HOW INTEGER SEQUENCES FIND THEIR WAY INTO AREAS OUTSIDE PURE MATHEMATICS

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Abstract. The integer sequences A190249, A190250 and A190251 of the On-Line Encyclopedia OEIS are defined by mathematical conditions related to the golden ratio Φ. The botanical study of phyllotaxis models plants such as pineapples or sunflowers and describes the angular position of the nth leaf, scale, or seed by the fractional part of $n\Phi$. Front numbers and back numbers are distinguished by turning the model such that the polar axis faces forward. This paper shows that A295085, the sequence of front numbers, is the intertwining of A190249 and A190251. Moreover, A190250 is the sequence of back numbers.

The main theorem states that any generalized Fibonacci sequence eventually is a sequence of front numbers.

1. Arithmetics of the integer and fractional part

For real numbers x, y the integer part $\lfloor x \rfloor$ is the greatest integer not exceeding x, and $\langle x \rangle$ denotes the fractional part $x - \lfloor x \rfloor$. Of course, $x = \lfloor x \rfloor + \langle x \rangle$. For an integer k we have

$$\lfloor k + x \rfloor = k + \lfloor x \rfloor,$$

i.e., the integer part is additive with respect to integers. However, it is not additive in general since

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \langle x \rangle + \langle y \rangle \rfloor. \tag{1}$$

The fractional part is invariant with respect to integers,

$$\langle k + x \rangle = \langle x \rangle, \tag{2}$$

and thus,

$$\langle kx \rangle = \langle k \langle x \rangle \rangle. \tag{3}$$

Furthermore,

$$\langle -x \rangle = 1 - \langle x \rangle. \tag{4}$$

The following implications are useful in the proof of the upcoming Theorem 4.1:

$$\langle x \rangle + \langle y \rangle < 1 \quad \text{implies} \quad \langle x + y \rangle = \langle x \rangle + \langle y \rangle \tag{5}$$

and

$$\langle x \rangle + \langle y \rangle > 1 \quad \text{implies} \quad 1 - \langle x + y \rangle = 1 - \langle x \rangle + 1 - \langle y \rangle \tag{6}$$

since, by (2), $\langle x + y \rangle = \langle \langle x \rangle + \langle y \rangle \rangle = \langle x \rangle + \langle y \rangle - \lfloor \langle x \rangle + \langle y \rangle \rfloor$.

2. The function A190248

The number $\Phi = (\sqrt{5} + 1)/2 \approx 1.618$ is the golden ratio. Note that

$$\Phi^2 = 1 + \Phi. \tag{7}$$

Applying (7) twice yields $\Phi^3 = \Phi(1 + \Phi) = \Phi + (1 + \Phi)$ and hence,

$$\Phi^3 = 1 + 2\Phi. \tag{8}$$

Let n be a positive integer and $f(n)$ the value at the n^{th} position of the sequence A190248 in [4] defined as

$$f(n) = \lfloor n\Phi + n\Phi^2 + n\Phi^3 \rfloor - \lfloor n\Phi \rfloor - \lfloor n\Phi^2 \rfloor - \lfloor n\Phi^3 \rfloor$$

and shown in Table 1. From (1) generalized to three terms we obtain

$$f(n) = \lfloor \langle n\Phi \rangle + \langle n\Phi^2 \rangle + \langle n\Phi^3 \rangle \rfloor.$$

Thus, $f(n) = \lfloor \langle n\Phi \rangle + \langle n(1+\Phi) \rangle + \langle n(1+2\Phi) \rangle \rfloor = \lfloor \langle n\Phi \rangle + \langle n\Phi \rangle + \langle 2n\Phi \rangle \rfloor$, where for the first equation we have used (7) and (8) and for the second equation we have used (2). Hence, by (3),

$$f(n) = \lfloor 2\langle n\Phi \rangle + \langle 2\langle n\Phi \rangle \rangle \rfloor. \tag{9}$$

As $\langle x \rangle < 1/2$ implies $\langle 2\langle x \rangle \rangle = 2\langle x \rangle$, and by (9), we have

$$\langle n\Phi \rangle < 1/2 \text{ implies } f(n) = \lfloor 4\langle n\Phi \rangle \rfloor. \tag{10}$$

As $\langle x \rangle > 1/2$ implies $\langle 2\langle x \rangle \rangle = 2\langle x \rangle - 1$, and by (9), we have

$$\langle n\Phi \rangle > 1/2 \text{ implies } f(n) = \lfloor 4\langle n\Phi \rangle \rfloor - 1. \tag{11}$$

Lemma 2.1. *Let $f(n)$ be the value of the n^{th} position of OEIS sequence A190248. Then*
$f(n) = 0 \Leftrightarrow \langle n\Phi \rangle < 1/4$,
$f(n) = 1 \Leftrightarrow 1/4 < \langle n\Phi \rangle < 3/4$,
$f(n) = 2 \Leftrightarrow \langle n\Phi \rangle > 3/4$.

Proof. By (10), the 1st and 2nd, and by (11), the 3rd and 4th of the following implications hold

$$\begin{aligned} \langle n\Phi \rangle < 1/4 \quad &\Rightarrow \quad f(n) = 0, \\ 1/4 < \langle n\Phi \rangle < 1/2 \quad &\Rightarrow \quad f(n) = 1, \\ 1/2 < \langle n\Phi \rangle < 3/4 \quad &\Rightarrow \quad f(n) = 1, \\ \langle n\Phi \rangle > 3/4 \quad &\Rightarrow \quad f(n) = 2. \end{aligned}$$

This covers all the cases and, thus, finishes the proof. □

The next section interprets Lemma 2.1 in areas outside pure mathematics.

3. The phyllotactic front

Chapter 11.5 in [1] is about phyllotaxis and discusses the role of $\langle n\Phi \rangle$ in the model of pineapples. It is considered as an angle of rotation. Thus, the intervals in Lemma 2.1 distinguish whether the angle in degree is smaller than 90°, between 90° and 270°, or larger than 270°.

Definition 3.1. *A positive integer n such that $1/4 < \langle n\Phi \rangle < 3/4$ is called a* back number*, otherwise it is a called a* front number*.*

The sequence of front numbers, 2, 3, 5, 8, 10, 11, 13, 16, 18, 21,. . . , is registered as A295085 in [4]. The front numbers provide masks that are useful for labeling plants for which the model of golden spiral phyllotaxis fits; see Figure 3 in [2].

The sequences A190249, A190250, A190251, are respectively defined as positions of 0, 1, 2 in A190248 as shown in Table 1.

Lemma 2.1 yields

Corollary 3.2. *The sequence of front numbers is the intertwining of A190249 and A190251. The sequence of back numbers coincides with A190250.*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(n)$	1	0	2	1	0	1	1	2	1	0	2	1	0	1	1	2
A190249	2	5	10	13	18	23	26	31	34	36	39	44	47	52	57	60
A190250	1	4	6	7	9	12	14	15	17	19	20	22	25	27	28	30
A190251	3	8	11	16	21	24	29	32	37	42	45	50	53	55	58	63

Table 1. Initial values of f and the position sequences of value 0 (A190249), value 1 (A190250), and value 2 (A190251).

4. Application to generalized Fibonacci sequences

A non-trivial sequence $(G_n)_{n\geq 0}$ of non-negative integers such that $G_n = G_{n-1} + G_{n-2}$ for all $n \geq 2$ is called a *generalized Fibonacci sequence*. Non-trivial means that the constant sequence of 0's is excluded. The Fibonacci sequence is determined from $F_0 = 0$ and $F_1 = 1$. It is useful to prefix $F_{-1} = 1$.

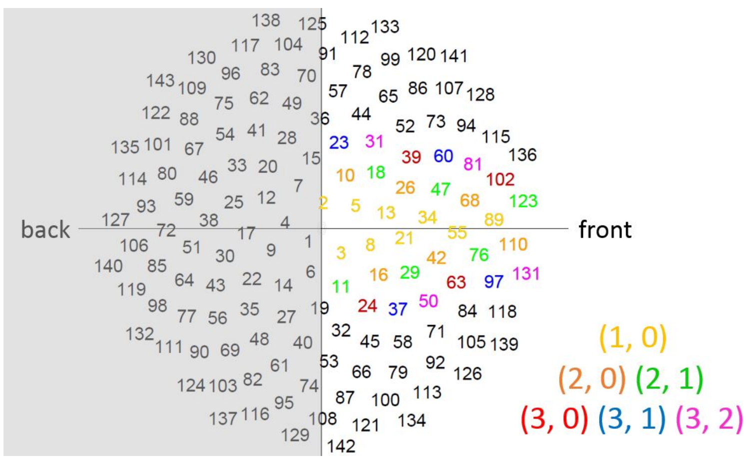

Figure 1. Labels $n = 1, 2, 3, \ldots$ spiral outward with angular coordinate $\langle n\Phi\rangle$ and radial coordinate $\sqrt{n}$. Back numbers are on the left side (grey area) and front numbers are on the right side. Colored front numbers belong to generalized Fibonacci sequences as indicated in the color-map-triangle where (k, l) represents the sequence $G_0 = k, G_1 = l$ with second order recurrence $G_i = G_{i-1} + G_{i-2}$ for $i \geq 2$.

According to Exercise 31 in [3], p. 85, for $n \geq 0$ we have

$$\langle F_n\Phi\rangle = \begin{cases} \psi^n & \text{if } n \text{ is odd,} \\ 1 - \psi^n & \text{if } n \text{ is even,} \end{cases} \tag{12}$$

where $\psi = 1/\Phi \approx 0.618$. Note that $\psi^3 < 1/4$. Thus, Fibonacci numbers F_n are front numbers for $n \geq 3$; see Figure 1.

The Lucas sequence is determined from $L_0 = 2$ and $L_1 = 1$. Lucas numbers $L_n \geq 11$ are front numbers; see Figure 1. We show that any generalized Fibonacci sequence is eventually a sequence of front numbers.

Theorem 4.1. *For a generalized Fibonacci sequence $(G_n)_{n\geq 0}$ there exists n_0 such that G_n is a front number for all $n \geq n_0$.*

Proof. It suffices to show

$$\lim_{n\to\infty} \delta(G_n\Phi) = 0 \tag{13}$$

where $\delta(x) = \min(\langle x\rangle, 1 - \langle x\rangle)$. We first give a decomposition in (15) and then proceed with various cases.

Equation (8) from the List of Formulae in the appendix of [5] states $G_n = G_0F_{n-1} + G_1F_n$ and, hence,

$$G_n = (G_0 - G_1)F_{n-1} + G_1F_{n+1}. \tag{14}$$

Without loss of generality, we assume $G_0 > G_1$ (otherwise, $G_1 - G_0$ could be prefixed). Put $a = G_0 - G_1$ and $b = G_1$ and note that $a > 0$. By (14),

$$G_n = aF_{n-1} + bF_{n+1}. \tag{15}$$

Case 1. Assume $b = 0$ and let n be large such that $\psi^{n-1} < 1/a$.

Case 1.1. Say n is even. We show

$$\langle G_n\Phi\rangle = a\psi^{n-1}.$$

By (15), $\langle G_n\Phi\rangle = \langle aF_{n-1}\Phi\rangle$ and it remains to show $\langle aF_{n-1}\Phi\rangle = a\psi^{n-1}$. By (3), $\langle aF_{n-1}\Phi\rangle = \langle a\langle F_{n-1}\Phi\rangle\rangle$ and, by (12), $\langle a\langle F_{n-1}\Phi\rangle\rangle = \langle a\psi^{n-1}\rangle$ and, finally, $\langle a\psi^{n-1}\rangle = a\psi^{n-1}$, since n is large such that $a\psi^{n-1} < 1$.

Case 1.2. Say n is odd. We show

$$1 - \langle G_n\Phi\rangle = a\psi^{n-1}.$$

By (15), (3), (12) and (2), $\langle G_n\Phi\rangle = \langle aF_{n-1}\Phi\rangle = \langle a\langle F_{n-1}\Phi\rangle\rangle = \langle a(1 - \psi^{n-1})\rangle = \langle -a\psi^{n-1}\rangle$ and it remains to show $1 - \langle -a\psi^{n-1}\rangle = a\psi^{n-1}$. By (4), $1 - \langle -a\psi^{n-1}\rangle = \langle a\psi^{n-1}\rangle$ and, finally, $\langle a\psi^{n-1}\rangle = a\psi^{n-1}$, since n is large such that $a\psi^{n-1} < 1$.

Thus, in both subsets we have $\delta(G_n\Phi) \le a\psi^{n-1}$ and the value converges to zero as n goes to infinity. This takes care of (13) for Case 1.

Case 2. Assume $b > 0$ and let n be large such that $\psi^{n-1} < 1/(a + b\psi^2)$.

We have $\psi^{n-1} < 1/a$ and $\psi^{n+1} < 1/b$.

Case 2.1. Say n is even. We show

$$\langle G_n\Phi\rangle = a\psi^{n-1} + b\psi^{n+1}.$$

As above, $\langle aF_{n-1}\Phi\rangle = a\psi^{n-1}$ and, similarly, $\langle bF_{n+1}\Phi\rangle = b\psi^{n+1}$. Thus, $\langle aF_{n-1}\Phi\rangle + \langle bF_{n+1}\Phi\rangle = a\psi^{n-1} + b\psi^{n+1} = \psi^{n-1}(a + b\psi^2) < 1$. By (15) and (5), $\langle G_n\Phi\rangle = \langle aF_{n-1}\Phi\rangle + \langle bF_{n+1}\Phi\rangle = a\psi^{n-1} + b\psi^{n+1}$.

Case 2.2. Say n is odd. We show

$$1 - \langle G_n\Phi\rangle = a\psi^{n-1} + b\psi^{n+1}.$$

As above, $1 - \langle aF_{n-1}\Phi\rangle = a\psi^{n-1}$ and, similarly, $1 - \langle bF_{n+1}\Phi\rangle = b\psi^{n+1}$. Thus, $\langle aF_{n-1}\Phi\rangle + \langle bF_{n+1}\Phi\rangle = 1 - a\psi^{n-1} + 1 - b\psi^{n+1} = 2 - \psi^{n-1}(a + b\psi^2) > 1$. By (15) and (6), $1 - \langle G_n\Phi\rangle = 1 - \langle aF_{n-1}\Phi\rangle + 1 - \langle bF_{n+1}\Phi\rangle = a\psi^{n-1} + b\psi^{n+1}$.

Thus, in both subsets we have $\delta(G_n\Phi) \le a\psi^{n-1} + b\psi^{n+1}$ and the value converges to zero. This takes care of (13) for Case 2.

In any case n_0 can be taken as the smallest n such that $a\psi^{n-1} + b\psi^{n+1} = \psi^{n-1}(a + b\psi^2) < 1/4$ or, equivalently, $\psi^{n-1} < 1/(4(a + b\psi^2))$. □

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