MATRICES IN THE HOSOYA TRIANGLE

MATTHEW BLAIR, RIGOBERTO FLÓREZ, AND ANTARA MUKHERJEE

Abstract. In this paper, we use well-known results from linear algebra as tools to explore some properties of products of Fibonacci numbers. Specifically, we explore the behavior of the eigenvalues, eigenvectors, characteristic polynomials, determinants, and the norm of non-symmetric matrices embedded within the Hosoya triangle. We discovered that most of these objects either embed themselves within the Hosoya triangle, or they give rise to Fibonacci identities.

We also study the nature of these matrices when their entries are taken mod 2. As a result, we found an infinite family of non-connected graphs. Each graph in this family has a complete graph with loops attached to each of its vertices as a component and the other components are isolated vertices. The Hosoya triangle allowed us to show the beauty of both the algebra and geometry.

1. Introduction

The Hosoya triangle [12] is a triangular array where the entries are products of Fibonacci numbers. Our main purpose of this paper is to show some connection between this triangle and several aspects of geometry, graph theory, and the Fibonacci number sequence; all of them by means of linear algebra techniques. Several authors have been geometrically representing the beauty of some elementary concepts of algebra, number theory, and combinatorics bridging them with Fibonacci numbers using the Hosoya triangle [1, 5, 7, 8, 9, 12, 13, 14].

The recurrence relations of Fibonacci numbers provide an interesting way to study properties of matrices with these entries. For example, in 2002 Lee et al. [15] studied matrices that have squares of Fibonacci numbers in the diagonal and the rest of the entries are generalized Fibonacci numbers. In particular, they studied the Cholesky factorizations and the eigenvalues of these matrices. In 2008 Stanimirović et al. [19] studied the inverses of generalized Fibonacci and Lucas matrices.

In this paper, we study the nature of symmetric and non-symmetric matrices embedded within the Hosoya triangle (matrices with products of Fibonacci numbers as entries). We use well-known results in linear algebra as tools to explore different patterns within this triangle. We present four infinite families of matrices classifying them as rank one matrices that are symmetric and non-symmetric, skew-triangular matrices, and antidiagonal matrices where the eigenvalues and eigenvectors of the matrices in each family satisfy a "closure property" in the set of Fibonacci numbers and partially in the Hosoya triangle.

The primary results of this paper involve the first family of matrices (matrices of rank one). These matrices have the property that they are products of two vectors $\mathbf{u}$ and $\mathbf{v}^T$. The entries of the vectors are consecutive Fibonacci numbers —in fact, the vectors $\mathbf{u}$ and $\mathbf{v}$ are located on the sides of the Hosoya triangle. Some of the rank one matrices are symmetric and the others are persymmetric matrices.

Several of the results in this paper were found by the first author while working on his undergraduate research project under the guidance of the second and third authors (who followed the guidelines given in [6]).

We connect the first family of matrices with graph theory by observing the fractal generated by taking the entries of the matrices mod 2. Therefore, the matrices of our first family give rise to the adjacency matrices of undirected and non-connected graphs.

The second family satisfies the property that the non-zero eigenvalues are Fibonacci numbers convolved with themselves (see [3, 16] or [18] at A001629). The matrices of the third family have the feature that the product of their eigenvalues (that are not necessarily integers) is a product of Fibonacci numbers.

2. Background and basic definitions

The *Hosoya triangle*, denoted by $\mathcal{H}$, is a triangular array where the entry in position k (taken from left to right) of the rth row is equal to $H_{r,k} := F_k F_{r-k+1}$, where $1 \le k \le r$ (see Table 1 and Figure 1). The classic definition of Hosoya triangle, tables, and other results related to Hosoya triangle can be found in [5, 7, 12, 13, 14] and [18] at A058071.

$$\begin{array}{ccccccccccccc}
 & & & & & & H_{1,1} & & & & & & \\
 & & & & & H_{2,1} & & H_{2,2} & & & & & \\
 & & & & H_{3,1} & & H_{3,2} & & H_{3,3} & & & & \\
 & & & H_{4,1} & & H_{4,2} & & H_{4,3} & & H_{4,4} & & & \\
 & & H_{5,1} & & H_{5,2} & & H_{5,3} & & H_{5,4} & & H_{5,5} & & \\
 & H_{6,1} & & H_{6,2} & & H_{6,3} & & H_{6,4} & & H_{6,5} & & H_{6,6} & \\
H_{7,1} & & H_{7,2} & & H_{7,3} & & H_{7,4} & & H_{7,5} & & H_{7,6} & & H_{7,7}
\end{array}$$

Table 1. Hosoya triangle $\mathcal{H}$.

An nth *diagonal* in Hosoya's triangle is the collection of all Fibonacci numbers multiplied by F_n. The following two sets are called *slash diagonal* and *backslash diagonal*, respectively,

$$\{H_{n+i-1,n}\}_{i=1}^{\infty} = \{F_i F_n | i \in \mathbb{N}\} \text{ and } \{H_{m+i-1,i}\}_{i=1}^{\infty} = \{F_m F_i | i \in \mathbb{N}\}.$$

We define matrices using the backslash diagonals of $\mathcal{H}$. For m, n, and t, all positive integers with $m, t \le n$, we define in (1) the square $t \times t$ *slash matrix* $B(m, n, t)$ with upper left hand corner at position (m, n) in the Hosoya triangle (the *backslash matrix* is defined similarly). Let $s = (t - 1)$ and $r_i = (m + n - 1) + i$ for $i = 0, 1, 2, \ldots, s$, then

$$B(m,n,t) = \begin{bmatrix} H_{r_0,m} & H_{r_0-1,m} & H_{r_0-2,m} & \cdots & H_{r_0-s,m} \\ H_{r_1,m+1} & H_{r_1-1,m+1} & H_{r_1-2,m+1} & \cdots & H_{r_1-s,m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{r_s,m+s} & H_{r_s-1,m+s} & H_{r_s-2,m+s} & \cdots & H_{r_s-s,m+s} \end{bmatrix}. \tag{1}$$

For example, Figure 1 Part (a) and Part (b) depicts $B(3, 7, 5)$ and $B(1, 7, 7)$, respectively. Note that the first entry of $B(m, n, t)$ is the point in the intersection of the m-th backslash diagonal and the n-th slash diagonal. In particular, the entry (point) in position $(1, 1)$ of $B(3, 7, 5)$ (which is represented by $H_{r_0,m}$) can be determined by writing $r_0 = 9$ and $m = 3$ and using that $H_{r,k} = F_k F_{r-k+1}$. Therefore, $H_{9,3} = F_3 F_7 = 26$. This technique may be used to find all entries of the matrix.

A particular case of the matrices $B(m, n, t)$ when $m = 1$ and $t = n$ are *persymmetric matrices* (matrices that are symmetric with respect to the antidiagonal), we denote these matrices simply by $B(n)$ (or for brevity by B). Therefore, persymmetric matrices $B(n)$ are square matrices in the Hosoya triangle $\mathcal{H}$, that are symmetric along the antidiagonal (see Figure 1(b) below).

Throughout the paper we use $\mathrm{tr}(A)$ to represent the trace of a square matrix A while A^T represents the transpose of the square matrix A.

3. Non-symmetric matrices of rank one in the Hosoya triangle

In this section, we study the first family where every matrix is of rank one (matrices that are products of two vectors) and present one of the main results of this paper. In this result, we give a closed formula for the trace of these matrices. We also present a result on the eigenvectors and the eigenvalues of matrices found in the Hosoya triangle.

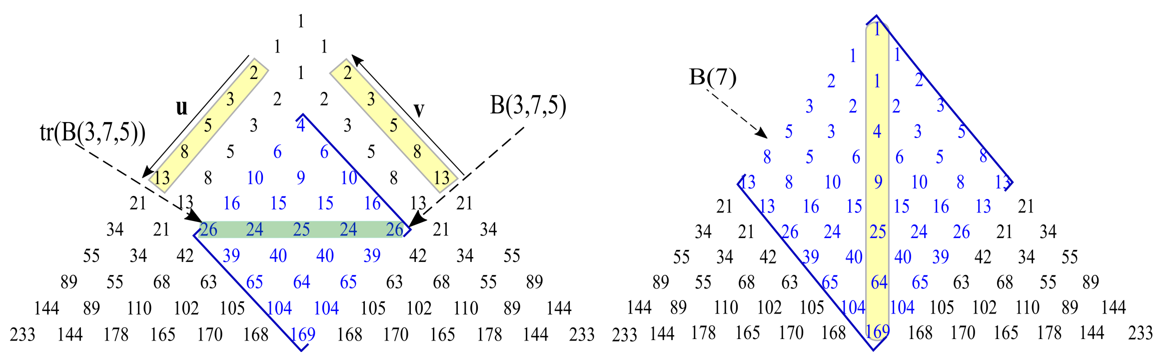


Figure 1. (a) $B(3,7,5)$ in the Hosoya triangle (b) $B(7)$ in the Hosoya triangle.

In this part of the section, we describe some properties of diagonalizable matrices in the Hosoya triangle. In addition to the linear algebraic properties in this section, we present an identity in Corollary 3 below that generalizes the convolutions given in [3, 16] and [18, A06733].

Lemma 1. *Let $m, n,$ and t be fixed positive integers. If L_k is the kth Lucas number, then the following hold:*

(a) $\displaystyle\sum_{i=0}^{t} F_{m+i}F_{n-i} = \frac{(t+1)L_{m+n} - \sum_{i=0}^{t}(-1)^{n-i}L_{m-n+2i}}{5},$

(b) $\displaystyle\sum_{i=0}^{t-1}(-1)^{n-i-1}L_{m-n+2i} = (-1)^{n-t}F_{m-n+2t-1} + (-1)^{m-1}F_{n-m+1}.$

Proof. To prove Part (a), we simplify the expression $F_{m+i}F_{n-i}$ using the Binet formula for Fibonacci numbers [13] to obtain

$$\sum_{i=0}^{t} F_{m+i}F_{n-i} = \frac{(t+1)L_{m+n} - \sum_{i=0}^{t}(-1)^{n-i}L_{m-n+2i}}{5}.$$

This completes the proof of Part (a).

The proof of Part (b) follows easily using mathematical induction on t, and we omit it. □

Proposition 2. *If m, n, t are fixed positive integers with $m, t \leq n$ and L_k represents Lucas numbers for $k \geq 0$, then $\mathrm{tr}(B(m, n, t))$ is given by*

$$\mathrm{tr}(B(m,n,t)) = \frac{tL_{m+n} + (-1)^{n-t}F_{m-n+2t-1} + (-1)^{m-1}F_{n-m+1}}{5}.$$

Proof. From (1) above we have that $\operatorname{tr}(B(m,n,t)) = \sum_{i=0}^{t-1} H_{r_i-i,m+i}$, where $r_i = (m+n-1)+i$ for $i = 0, 1, \ldots, (t-1)$. Since $H_{r,k} = F_k F_{r-k+1}$, we have $\operatorname{tr}(B(m,n,t)) = \sum_{i=0}^{t-1} F_{m+i} F_{n-i}$. The conclusion follows from Lemma 1. □

Corollary 3. *If $m, n,$ and t are fixed positive integers, then*

$$\sum_{i=0}^{t-1} F_{m+i}F_{n-i} = \frac{tL_{m+n} + (-1)^{n-t}F_{m-n+2t-1} + (-1)^{m-1}F_{n-m+1}}{5}.$$

The following Lemma is an easy exercise in linear algebra.

Lemma 4. *Let $A = \mathbf{u} \cdot \mathbf{v}^T$ for non-zero column vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^t$. Then the following hold:*

(a) A is of rank 1.
(b) The characteristic equation of A is given by $x^{t-1}(x - \operatorname{tr}(A)) = 0$.
(c) $\mathbf{u}$ is the eigenvector associated with $\operatorname{tr}(A)$.

Note that $\mathbf{u}$ is not orthogonal to $\mathbf{v}$. (A proof of Proposition 5 is straightforward geometrically with the following ideas.) From linear algebra we know that a matrix A is diagonalizable $\iff$ $\mathbf{v}^T\mathbf{u} \neq 0 \iff \mathbf{u}$ is not orthogonal to $\mathbf{v}$. The orthogonal complement of $\mathbf{v}$ is a basis of the null space. In particular, if W is the set of eigenvectors of $B(m,n,t)$ given by $\{\mathbf{u}, \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{t-1}}\}$ then $\mathbf{v_i}$ is orthogonal to $\mathbf{u}$ for all i.

Proposition 5 is a special case of the results in Lemma 4, however we provide an algebraic proof for it below. Now we establish certain notations needed to prove Proposition 5.

Proposition 5. *Let $B' = B(m,n,t)$ be a backslash matrix embedded in the Hosoya triangle $\mathcal{H}$. Then the following hold:*

(1) the eigenvalues of B' are $\lambda_1 = \operatorname{tr}(B')$ and $\lambda_2 = 0$ with algebraic multiplicity 1 and $(t-1)$ respectively.
(2) The matrix B' is diagonalizable and if $s = t-1$, then the eigenvectors of B' are given by,

$$\mathbf{u} = \begin{bmatrix} F_m \\ F_{m+1} \\ F_{m+2} \\ \vdots \\ F_{m+s} \end{bmatrix}, \mathbf{v_1} = \begin{bmatrix} -F_{n-1} \\ F_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{v_s} = \begin{bmatrix} -F_{n-s} \\ 0 \\ 0 \\ \vdots \\ F_n \end{bmatrix}.$$

Proof. By the definition of B' and $H_{r,k} = F_k F_{r-k+1}$ we have

$$B' = \begin{bmatrix} F_mF_n & F_mF_{n-1} & F_mF_{n-2} & \cdots & F_mF_{n-s} \\ F_{m+1}F_n & F_{m+1}F_{n-1} & F_{m+1}F_{n-2} & \cdots & F_{m+1}F_{n-s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{m+s}F_n & F_{m+s}F_{n-1} & F_{m+s}F_{n-2} & \cdots & F_{m+s}F_{n-s} \end{bmatrix}.$$

Now from Lemma 4 we have that $B' = \mathbf{u}\mathbf{v}^T$ where $\mathbf{u}$ is as given in the statement of Part (2) and $\mathbf{v}^T = [F_n, F_{n-1}, F_{n-2}, \ldots, F_{n-s}]$. So the eigenvalues are $\lambda_1 = \operatorname{tr}(B')$ and $\lambda_2 = 0$ with algebraic multiplicity 1 and $(t-1)$ respectively.

Proof of Part (2). It is clear that $\mathbf{u}$ is an eigenvector associated with $\lambda = \operatorname{tr}(B')$ (see the discussion above). In order to find the eigenvectors associated with $\lambda = 0$, it is enough to find a basis for the null space of B'. Note that in the following part we use $E_t(k)$ to represent the elementary matrix obtained by multiplying row t of the identity matrix with a constant $k \neq 0$. Since

$$E_s\left(\frac{1}{F_{m+s}}\right)E_{s-1}\left(\frac{1}{F_{m+s-1}}\right)\cdots E_2\left(\frac{1}{F_{m+1}}\right)E_1\left(\frac{1}{F_m}\right)B'$$

is equal to the matrix

$$\begin{bmatrix} F_n & F_{n-1} & F_{n-2} & \cdots & F_{n-s} \\ F_n & F_{n-1} & F_{n-2} & \cdots & F_{n-s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_n & F_{n-1} & F_{n-2} & \cdots & F_{n-s} \end{bmatrix},$$

it is easy to see that $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_s}\}$ is a basis for the null space of B'. This completes the proof of Part (2). □

Notice, in general, if in the Hosoya triangle $\mathcal{H}$, instead of $H_{1,1} = H_{2,1} = H_{2,2} = H_{3,1} = 1$, we consider $H_{1,1} = a^2; H_{2,1} = ab; H_{2,2} = ab; H_{3,2} = b^2$ with $a, b \in \mathbb{Z}$ (see examples in [18], A284115, A284129, A284126, A284130, A284127, A284131, A284128) then the matrix $B(m, n, t)$ defined over this general recursive sequence satisfies the following properties: first $B(m, n, t)$ has rank 1; second, there are vectors $\mathbf{u}$ and $\mathbf{v}$ from the sides of the general triangle such that $B(m, n, t) = \mathbf{u} \cdot \mathbf{v}^T$ and third, that the only non-zero eigenvalue of the matrix is given by the trace $\operatorname{tr}(B(m, n, t))$ which is equal to $\mathbf{v}^T \cdot \mathbf{u}$ (similar to the vectors $\mathbf{u}$ and $\mathbf{v}$ shown in Figure 1(a)).

Recall that $B(n)$ or B represent the persymmetric matrices defined in Section 2. If $\mathbf{u}$, $\mathbf{v_i}$ for $1 \leq i \leq (n-1)$, are the eigenvectors of B (as given in Proposition 5 Part (2)), then the matrix of eigenvectors of B is

$$Q_n = [\mathbf{u}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}]. \tag{2}$$

If $\operatorname{tr}(B)$ is the non-zero eigenvalue associated to $\mathbf{u}$ (recall that the eigenvalues associated to $\mathbf{v_i}$ for $1 \leq i \leq (n-1)$ are all zero) and $T_n = [t_{ij}]$ with $t_{11} = 1$ as the only non-zero entry, then the diagonal matrix of B is given by $D_n = \operatorname{tr}(B)T_n$.

Corollary 6. *If Q_n, B, and T_n are as described above, then for $k > 0$, the following holds:*

$$B^k Q_n = Q_n\left(\frac{nL_{n+1} + 2F_n}{5}\right)^k T_n.$$

Proof. From Proposition 5 Part (2) we know that B is diagonalizable. This and (2) imply that $B = Q_n D_n Q_n^{-1}$. Therefore $B^k = Q_n D_n^k Q_n^{-1}$. This in turn implies that $B^k Q_n = Q_n(\operatorname{tr}(B))^k T_n$. To complete the proof we recall from [3, 16] that

$$\operatorname{tr}(B) = \sum_{i=1}^{n} F_{n-i}F_i = \frac{nL_{n+1} + 2F_n}{5}.$$

□

We also observe that by using elementary row operations to compute the determinant of matrix $Q_t = [\mathbf{u}, \mathbf{v_1}, \ldots, \mathbf{v_s}]$ comprising of all eigenvectors of $B(m, n, t)$ where $s = t - 1$, we obtain

$$\det(Q_t) = F_n^{t-2}\sum_{i=0}^{s} F_{m+i}F_{n+i}.$$

In other words, $\det(Q_t) = F_n^{t-2}\operatorname{tr}(B(m, n, t))$.

4. Symmetric matrices of rank one in the Hosoya triangle

In this section, we study a family of symmetric matrices where every matrix is of rank one. We first define the matrices in $\mathcal{H}$ and then present some results. In the first of these results, we give a closed formula for the trace of these matrices. We also present a result on the eigenvectors and the eigenvalues of these symmetric matrices found in the Hosoya triangle.

For positive integer n, we define in (3) the square matrix S of order n in the following way:

$$S = \begin{bmatrix} H_{1,1} & H_{2,1} & H_{3,1} & \cdots & H_{n,1} \\ H_{2,2} & H_{3,2} & H_{4,2} & \cdots & H_{n+1,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n,n} & H_{n+1,n} & H_{n+2,n} & \cdots & H_{2n-1,n} \end{bmatrix}. \tag{3}$$

Proposition 7. *If n is a positive integer and S is the matrix defined in (3), then* $\operatorname{tr}(S) = F_n F_{n+1}$.

Proof. Observe that the $\operatorname{tr}(S) = H_{1,1} + H_{3,2} + H_{4,3} + \cdots + H_{2n-1,n}$. By the definition of the entries of the Hosoya triangle $H_{r,k} = F_k F_{r-k+1}$, we have that $\operatorname{tr}(S) = F_1^2 + F_2^2 + \cdots + F_n^2$. Finally, using a well-known identity on finite sums of squares of Fibonacci numbers (see [13]), we have that $\operatorname{tr}(S(n)) = F_n F_{n+1}$. □

Proposition 8. *Let S be the symmetric matrix of order n in the Hosoya triangle $\mathcal{H}$ (see (3)). Then the following hold:*

(1) the eigenvalues of S are $\lambda_1 = \operatorname{tr}(S)$ and $\lambda_2 = 0$ with algebraic multiplicity 1 and $(n-1)$ respectively.

(2) The matrix S is diagonalizable and the eigenvectors of S are given by,

$$\mathbf{u} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{bmatrix}, \mathbf{v_1} = \begin{bmatrix} -F_{n-1} \\ F_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -F_{n-2} \\ 0 \\ F_n \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{v_{n-1}} = \begin{bmatrix} -F_2 \\ 0 \\ 0 \\ \vdots \\ F_n \end{bmatrix}.$$

Proof. The proof of Part (1) follows directly from Lemma 4.

For the proof of Part (2), we first observe the vector $\mathbf{u}$ described in the statement is the eigenvector associated with the eigenvalue $\lambda = \operatorname{tr}(S)$ (see proof of Proposition 5 Part (2)). In order to complete the proof, it is enough to find a basis for the null-space of the matrix. Following the steps as seen in Proposition 5 Part (2), we obtain the basis $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_{n-1}}\}$ for the null space of S. This completes the proof of Part(2). □

If $\mathbf{u}$, $\mathbf{v_i}$ for $i = 1, \dots, n-1$ are the eigenvectors of S (as given in Proposition 8 Part (2)), then the matrix of eigenvectors of S is

$$Q'_n = [\mathbf{u}, \mathbf{v_1}, \dots, \mathbf{v_{n-1}}]. \tag{4}$$

If $\operatorname{tr}(S)$ is the non-zero eigenvalue associated to $\mathbf{u}$ (recall that the eigenvalues associated to $\mathbf{v_i}$ for $i = 1, \dots, n-1$ are all zero) and $T_n = [t_{ij}]$ with $t_{11} = 1$ as the only non-zero entry, then the diagonal matrix of S is given by $D'_n = \operatorname{tr}(S) T'_n$.

Corollary 9. *If Q'_n, S, and T'_n are as described above, then for $k > 0$ the following holds:*

$$S^k Q'_n = Q'_n \left(F_n F_{n+1}\right)^k T'_n.$$

The proof of this corollary is very similar to the proof of Corollary 6.

5. Norm of matrices

Suppose that B is the persymmetric matrix $B(n)$ as defined in Section 2. In this section we present some results on the norm of matrix B.

From linear algebra we know that if the matrix A has rank one, the spectral radius of the matrix A is given by $\rho(A) = \lambda$ where λ is the only non-zero eigenvalue of A. If A is rank one then it is also true that $A^T A$ is of rank one. Therefore, the spectral norm of any matrix A is given by $||A||_2 = \sqrt{\rho(A^T A)} = \sqrt{\lambda}$ where $\rho(A^T A)$ is the spectral radius of $A^T A$ and λ is the only non-zero eigenvalue of $A^T A$ (see [10, 20]). Using this property of rank one matrices, we present a result on the spectral norm of the persymmetric matrix B in Proposition 10.

Another interpretation of the norm of a matrix is given by recalling that the matrix norm of A measures how much a vector X can be extended by applying matrix A on it. Applying these concepts to the matrices in the Hosoya triangle, we obtain that the norm is again a point within the triangle. So, this norm provides a good geometric interpretation in the Hosoya triangle of two well-known Fibonacci identities. One can refer back to Figures 1(b) and 2(b) to explore the geometric significance of the value of those norms.

Proposition 10. *Let B be a persymmentric matrix in $\mathcal{H}$. If $A = B^T B$, then*

(1) A has exactly one non-zero eigenvalue λ.

(2) If λ is the non-zero eigenvalue of A, then λ is the product of the sum of antidiagonal elements of B with the sum of antidiagonal elements of B^T. Thus, $\lambda = \left(\sum_{i=1}^{n} H_{2i+1,i}\right)^2 = (F_n F_{n+1})^2$.

(3) The eigenvalue $\lambda = \operatorname{tr}(A) = \sum_{i,j=1}^{n} (F_i F_j)^2 = (F_n F_{n+1})^2$.

(4) If λ is the non-zero eigenvalue of A, then

$$\sqrt{\lambda} = ||B||_2 = F_n F_{n+1} = \sqrt{\sum_{i,j=1}^{n} (F_i F_j)^2} = \sum_{i=1}^{n} F_i^2.$$

(5) $\sqrt{\sum_{i,j=1}^{n} F_i F_j} = \sum_{i=1}^{n} F_i = F_{n+2} - 1 = ||B||_\infty / F_n$.

(6) $\sqrt{\lambda} = H_{2n,n}$.

Proof. The proof of Part (1) is straightforward since the rank of matrix A is 1.

Proof of Part (2). We know that $A = B^T B = (\mathbf{u}\cdot\mathbf{v}^T)^T(\mathbf{u}\cdot\mathbf{v}^T) = \mathbf{v}\cdot(\mathbf{u}^T\cdot\mathbf{u})\cdot\mathbf{v}^T$. Since $(\mathbf{u}^T\cdot\mathbf{u})$ is a real number we have $A = (\mathbf{u}^T\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{v}^T$. Therefore, the eigenvalues of A are actually the eigenvalues of $\mathbf{v}\cdot\mathbf{v}^T$ multiplied by $\mathbf{u}^T\cdot\mathbf{u}$. We know that the non-zero eigenvalue of $\mathbf{v}\cdot\mathbf{v}^T$ is given by $\operatorname{tr}(\mathbf{v}\cdot\mathbf{v}^T)$.

Since

$$\mathbf{u}^T\cdot\mathbf{u} = [F_n\, F_{n-1}\, \cdots\, F_2\, F_1]\begin{bmatrix} F_n \\ F_{n-1} \\ F_{n-2} \\ \vdots \\ F_1 \end{bmatrix} = \sum_{i=1}^{n} F_i^2 = F_n F_{n+1} \text{ and } \operatorname{tr}(\mathbf{v}\cdot\mathbf{v}^T) = \sum_{i=1}^{n} F_i^2 = F_n F_{n+1},$$

we have that the eigenvalue λ of A is $(F_n F_{n+1})^2$. This completes the proof of Part (2).

The proof of Part (3) is easy using linear algebra techniques and identities involving Fibonacci numbers.

Since A is a normal matrix, the proof of Part (4) follows from the Schur inequality [20] (an alternative proof can be found using parts (2) and (3) or basic algebra).

The proofs of Parts (5) and (6) are straightforward using linear algebra techniques and identities on Fibonacci numbers, therefore we omit those details. □

There are some geometrical interpretation of Proposition 10. Parts (6) and (4) tell us that adding the square of every entry of the matrix gives the sum of all points in the antidiagonal which is a point of the Hosoya triangle. Part (5) tells that the sum of all entries in the matrix is the difference of two points in the Hosoya triangle. Part (6) also shows that the singular value of B (see [10]) is again an entry of the Hosoya triangle.

Notice that if $C = B \circ B$ is the Hadamard product [11] of the matrix B with itself, then the only non-zero eigenvalue of C is $\lambda = \operatorname{tr}(C)$.

6. Graphs associated with the Hosoya triangle

Several authors have been interested in graphs generated by considering the Pascal triangle entries $\bmod 2$. The first example is the well-known Sierpiński triangle. Other examples can be found in Koshy [14, Chapter 9]. There he discussed Pascal graphs in the Pascal Binary Triangle (see also [4, 2, 17]). We explore similar properties for the family of rank one persymmetric matrices in $\mathcal{H}$.

6.1. Adjacency graphs of persymmetric matrices in the Hosoya triangle.

We consider the adjacency matrix constructed by taking each entry of the persymmetric matrix $B(n)$ modulo 2 where $n \equiv 2 \bmod 3$. This gives rise to a family of adjacency matrices of undirected and non-connected graphs. The graphs are composed of a complete graph with loops attached to each of its vertices as a component, and the other components are some isolated vertices (see Table 2).

Proposition 11. *If $k \geq 0$ and $n = 3k+2$, then the graph of the adjacency matrix corresponding to $B(n) \bmod 2$ is a complete graph on $2(k+1)$ vertices with loops at every vertex and k isolated vertices.*

Proof. It is known that the Fibonacci number $F_n \equiv 0 \bmod 2 \iff 3 \mid n$. This and the definition of $B(n)$ imply that every third row and every third column of $B(n)$ are formed by even numbers and that the remaining rows and columns are formed by odd numbers only. Thus, if b_{ij} is an entry of $B(n)$, then $b_{ij} \equiv 0 \bmod 2 \iff i \equiv 0 \mod 3$ or $j \equiv 0 \mod 3$. This and $n = 3k+2$ imply that $B(n) \bmod 2$ contains k columns and k rows with zeros as entries. The remaining 2(k+1) rows and columns have ones as entries. These two features of $B(n) \bmod 2$ give us a complete graph on $2(k+1)$ vertices with loops at every vertex and k isolated vertices. This completes the proof. □

7. Antidiagonal matrices and skew-triangular matrices in the Hosoya triangle

In this section, we study the family of antidiagonal matrices where the entries of the antidiagonal are points from the "median" of $\mathcal{H}$ (see Figure 2(a)). Let A be an $n \times n$ matrix in this family. We prove that the eigenvalues of A are again entries of $\mathcal{H}$ as well as the entries of its eigenvectors (except maybe by the sign of those entries). The eigenvectors of A form the rows of a new square matrix E where non-zero entries of E are in the diagonal and antidiagonal. The diagonal of E is formed by all points in a horizontal line of $\mathcal{H}$, while the antidiagonal of E is the same antidiagonal of A seen in Figure 2(b). Note that every first entry of a row of A is located in the nth backslash diagonal of $\mathcal{H}$, while every first entry of a row of E is located in the first backslash diagonal of $\mathcal{H}$.

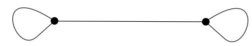
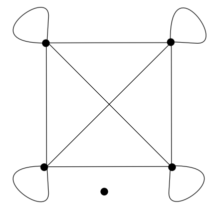
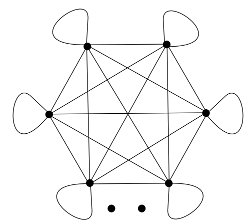
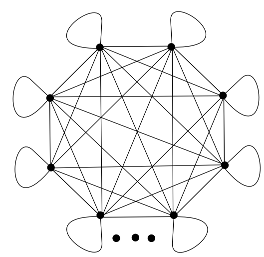
$3k+2$	Matrix	Graph
2	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	
5	$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$	
8	$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$	
11	$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$	

Table 2. Adjacency Graphs of $B(n)$ mod 2 in $\mathcal{H}$.

The matrix E can be seen geometrically as a cross in $\mathcal{H}$ where the only non-zero entries of E are the first n entries of the "median" of $\mathcal{H}$ and the entries of the nth row of $\mathcal{H}$. However, some eigenvectors of A have negative entries, but in $\mathcal{H}$ all entries are positive, so our representation is not a perfect geometric representation. Therefore, to have a good geometric representation of the eigenvectors of A we introduce a convention (only for this type of eigenvector). We are going to assume that negative entries of the eigenvector of A are represented by points on the left side of the "median" of $\mathcal{H}$.

7.1. **Eigenvalues of antidiagonal matrices.** In this section, we discuss properties of the second family of matrices mentioned in Section 1. This family includes antidiagonal matrices obtained from persymmetric matrices defined in the following way:

$$A = [a_{ij}]_{1 \le i,j \le n}, \qquad \text{where} \qquad a_{ij} = \begin{cases} F_i^2, & \text{if } j = n-i+1; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

The properties we describe for the antidiagonal matrix A involve finding the eigenvalues and eigenvectors. These results are true for antidiagonal matrices in general. We therefore first state and provide the proof of the general case.

Proposition 12. *If $A = [a_{i,j}]_{1 \le i,j \le n}$ is an antidiagonal matrix with the entries $a_{i,j} \in \mathbb{R}$, then the followiing results hold:*

(a) The eigenvalues of A are $\lambda_i = \pm\sqrt{a_{i,n-i+1}a_{n-i+1,i}}$ for $1 \le i \le n/2$ when $n = 2k$ and $1 \le i \le (n+1)/2$ when $n = 2k-1$.

(b) The eigenvectors of A for $n = 2k$ are given by $x_i = [\alpha_1, \alpha_2, \ldots, \alpha_{2k}]$ for $1 \le i,j \le k$ where

$$\alpha_t = \begin{cases} \pm\sqrt{a_{i,n+1-j}}, & \text{if } t = j \text{ and } j = i; \\ \pm\sqrt{a_{n+1-j,i}}, & \text{if } t = n+1-j \text{ and } i = j; \\ 0, & \text{if } i \ne j. \end{cases}$$

(c) The eigenvectors of A for $n = 2k-1$ are given by $y_i = [\beta_1, \beta_2, \ldots, \beta_{2k-1}]$ for $1 \le i,j \le k-1$ and $y_k = [0, \ldots, 0, 1, 0 \ldots, 0]$, where

$$\beta_t = \begin{cases} \pm\sqrt{a_{i,n+1-j}}, & \text{if } t = j \text{ and } j = i; \\ \pm\sqrt{a_{n+1-j,i}}, & \text{if } t = n+1-j \text{ and } i = j; \\ 0, & \text{if } i \ne j. \end{cases}$$

Proof. To prove Part (a) we note that for the antidiagonal matrix $A = [a_{i,j}]_{1 \le i,j \le n}$, the characteristic polynomial of A is given by $\det(A - xI_n) = 0$, where I_n is the identity matrix of order n. Since A is an antidiagonal matrix, simplifying the above equation we obtain:

$$p(x) = \prod_{i=1}^{n}(x^2 - a_{i,n}a_{n-i+1,i}), \text{ for all } n.$$

The proof of this part now follows directly.

We now prove Part (b). The eigenvector corresponding to the eigenvalue $\lambda = \sqrt{a_{i,n}a_{n-i+1,i}}$ for $1 \le i \le k$ can be found by solving the system of equations $(A - \sqrt{a_{i,n}a_{n-i+1,i}}\, I_n)\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ and $\mathbf{0}$ is the zero vector.

Using row operations on $(A - \sqrt{a_{i,n}a_{n-i+1,i}}\, I_n)$, the system of equations given above simplifies to $x_i - \sqrt{(a_{i,n}/a_{n-i+1,i})}\, x_{n-i+1} = 0$ and $x_j = 0$ when $i \ne j$. Therefore, the eigenvector corresponding to the eigenvalue $\sqrt{a_{i,n}a_{n-i+1,i}}$ is given by $[\alpha_1, \alpha_2, \ldots, \alpha_{2k}]$, where

$$\alpha_t = \begin{cases} \pm\sqrt{a_{i,n+1-j}}, & \text{if } t = j \text{ and } j = i; \\ \pm\sqrt{a_{n+1-j,i}}, & \text{if } t = n+1-j \text{ and } i = j; \\ 0, & \text{if } i \ne j. \end{cases}$$

The analysis for the eigenvector associated to $\lambda = -\sqrt{a_{i,n}a_{n-i+1,i}}$ is similar. This completes the proof of Part (b).

The proof of Part (c) is similar to the proof of Part (b), so we omit it. $\square$

The proof of the following corollary follows directly from the Proposition 12.

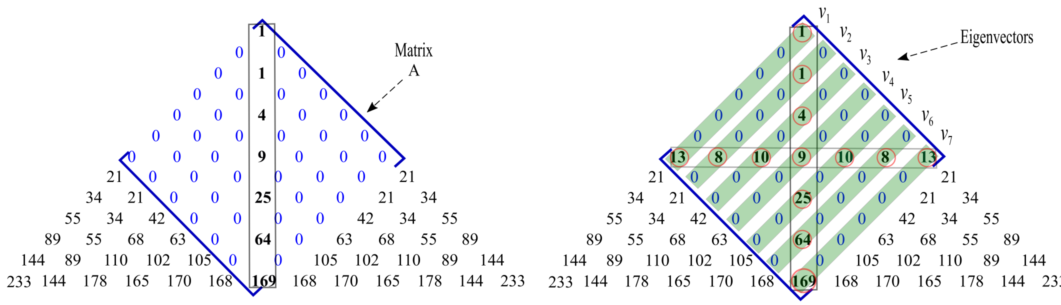


FIGURE 2. (a) Anti-diagonal Matrix A. (b) Eigenvectors of A.

Corollary 13. *If A is a matrix as defined in (5), then the following hold:*

(1) The rank of A equals n.
(2) If $n = 2k$ and $1 \le i \le k$, then the eigenvalues of A are

$$\lambda_j = \begin{cases} F_i F_{n-i+1}, & \text{if } j = i; \\ -F_i F_{n-i+1}, & \text{if } j = k + i. \end{cases}$$

(3) If $n = 2k - 1$ and $1 \le i \le k$, then the eigenvalues of A are

$$\lambda_j = \begin{cases} F_k^2 & \text{if } j = k; \\ F_i F_{n-i+1}, & \text{if } j = i; \\ -F_i F_{n-i+1}, & \text{if } j = k + i. \end{cases}$$

(4) If $n = 2k$ and $1 \le i \le k$, then the eigenvectors of A are $x_i = [\alpha_1, \alpha_2, \ldots, \alpha_n]$, where

$$\alpha_t = \begin{cases} \pm F_j, & \text{if } t = j \text{ and } j = i; \\ \pm F_{n-j+1}, & \text{if } t = n - j + 1 \text{ and } i = j; \\ 0, & \text{if } i \ne j. \end{cases}$$

(5) If $n = 2k - 1$ and $1 \le i \le k - 1$, then the eigenvectors of A are $y_i = [\beta_1, \beta_2, \ldots, \beta_n]$ and $y_k = [0, 0, \ldots, 1, 0 \ldots, 0]$, where

$$\beta_t = \begin{cases} \pm F_j, & \text{if } t = j \text{ and } j = i; \\ \pm F_{n-j}, & \text{if } t = n - j \text{ and } i = j; \\ 0, & \text{if } i \ne j. \end{cases}$$

Example. If we have the matrix A seen in Figure 2(a), then the eigenvalues of A are $\pm F_1 F_7$, $\pm F_2 F_6$, $\pm F_3 F_5$, and F_4^2. The eigenvectors of A are given by [0,0, 0,1,0, 0, 0] and

$$\begin{cases} [F_1^2, 0, 0, 0, 0, 0, F_7 F_1] \\ [F_1^2, 0, 0, 0, 0, 0, -F_7 F_1] \end{cases}, \begin{cases} [0, F_2^2, 0, 0, 0, F_6 F_2, 0] \\ [0, F_2^2, 0, 0, 0, -F_6 F_2, 0], \end{cases} \begin{cases} [0, 0, F_3^2, 0, F_5 F_3, 0, 0] \\ [0, 0, F_3^2, 0, -F_5 F_3, 0, 0]. \end{cases}$$

Comment: The matrix A is diagonalizable. In fact, when $n = 2k$, we have

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & F_1^2 \\ 0 & 0 & \cdots & F_2^2 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ F_n^2 & 0 & \cdots & 0 & 0 \end{bmatrix} P = P \begin{bmatrix} F_1F_{2k} & 0 & 0 & \cdots & 0 & 0 \\ 0 & -F_1F_{2k} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & F_kF_{k+1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & -F_kF_{k+1} \end{bmatrix}$$

where

$$P = \begin{bmatrix} F_1 & F_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & F_2 & F_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & F_k \\ 0 & 0 & 0 & 0 & \cdots & F_{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & F_{2k-1} & -F_{2k-1} & \cdots & 0 \\ F_{2k} & -F_{2k} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Similarly, it is easy to verify that A is diagonalizable when $n = 2k - 1$.

Another interesting property of A is that if

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & F_1^2 \\ 0 & 0 & \cdots & F_2^2 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ F_n^2 & 0 & \cdots & 0 & 0 \end{bmatrix}, \text{ then } A^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{1}{F_n^2} \\ 0 & 0 & \cdots & \frac{1}{F_{n-1}^2} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ \frac{1}{F_1^2} & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

7.2. **Determinants of skew-triangular matrices in the Hosoya triangle.** In this section we define skew-triangular matrices in $\mathcal{H}$. This family of matrices does not necessarily have integers as eigenvalues. So, we analyze their determinants to obtain some information about their eigenvalues. We also discuss some properties of the determinants (see the determinant in (7) on page 27). The determinant of an antidiagonal matrix is well-known in linear algebra. Here we use this tool to show that the determinant of a member of the subfamily of matrices with entries in $\mathcal{H}$ is a product of points of this triangle. The geometry of the triangle helps us to see these properties clearly. We now define the family $T(n, k)$ of skew-triangular matrices $\mathcal{H}$. If $k = 2, 3, \dots, n + 1$, then

$$T(n, k) = [a_{ij}]_{1 \le i,j \le n} \quad \text{where} \quad a_{ij} = \begin{cases} F_iF_{n-j+1}, & \text{if } k \le i + j \le n + 1; \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

From linear algebra, we know that if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of a matrix A, then the determinant of A is given by $\det(A) = \prod_{i=1}^{n} \lambda_i$ (see [20]). It is easy to verify that for a fixed n and $i \neq j \in \{2, \dots, n + 1\}$, the matrices $T(n, i)$ and $T(n, j)$ do not necessarily have the same eigenvalues (in most of the cases the eigenvalues are not integers). Proposition 14 shows that if n is fixed, the product of the eigenvalues of $T(n, k)$ is equal to the product of eigenvalues of $T(n, n + 1)$ for $1 < k \le n$. This is the product of points located in the "median" of $\mathcal{H}$. This result is true for any skew-triangular matrix in general and the proof is straightforward using cofactors or using Leibnitz's formula of the sum over all permutations of the numbers $1, 2, \dots, n$.

Proposition 14. *If $2 \le k \le n+1$, then for every $n \ge 2$ we have*

$$\det(T(n,k)) = \begin{cases} \prod_{i=1}^{n} F_i^2, & \text{if } n \equiv 0 \text{ or } 1 \bmod 4; \\ -\prod_{i=1}^{n} F_i^2, & \text{if } n \equiv 2 \text{ or } 3 \bmod 4. \end{cases}$$

Note that $\det(T(n,k)) = \det(A)$ where A is the $n \times n$ antidiagonal matrix defined in Section 7 in page 22. This result can be extended to matrices which have entries $a_{ij} = F_i F_{n-j+1}$ such that $n-k \le i+j \le n+1$ and k is either 1 or 2 or any positive integer less than $n-2$. For example, if $n = 4$ and $\det(A)$ is denoted by $|A|$ then it holds that

$$\begin{vmatrix} F_1F_4 & F_1F_3 & F_1F_2 & F_1^2 \\ F_2F_4 & F_2F_3 & F_2^2 & \\ F_3F_4 & F_3^2 & & \\ F_4^2 & & 0 & \end{vmatrix} = \begin{vmatrix} 0 & F_1F_3 & F_1F_2 & F_1^2 \\ F_2F_4 & F_2F_3 & F_2^2 & \\ F_3F_4 & F_3^2 & & \\ F_4^2 & & 0 & \end{vmatrix} = \cdots = \begin{vmatrix} 0 & 0 & 0 & F_1^2 \\ 0 & 0 & F_2^2 & \\ 0 & F_3^2 & & \\ F_4^2 & & 0 & \end{vmatrix} = \prod_{i=1}^{4} F_i^2. \tag{7}$$

8. Acknowledgement

This work is dedicated to Jaime Chica who many years ago imparted to the second author an understanding of the beauty of linear algebra. All the authors of this paper were partially supported by The Citadel Foundation. The authors would like to thank Dr. Mei Chen for the numerous insightful discussions they had with her on various topics from linear algebra; these discussions helped improve the paper. The authors are also thankful to the anonymous referee(s) and the editor for their helpful comments which made the paper better.

References

[1] B. A. Bondarenko, Obobshchennye treugol'niki i piramidy Paskalya, ikh fraktali, grafy i prilozheniya [Generalized Pascal triangles and pyramids, their fractals, graphs and applications] "Fan", Tashkent, 1990.

[2] B. Baker Swart, R. Flórez, D. A. Narayan, and G. L. Rudolph, Extrema property of the k-ranking of directed paths and cycles, *AKCE Int. J. Graphs Comb.*, **13** (2016), no. 1, 38–53.

[3] É. Czabarka, R. Flórez, and L. Junes, A discrete convolution on the generalized Hosoya triangle, *J. Integer Seq.*, **18** (2018), Article 15.1.6.

[4] N. Deo and M. J. Quinn, Pascal graphs and their properties, *The Fibonacci Quarterly*, **21** (1983), no. 3, 203–214.

[5] R. Flórez, R. Higuita, and L. Junes, GCD property of the generalized star of David in the generalized Hosoya triangle, *J. Integer Seq.*, **17** (2014), Article 14.3.6, 17 pp.

[6] R. Flórez and A. Mukherjee, Solving open problems with students as a first research experience, *Teach. Math. Its Appl.*, **10** (2017), 1–7.

[7] R. Flórez and L. Junes, GCD properties in Hosoya's triangle, *The Fibonacci Quarterly*, **50** (2012), 163–174.

[8] R. Flórez, R. Higuita, and A. Mukherjee, The Geometry of some Fibonacci identities in the Hosoya triangle. Submitted. https://arxiv.org/abs/1804.0248

[9] M. Griffiths, Fibonacci diagonals, *The Fibonacci Quarterly*, **10** (2011), no. 1, 51–56.

[10] G. H. Golub and C. F. Van Loan, *Matrix computations*, Fourth edition, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, MD, 2013

[11] R. A. Horn and C. R. Johnson, *Topics in matrix analysis. Corrected reprint of the 1991 original*, Cambridge University Press, 1994.

[12] H. Hosoya, Fibonacci Triangle, *The Fibonacci Quarterly*, **14** (1976), no. 3, 173–178.

[13] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.

[14] T. Koshy, *Triangular Arrays with Applications*, Oxford University Press, 2011.

[15] G. Lee, J. Kim, and S. Lee, Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices, *The Fibonacci Quarterly*, **40** (2002), no. 3, 203–211.

[16] P. Moree, Convoluted convolved Fibonacci numbers, *J. Integer Seq.*, **7** (2004), Article 04.2.2.

[17] D. Romik, Shortest paths in the tower of Hanoi graph and finite automata, *SIAM J. Discrete Math.*, **20** (2006), 610–622.

[18] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://oeis.org/.

[19] P. Stanimirović, J. Nikolov, and I. Stanimirović, Generalizations of Fibonacci and Lucas matrices, *Discrete Appl. Math.*, **156** (2008), no. 14, 2606–2619.

[20] F. Zhang, *Matrix theory. Basic results and techniques*, second edition, universitext, Springer, 2011.

2010 *Mathematics Subject Classification*: Primary 11B39, 15A03, 15A18; Secondary 11B83.
Keywords: Fibonacci number, Hosoya triangle, matrix, eigenvalue, and eigenvector.

Department of Mathematical Sciences, The Citadel, Charleston, SC, U.S.A.
E-mail address: mblair1@citadel.edu

Department of Mathematical Sciences, The Citadel, Charleston, SC, U.S.A.
E-mail address: rflorez1@citadel.edu

Department of Mathematical Sciences, The Citadel, Charleston, SC, U.S.A.
E-mail address: mukherjeea1@citadel.edu