 x^2 + T_2 x - D = 0, \tag{2.2}$$

which will give the eigenvalues of the matrix A. We can certainly use Cardano-Tartaglia formulas for the roots, namely,

$$\begin{aligned} x_1 &= \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}} + \frac{T_1}{3} \\ x_2 &= -\frac{\sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}}{2} - \frac{T_1}{3} + \frac{\imath\sqrt{3}(\sqrt[3]{r + \sqrt{r^2 - q^3}} - \sqrt[3]{r - \sqrt{r^2 - q^3}})}{2} \\ x_3 &= -\frac{\sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}}{2} - \frac{T_1}{3} - \frac{\imath\sqrt{3}(\sqrt[3]{r + \sqrt{r^2 - q^3}} - \sqrt[3]{r - \sqrt{r^2 - q^3}})}{2}, \end{aligned}$$

where $r = \frac{9T_1T_2 + 27D + 2T_1^3}{54}, q = \frac{3T_2 + T_1^2}{9}$. However, we will review here a seemingly not so well known method that displays a better form for these roots (albeit, if the coefficients are complex, there are a few subtle issues, which we will point out later). The method is scattered in several places (see, for instance [4], for a particular case), but we have not been able to find a suitable reference for it.

We attempt to find an affine transformation on the variable x, say $x = \alpha y + \beta$ such that (2.2) transforms into an equation of the form

$$y^3 - 3y - \gamma = 0, \tag{2.3}$$

and it turns out that we can take $\gamma = \det(B)$, where $B := \frac{A - \beta I_3}{\alpha}$. The above equation is reminiscent of the trigonometric identity $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$. The advantage is that the eigenvalues of B will render (via the affine transformation) the eigenvalues of A (a result well known in Linear Algebra).

We first assume that $T_1^2 + 3T_2 \neq 0$ (it will be quite apparent later on why we impose this condition). Replacing $x = \alpha y + \beta$ in (2.2) and multiplying by α^{-3}, we obtain

$$x^3 + \frac{3\beta - T_1}{\alpha} x^2 + \frac{3\beta^2 - 2\beta T_1 - T_2}{\alpha^2} x + \frac{\beta^3 - D - \beta^2 T_1 - \beta T_2}{\alpha^3} = 0.$$

Imposing the mentioned conditions, we get the system

$$\begin{aligned} 3\beta - T_1 &= 0 \\ 3\beta^2 - 2\beta T_1 - T_2 &= -3\alpha^2, \end{aligned}$$

from which we get the solutions $\beta = T_1/3$, $\alpha = \frac{1}{3}\sqrt{T_1^2 + 3T_2} = \sqrt{\mathrm{Tr}((A-\beta I_3)^2/6)}$ (we do not take both values of α, rather one will suffice). Now, we can set $y = 2\cos\theta$ in equation (2.3), which implies $\cos(3\theta) = \det(B)/2$ and consequently we obtain the eigenvalues

$$y_j = 2\cos\left(\frac{1}{3}\cos^{-1}\left(\det(B)/2\right) + \frac{2\,j\,\pi}{3}\right), 0 \le j \le 2,$$

and so, for our original equation, we get the eigenvalues of A

$$\lambda_j = 2\,\alpha\cos\left(\frac{1}{3}\cos^{-1}\left(\det(B)/2\right) + \frac{2\,j\,\pi}{3}\right) + \beta, 0 \le j \le 2. \tag{2.4}$$

If the argument of the inverse cosine is complex, the inverse cosine is a multivalued function, requiring a branch cut (recall that, by definition, $\cos^{-1} z = \frac{\pi}{2} + i\ln(\sqrt{1-z^2} + i\,z)$), so, for all values j we should use the same branch.

Certainly, it may happen that $\alpha = 0$ so, $T_1^2 + 3T_2 = 0$, which is equivalent to

$$a^2 + bd + ae + e^2 + f^2 + cg + aj + ej + j^2 = 0.$$

Under these conditions, we use Cardano formulas and get the eigenvalues

$$\begin{aligned}
\lambda_0 &= \frac{T_1}{3} - \frac{1}{3}\sqrt[3]{T_1^3 - 27D}\\
\lambda_1 &= \frac{T_1}{3} + \frac{1}{6}\left(1 - \imath\sqrt{3}\right)\\
\lambda_2 &= \frac{T_1}{3} + \frac{1}{6}\left(1 + \imath\sqrt{3}\right)\sqrt[3]{T_1^3 - 27D}.
\end{aligned} \tag{2.5}$$

We consider several cases.

Case 1. $\lambda_0 \ne \lambda_1 \ne \lambda_2 \ne \lambda_0$. In this case, regardless of whether the eigenvalues are given by (2.4) or (2.5), the powers of A are

$$A^n = U\lambda_0^n + V\lambda_1^n + W\lambda_2^n,$$

for some matrices U, V, W determined by the initial conditions, for $n = 1, 2, 3$.

Case 2. $\lambda_0 \ne \lambda_1 = \lambda_2$. It is easy to see that $\alpha = 0$ cannot occur in this case. If $\alpha \ne 0$, then we can only have $\lambda_0 = \lambda_2$, or $\lambda_1 = \lambda_2$. In the first case we get $\det(B) = -2$ and consequently, the eigenvalues are $\frac{\alpha}{2} + \beta, -\alpha + \beta, \frac{\alpha}{2} + \beta$. Then

$$A^n = U(-\alpha+\beta)^n + (V + Wn)\ \left(\frac{\alpha}{2} + \beta\right)^n,$$

for some matrices U, V, W. In the second case, $\det(B) = 2$ and the eigenvalues are $1, -\frac{\alpha}{2} + \beta, -\frac{\alpha}{2} + \beta$, and consequently,

$$A^n = U(\alpha+\beta)^n + (V + Wn)\left(-\frac{\alpha}{2} + \beta\right)^n,$$

for some matrices U, V, W.

Case 3. $\lambda_0 = \lambda_1 = \lambda_2 (=: \lambda)$.

In this case, if $\alpha \ne 0$, one can easily show by attempting to solve the corresponding system that there are no values of $\det(B)$ for this to happen. If $\alpha = 0$, then we must have $T_1 = 3\sqrt[3]{D}$, and the eigenvalues will become $\lambda_0 = \lambda_1 = \lambda_2 = \sqrt[3]{D}$. Thus, we must have

$$A^n = (U + Vn + Wn^2)\sqrt[3]{D},$$

for some matrices U, V, W.

The theorem is shown. $\square$

Remark 2.3. *To further explain the statement of Theorem 2.2, we assume that we are in the third case, say (that is, the eigenvalues of A are all equal to $\sqrt[3]{D}$). We can find easily the matrices U, V, W by solving the matrix system*

$$\begin{aligned} &U\,\sqrt[3]{D} = A^0 = I\\ &(U+V+W)\,\sqrt[3]{D} = A\\ &(U+2V+4W^2)\,\sqrt[3]{D} = A^2, \end{aligned}$$

from which we derive

$$\begin{aligned} U &= \frac{1}{\sqrt[3]{D}} I,\\ V &= -\frac{1}{2\sqrt[3]{D}}\left(A^2-4A+3I\right),\\ W &= \frac{1}{2\sqrt[3]{D}}\left(A^2-2A+I\right). \end{aligned}$$

For the next result only, we let

$$\begin{aligned} \alpha &:= abd-a^2e-bde+ae^2-acg+efh+a^2j-e^2j+cgj-fhj-aj^2+ej^2;\\ \beta &:= -b^2d+abe+bcg+ceh-abj-bej-chj+bj^2;\\ \gamma &:= -b^2d+abe+ach+bfh-abj-bej-chj+bj^2;\\ \delta &:= bcd-ace+ce^2-bef-c^2g+acj-cej+bfj;\\ \epsilon &:= -ace+ce^2+abf-bef-c^2g+cfh+acj-cej;\\ \zeta &:= bd^2-ade-cdg-efg+adj+dej+fgj-dj^2;\\ \eta &:= -bd^2+ade+afg+dfh-adj-dej-fgj+dj^2;\\ \theta &:= acd-a^2f-bdf+aef+f^2h-cdj+afj-efj;\\ \rho &:= acd-cde-a^2f+aef-cfg+f^2h+afj-efj;\\ \sigma &:= bdg-aeg+e^2g-cg^2-deh+agj-egj+dhj;\\ \tau &:= -aeg+e^2g-cg^2+adh-deh+fgh+agj-egj;\\ \upsilon &:= abg-beg-a^2h+aeh-cgh+fh^2+ahj-ehj;\\ \omega &:= abg-a^2h-bdh+aeh+fh^2-bgj+ahj-ehj. \end{aligned}$$

Theorem 2.4. *The entries of the power matrix A^n satisfy the recurrences*

$$\begin{aligned} \alpha b_n &= -\beta a_n+\gamma e_n+(\beta-\gamma)j_n,\\ \alpha c_n &= -\delta a_n+(\epsilon-\delta)e_n-\epsilon j_n,\\ \alpha d_n &= \zeta a_n+\eta e_n-(\zeta+\eta)j_n,\\ \alpha f_n &= (\rho-\theta)a_n+\theta e_n-\rho j_n,\\ \alpha g_n &= \sigma a_n+(\tau-\sigma)e_n-\tau j_n,\\ \alpha h_n &= (\upsilon-\omega)a_n+\omega e_n-\upsilon j_n. \end{aligned}$$

Proof. We will use the simple observation that $A^n \cdot A = A \cdot A^n$. This will render the system

$$\begin{pmatrix} -db_n-gc_n+bd_n+cg_n & -ba_n+(a-e)b_n-hc_n+be_n+ch_n & -ca_n-fb_n+(a-j)c_n+bf_n+cj_n\\ da_n-(a-e)d_n-de_n-gf_n+fg_n & db_n-bd_n-hf_n+fh_n & dc_n-cd_n-fe_n+(e-j)f_n+fj_n\\ ga_n+hd_n-(a-j)g_n-dh_n-gj_n & gb_n+he_n-bg_n-(e-j)h_n-hj_n & gc_n+hf_n-cg_n-fh_n \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which, by solving it diligently, we get our claim (we can also express any entry of the nth power A^n in terms of a_n, b_n, c_n, obtaining other relationships among the entries of A^n, but we preferred, for symmetry purposes, to express the solutions of the previous system in terms of a_n, e_n, j_n). □

Certainly, these results will render Larcombe's result [2], since we can pad a 2×2 matrix with zeroes in the last row and column, regarding it as a 3×3 matrix.

Corollary 2.5. *For a 2×2 matrix $A = \begin{pmatrix} a & b \\ d & e \end{pmatrix}$ with powers $A^n = \begin{pmatrix} a_n & b_n \\ d_n & e_n \end{pmatrix}$, we have $b_n/d_n = b/d$.*

Proof. From Theorem 2.4, by taking $c = f = g = h = j = 0$ and simplifying the quotient b_n/d_n, we obtain the invariance result. □

When A is real and symmetric, the previous theorem will take a simpler form. Let

$$\begin{aligned}
\alpha &: = -ab^2 + ac^2 + a^2e + b^2e - ae^2 - ef^2 - a^2j - c^2j + e^2j + f^2j + aj^2 - ej^2;\\
\beta &: = b^3 - abe - acf - bf^2 + abj + bej + cfj - bj^2;\\
\gamma &: = -b^3 + bc^2 + abe + cef - abj - bej - cfj + bj^2;\\
\delta &: = -b^2c + c^3 + ace - ce^2 + bef - acj + cej - bfj;\\
\rho &: = -c^3 - ace + ce^2 + abf - bef + cf^2 + acj - cej;\\
\sigma &: = -abc + a^2f + b^2f - aef - f^3 + bcj - afj + efj;\\
\tau &: = abc - bce - a^2f - c^2f + aef + f^3 + afj - efj.
\end{aligned}$$

Further, let $m = \frac{1}{3}\mathrm{Tr}(A)$ and $q = \frac{1}{2}\det(A - mI)$, $6p = (a-m)^2 + (e-m)^2 + (j-m)^2 + b^2 + c^2 + d^2 + f^2 + g^2 + h^2$ is the sum of the squares of the elements of $A - mI$, and $\phi = \frac{1}{3}\tan^{-1}\frac{\sqrt{p^3-q^2}}{q}, 0 \le \phi \le \pi$.

Theorem 2.6. *The entries of powers of the (real and) symmetric matrix A^n satisfy the recurrences*

$$\begin{aligned}
\alpha b_n &= \gamma a_n + \beta e_n - (\beta+\gamma) j_n,\\
\alpha c_n &= \delta a_n - (\delta+\rho) e_n + \rho j_n,\\
\alpha f_n &= -(\sigma+\tau) a_n + \sigma e_n + \tau j_n.
\end{aligned}$$

More precisely,

$$A^n = \begin{cases} m^n I_3 & \text{if } p = q = 0, \text{ that is, } A \text{ is diagonal}\\ (U+Vn)(m-\sqrt{p})^n + W(m+2\sqrt{p})^n & \text{if } \phi = 0, 2\pi/3\\ (U+Vn)(m+\sqrt{p})^n + W(m-2\sqrt{p})^n & \text{if } \phi = \pi/3, \pi\\ U\lambda_0^n + V\lambda_1^n + W\lambda_2^n & \text{if } \phi \ne 0, \pi/3, 2\pi/3, \pi, \end{cases}$$

for some matrices U, V, W obtained using the initial conditions A^n, for $n = 0, 1, 2$ (see (2.6) below for the definition of ϕ).

Proof. To show the first claim, we proceed as we did in the proof of Theorem 2.4 (or take the particular case $b = d, c = g, f = h$ in Theorem 2.4, although in this instance our definitions for $\alpha, \beta, \ldots$ will change).

To show the second claim, we use Smith's simplification (see [4]) of the eigenvalues of the 3×3 real symmetric matrix, namely

$$\begin{aligned}\lambda_0 &= m + 2\sqrt{p}\,\cos\phi\\ \lambda_1 &= m - \sqrt{p}\,(\cos\phi + \sqrt{3}\sin\phi)\\ \lambda_2 &= m - \sqrt{p}\,(\cos\phi - \sqrt{3}\sin\phi),\end{aligned} \tag{2.6}$$

where $m = \frac{1}{3}\mathrm{Tr}(A)$ and $q = \frac{1}{2}\det(A - mI)$, $6p = (a-m)^2 + (e-m)^2 + (j-m)^2 + b^2 + c^2 + d^2 + f^2 + g^2 + h^2$ is the sum of the squares of the elements of $A - mI$, and $\phi = \frac{1}{3}\tan^{-1}\frac{\sqrt{p^3 - q^2}}{q}, 0 \le \phi \le \pi$.

Certainly, since A is symmetric, hence Hermitian, all eigenvalues must be real, so $p^3 \ge q^2$. Thus, if $p = 0$, then q must be zero. We distinguish two cases. If $p = q = 0$, then A is diagonal (in the definition of ϕ, the argument of the inverse tangent is indeterminate, but the eigenvalues are all m, regardless of the value of ϕ).

We now assume that $p \neq 0$ (that is, A is not diagonal). Next, assume that we have an eigenvalue of multiplicity 2. If $\lambda_0 = \lambda_1$, then $\phi = 2\pi/3$, and the eigenvalues of A are $m - \sqrt{p}$, $m - \sqrt{p}$, $m + 2\sqrt{p}$. If $\lambda_0 = \lambda_2$, then $\phi = \pi/3$, and the eigenvalues are $m + \sqrt{p}$, $m - 2\sqrt{p}$, $m + \sqrt{p}$. If $\lambda_1 = \lambda_2$, then $\phi = 0, \pi$, and the eigenvalues are $m + 2\sqrt{p}$, $m - \sqrt{p}$, $m - \sqrt{p}$, respectively, $m - 2\sqrt{p}$, $m + \sqrt{p}$, $m + \sqrt{p}$. (Observe that in some of these cases we have $p^3 = q^2$.) The theorem follows. ☐

Remark 2.7. *To further explain the statement of the previous theorem, let us assume that we are in the second case and so, $\phi = 0$, or $2\pi/3$. We can find easily the matrices U, V, W by solving the matrix system*

$$\begin{aligned}&U + W = A^0 = I\\ &(U + V)(m - \sqrt{p}) + W(m + 2\sqrt{p}) = A\\ &(U + 2V)(m - \sqrt{p})^2 + W(m + 2\sqrt{p})^2 = A^2,\end{aligned}$$

from which we derive

$$\begin{aligned}U &= -\frac{1}{9p}\left(A^2 - 2(m - \sqrt{p})A + (m^2 - 2m\sqrt{p} - 8p)I\right),\\ V &= -\frac{1}{3m\sqrt{p} - 3p}\left(A^2 - (2m + \sqrt{p})A + (m^2 + m\sqrt{p} - 2p)I\right),\\ W &= \frac{1}{9p}\left(A^2 - 2(m - \sqrt{p})A + (m - \sqrt{p})^2 I\right).\end{aligned}$$

3. Block matrices

In this section, we will display yet another generalization approach to Larcombe's observation. We consider an $(n + t) \times (k + m)$ matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where the components are $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{t \times m}$, $D \in \mathbb{R}^{t \times k}$. Such a matrix M is called a *block matrix*. It should be well known, but regardless, it is also rather straightforward to show that if

$N = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$, with $A' \in \mathbb{R}^{m\times \ell}$, $B' \in \mathbb{R}^{m\times s}$, $C' \in \mathbb{R}^{k\times \ell}$, $D' \in \mathbb{R}^{k\times s}$, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} AA' + BC' & AB' + BD' \\ CA' + DC' & CB' + DD' \end{pmatrix}.$$

Let O be the identically zero matrix. While we can show the result in more generality, for simplicity we preferred to let $m = k = t = n$. By taking $m = k = t = n = 1$, we recover Larcombe's observation.

Theorem 3.1. *Let A, B, C, D be $n \times n$ matrices, let the M^n be the nth power of the matrix M, with block entries $M^n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$. The following hold:*

1. *If A, B, C, D commute with each other, then we have the invariance $B_nC = C_nB$.*
2. *If $A = O$ or $D = O$, then we have the invariance $B_nC = C_nB$.*
3. *If $C = O$, then $M^n = \begin{pmatrix} A^n & B\sum_{i=0}^{n-1} A^iD^{n-1-i} \\ 0 & D^n \end{pmatrix}$.*
4. *If $B = O$, then $M^n = \begin{pmatrix} A^n & O \\ C\sum_{i=0}^{n-1} A^iD^{n-1-i} & D^n \end{pmatrix}$.*

Proof. We use the simple observation (as Larcombe [2] did for 2×2 matrices) that $M \cdot M^n = M^n \cdot M$. Thus,

$$\begin{pmatrix} AA_n + BC_n & AB_n + BD_n \\ CA_n + DC_n & CB_n + DD_n \end{pmatrix} = \begin{pmatrix} A_nA + B_nC & A_nB + B_nD \\ C_nA + D_nC & C_nB + D_nD \end{pmatrix},$$

from which we get the system

$$\begin{aligned} AA_n + BC_n &= A_nA + B_nC, \\ AB_n + BD_n &= A_nB + B_nD, \\ CA_n + DC_n &= C_nA + D_nC, \\ CB_n + DD_n &= C_nB + D_nD. \end{aligned}$$

If the matrices A, B, C, D commute with each other, then using $AA_n = A_nA$ in the first identity we get the first claim. If $A = O$, then we use the first equation, and if $D = O$, then we use the last equation, thus obtaining the required identity of matrices, $B_nC = C_nB$ (hence of determinants, as well). The last two claims follow in the same manner and the proof is done. □

Remark 3.2. *We get for free from the above argument the following recurrences for the blocks of the powers M^n, $n \geq 1$,*

$$\begin{aligned} A_{n+1} &= AA_n + BC_n, \\ B_{n+1} &= AB_n + BD_n, \\ C_{n+1} &= CA_n + DC_n, \\ D_{n+1} &= CB_n + DD_n. \end{aligned}$$

4. Further comments

For 4×4 matrices we will only make some comments that shows that that case can be handled, as well. The Cayley-Hamilton theorem renders that a 4×4 matrix A will satisfy the equation

$$A^4 - T_1A^3 - T_2A^2 - T_3A + DI_4 = 0,$$

where $T_1 := \mathrm{Tr}(A)$, $T_2 := \frac{\mathrm{Tr}(A^2)-\mathrm{Tr}(A)^2}{2}$ are as before and $T_3 := \frac{\mathrm{Tr}(A)^3-3\mathrm{Tr}(A^2)\mathrm{Tr}(A)+2\mathrm{Tr}(A^3)}{6}$. Certainly, using what is known [1] about the solutions of the quartic, we can express the eigenvalues of the matrix A in terms of the parameters above, as well as the resolvent cubic's roots and consequently, the entries of any power of the matrix A can be found, albeit in a more complicated form.

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