Problem Proposals

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These twelve problems were posed by participants of the Eighteenth International Conference on Fibonacci Numbers and Their Applications, held at Dalhousie University, Halifax, Nova Scotia. Part 1 gives the proposals, and Part 2 gives notes and solutions received before December 1, 2018.

Part 1. Proposals

Problem 1, posed by Marjorie Bicknell Johnson:
Two Fibonacci numbers in a Pythagorean triple.

Are $(3,4,5)$ and $(5,12,13)$ the only Pythagorean triples that contain two Fibonacci numbers?

Problem 2, posed by Burghard Herrmann:
A winner problem.

Let $\gamma \in (0,1/2)$ be a noble number, and consider the simple phyllotactic system with divergence angle γ turns and plastochrone distance $h \in \mathbb{R}^+$ in the sense of Turing [1]. Show that the first and second principal numbers (called "the winners") are adjacent elements of the generalized Fibonacci sequence associated with γ provided that h is sufficiently small.

Here are some details. To say that γ is *noble* means that for the continued fraction of γ, with partial quotients c_n, we have, for some $n_0 \geq 1$,

$$\gamma = [0, c_1, \ldots, c_{n_0}, \bar{1}], \quad c_{n_0} \geq 2.$$

The *generalized Fibonacci sequence* (G_n) is defined for $n \geq 1$ as follows: $G_n = q_{n+n_0-2}$ where the denominators of convergents are recursively defined via $q_0 = 1$, $q_1 = c_1$, and $q_n = q_{n-2} + c_n q_{n-1}$ for $n \geq 2$. The "winners" with respect to γ and h are defined by the minimal distance

$$\delta(n) = \min(\|(\{n\gamma\}, nh)\|, \|(1 - \{n\gamma\}, nh)\|)$$

where $\|(x,y)\| = \sqrt{x^2+y^2}$ and $\{\}$ denotes fractional part. The problem is to show that there exists $h_0 > 0$ such that for every $h \in (0, h_0)$ there exists $n \geq 1$ such that *the winners* are G_n and G_{n+1}, i.e., for every positive integer m except G_n and G_{n+1},

$$\delta(m) \geq \max(\delta(G_n), \delta(G_{n+1})).$$

References

[1] P. T. Saunders, *Collected Works of A. M. Turing, vol. 4, Morphogenesis*, Saunders, North-Holland, 1992.

Problem 3, posed by Russell Hendel:
Identities.

First some background. In the Halifax Fibonacci conference and these Proceedings, the idea of a Tagiuri-generated family of identities has been introduced. The identities of a Tagiuri-generated family may have an arbitrarily large number of summands. Moreover, they are necessarily valid because each arises from

i) a trivially true initial identity, such as $P = 3P - 2P$, by

ii) application of the Tagiuri identity substitution, and the Tagiuri identity is also true.

Therefore the proof of an identity in a Tagiuri-generated family is the one-line statement that the identity is true because it is Tagiuri-generated by substituting the true Tagiuri identity to a trivially true start identity. In contrast to this one-line proof, an arbitrary identity in several dozen summands may appear time-consuming to prove.

The open problem is as follows: produce a method so that, given an arbitrary Fibonacci identity in one variable, it is possible to ascertain whether it belongs to some infinite family of identities such as a Tagiuri-generated family.

Problem 4, posed by William Webb and Curtis Cooper:
High-degree "nice" Fibonacci identities.

Two particularly simple identities are the fourth degree identity

$$1 + F_{n-2}F_{n-1}F_{n+1}F_{n+2} - F_n^4 = 0$$

and the fifth degree identity

$$F_n^2 F_{n+5}^3 - F_{n+1}^3 F_{n+6}^2 + (-1)^n L_{n+3}^3 = 0.$$

Both identities, when written as $A = 0$, have only three terms in the expression A. Restricting ourselves to Fibonacci and Lucas numbers, let

$$S_1 = \{F_{an+b},\ (-1)^n,\ 1\}$$

and

$$S_2 = \{F_{an+b},\ L_{cn+d},\ (-1)^n,\ 1\}.$$

A *term* in an identity is a product of elements in S_1 or S_2. The parameters a, b, c, d can be any fixed integer values.

For our purposes, an *allowed expression* A is a nontrivial linear combination of terms with no common factor from S_1 or S_2 except 1.

1. Find degree $d \geq 5$ identities using S_1 with 3 terms.
2. Find degree $d \geq 6$ identities using S_2 with 3 terms.
3. For a given degree d, what is the least number of terms in an identity?
4. For what degrees d are there no identities with 3 terms?
5. Generalize to arbitrary second order recurrences or higher order recurrences.
6. Find a systematic method to find simple identities of this kind.

Problem 5, posed by Clark Kimberling:
All coefficients positive.

For $r > 0$, define the sequence (c_n) by

$$c_0 + c_1 x + c_2 x^2 + \cdots = \frac{1}{\lfloor r \rfloor + \lfloor 2r \rfloor x + \lfloor 3r \rfloor x^2 + \cdots}.$$

Prove or disprove:

1. If $r \in [3/2, 5/3)$, then $c_k > 0$ if and only if k is even.
2. If $r \in [3/2, 5/3)$, then $\lim_{n\to\infty} c_{k+1}/c_k$ exists.
3. A minimal case: let (d_k) be the sequence (c_k) in the case $r = 8/5$. Then $|d_k| \le |c_k|$ for all r in $[3/2, 5/3)$.

Problem 6, posed by J. C. Saunders:
Viswanath's constant.

Consider a random Fibonacci sequence (f_n) defined by $f_1 = f_2 = 1$, and for all $n \ge 3$ by $f_n = f_{n-1} \pm f_{n-2}$. Viswanath showed in 2000 that

$$\lim_{n\to\infty} |f_n|^{1/n} = 1.13198824\ldots$$

with probability 1; viz., $+$ or $-$ chosen at each step with probability of $1/2$ for each. Determine whether this constant is algebraic, and find a closed form for it.

References

[1] É. Janvresse, B. Rittaud, and T. De La Rue, *Almost-sure growth rate of generalized random Fibonacci sequences*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques **46.1**, Institut Henri Poincaré, 2010.

[2] K. A. McLellan, *Periodic Coefficients and Random Fibonacci Sequences*, Ph.D. thesis, Dalhousie University, 2012.

[3] D. Viswanath, *Random Fibonacci sequences and the number 1.13198824...*, Math. Comp., **69.231** (2000), 1131–1155.

Problem 7, posed by Larry Ericksen:
Two sequences.

Let n be a nonnegative integer. Define $s_2(n)$ to be the base 2 digital sum of n. That is, $s_2(0) = 0$, $s_2(1) = 1$, and for $n \ge 1$,

$$s_2(2n) = s_2(n) \quad \text{and} \quad s_2(2n+1) = s_2(n) + 1.$$

Define $a(n)$ to be the nth Stern number. That is, $a(0) = 0$, $a(1) = 1$, and for $n \ge 1$,

$$a(2n) = a(n) \quad \text{and} \quad a(2n+1) = a(n) + a(n+1).$$

(1) Prove that for nonnegative integers n, $a(n) \ge s_2(n)$.
(2) Prove that for nonnegative integers n, $a(n) = s_2(n)$ if and only if $n = 2^i - 2^j$ for $i \ge j \ge 0$.
(3) Let r be a nonnegative integer. Prove that

$$\max_{2^r \le n < 2^{r+1}} (a(n) - s_2(n)) = F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor,$$

where F_r is the rth Fibonacci number.

Problem 8, posed by Sam Northshield:
Enumerating some algebraic rationals.

For $\alpha = 2\cos(\frac{\pi}{N+1})$, and $x_0 = \infty$, let

$$x_n := \alpha^2 \left(2\nu_N(n) + 1 - \frac{1}{x_{n-1}} \right).$$

Then $\{x_n : n \geq 1\} = \mathbb{Q}(\alpha)^+$. The cases $N = 2, 3$, and 5 were covered in my talk (where $\alpha^2 = 1, 2, 3$ respectively). I have recently shown the $N = 4$ case (where $\alpha = \phi$). What about $N > 5$?

Problem 9, posed by Sam Northshield:
Zeros on the unit circle.

Some of the sequences (a_n, b_n, c_n, d_n) arising in my talk had respective generating functions satisfying:

$$\frac{A(x^2)}{A(x)} = 1 + x + x^2,$$
$$\frac{B(x^3)}{B(x)} = 1 + \sqrt{2}x + x^2 + \sqrt{2}x^3 + x^4,$$
$$\frac{C(x^5)}{C(x)} = 1 + \sqrt{3}x + 3x^2 + \sqrt{3}x^3 + x^4 + \sqrt{3}x^5 + 2x^6 + \sqrt{3}x^7 + x^8,$$
$$\frac{D(x^4)}{D(x)} = 1 + \phi x + \phi x^2 + x^3 + \phi x^4 + \phi x^5 + x^6.$$

The first two polynomials have zeros on the unit circle. Empirically, so does the third.

Problem: show that the last two polynomials have all their zeros on the unit circle.

Problem 10, posed by Sam Northshield:
Infinite Fibonacci word.

Define
$$\nu_F(n) = \lfloor (n+1)\phi \rfloor - \lfloor n\phi \rfloor - 1,$$
this being the Fibonacci word, OEIS sequence A005614 = $(1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, \ldots)$. Let $R_0 = \phi$, and
$$R_n = 2\nu_F(n) + 1 - \frac{1}{R_{n-1}}$$
for $n > 0$. For example, for $n = 1, 2, \ldots, 8$, the numbers R_n are
$$4 - \phi,\ (8 - \phi)/11,\ (8 - \phi)/5,\ (26 - \phi)/11,\ (34 - \phi)/59,\ (24 - \phi)/19,\ (6 - \phi)/29,\ -2 - \phi.$$
Let $s = \sqrt{5}$. Prove or disprove:

(1) Every R_n has the form $(2a - 1 - s)/(2b)$, where a and b are integers; and

(2) if b divides $a^2 + a - 1$, then $(2a + 1 - s)/(2b) = R_n$ for some n.

Problem 11, posed by Dale Gerdemann:
Multiples of Lucas numbers.

For every n,
$$5F_n = F_{n+3} + F_{n-1} + F_{n-4},$$
and there are similar formulae for other multiples of Fibonacci numbers, which can be found using the greedy algorithm. Similar formulae can be found for other Lucas sequences. For example, (with $a_0 = 0$ and $a_1 = 1$, here and elsewhere in this problem), the sequence A007482 given by $a_n = 3a_{n-1} + 2a_{n-1}$ satisfies the formula
$$5a_n = a_{n+1} + a_n + a_{n-1} + 2a_{n-2}$$
and the sequence A007070 given by $a_n = 4a_{n-1} - 2a_{n-1}$ satisfies the formula
$$5a_n = a_{n+1} + a_n + a_{n-1} + a_{n-2}.$$

However for $a_n = 2a_{n-1} + 4a_{n-1}$ (A063727), the greedy algorithm fails to yield a general formula for $5a_n$. For example,

$$a_{11} = a_{11+1} + a_{11+0} + 2a_{11-1} + a_{11-2} + a_{11-3} + 2a_{11-4} + 3a_{11-6}$$

and

$$a_{12} = a_{12+1} + a_{12+0} + 2a_{12-1} + a_{12-2} + a_{12-3} + 2a_{12-4} + 3a_{12-6} + 2a_{12-9} + 2a_{12-10}.$$

What generalization can be made here? When is it possible to find greedy sum identities for multiples of Lucas numbers?

Problem 12, posed by Clark Kimberling and Peter Moses:
Find a formula.

Let x and y be positive real variables and $a_0 = 1, a_1 = 2$. Define

$$a_n = xa_{n-1} - ya_{n-2},$$

and let $y(x)$ = least real number y such that $a_n > 0$ for all $n \geq 0$. Find and verify a formula for $y(x)$, and generalize (e.g., vary the initial conditions, or formulate similar functions of one or more real variables using higher-order recurrences).

Part 2. Solutions and Notes

1. Notes for Problem 2

The following conjecture provides an extensive solution to Problem 2.

Conjecture. *Let $\gamma \in (0, 1/2)$ be a noble number and h a positive real number. The generalized Fibonacci sequence $G_1, G_2, \ldots$ as well as $\delta(m)$ for positive integers m are defined as in Problem 2. For $n \geq 3$ it holds that if $h \in [1/t_{n+1}, 1/t_n]$ then the winners are G_n and G_{n+1}, where $\Phi = (\sqrt{5} + 1)/2$ is the number of the golden ratio and*

$$t_n := \Phi^{n-1}(G_2 + G_1/\Phi)\sqrt{(G_{n+1}^2 - G_{n-1}^2)/\sqrt{5}}.$$

Moreover, if $G_2 < G_1\Phi^2\sqrt{5}$, then for $h \in [1/t_3, 1/t_2]$ the winners are G_2 and G_3.

Note that $t_2 < t_3 < t_4 < \ldots$, so that $1/t_2 > 1/t_3 > 1/t_4 > \ldots$ Thus, the conjecture solves the problem for $h_0 = 1/t_3$. Moreover, if $G_2 < G_1\Phi^2\sqrt{5}$, then the conjecture solves the problem for $h_0 = 1/t_2$.

The biologically most relevant divergence angles are $\gamma = [0, 2, \bar{1}]$ ($\approx 137.5°$) and $\gamma = [0, 3, \bar{1}]$ ($\approx 99.5°$). For these examples the maximal h_0 such that for $h \in [1/t_2, h_0]$ the winners are G_1 and G_2 has been determined as follows.

Example 1. $\gamma = [0, 2, \bar{1}]$ represents the golden angle of approximately 137.5°. From $G_1 = 1$ and $G_2 = 2$ it follows $t_2 = \Phi(2 + 1/\Phi)\sqrt{8/\sqrt{5}}$. It has been proved that for $h \geq 1/t_2$ ad infinitum the winners are G_1 and G_2.

Example 2. $\gamma = [0, 3, \bar{1}]$ represents the "Lucas angle" of approximately 99.5°. From $G_1 = 1$ and $G_2 = 3$ it follows $t_2 = \Phi(3+1/\Phi)\sqrt{3\sqrt{5}}$. It has been proved that $h_0 = 1/((3+1/\Phi)\sqrt{\sqrt{5}})$ is maximal such that for $h \in [1/t_2, h_0]$ the winners are G_1 and G_2.

There is a partial solution to Problem 2 ([1], Proposition 5) in which "winners" correspond to "Voronoi parastichy pairs".

References

[1] Y. Yamagishi and T. Sushida, *Archimedean Voronoi spiral tilings*, Journal of Physics A: Mathematical and Theoretical, **51.4** (2017), `https://iopscience.iop.org/article/10.1088/1751-8121/aa9ada`.

2. Notes for Problem 5

Here is an example. If $r = 3/2$, the series in question is simply

$$\frac{1}{1 + 4x^2 + 7x^4 + 10x^6 + \cdots + x(3 + 6x^2 + 9x^4 + 12x^6 + \cdots)},$$

which has the following expansion:

$$1 - 3x + 5x^2 - 9x^3 + 8x^4 - 36x^5 + \cdots .$$

The sequence of coefficients is given in OEIS by A279634, which also gives a guide to other examples for rational r. For irrational r, see A078140, in which r is the golden ratio.

3. First Solution for Problem 7, by Curtis Cooper and Larry Ericksen

1. Statement of the problem

A three-part problem was proposed by Larry Ericksen, describing the relationships between the Stern sequence and the base 2 digital sum function. First we restate the problem and then we will prove each part of the problem.

Problem 7a. *Let n be a nonnegative integer. Define $s_2(n)$ to be the base 2 digital sum of n. That is, $s_2(0) = 0$, $s_2(1) = 1$, and for $n \geq 1$,*

$$s_2(2n) = s_2(n) \text{ and } s_2(2n+1) = s_2(n) + 1.$$

Define $a(n)$ to be the nth Stern number. That is, $a(0) = 0$, $a(1) = 1$, and for $n \geq 1$,

$$a(2n) = a(n) \text{ and } a(2n+1) = a(n) + a(n+1).$$

(1) *Prove that for nonnegative integers n, $a(n) \geq s_2(n)$.*
(2) *Prove that for nonnegative integers n, $a(n) = s_2(n)$ if and only if $n = 2^i - 2^j$ for $i \geq j \geq 0$.*
(3) *Let r be a nonnegative integer. Prove that*

$$\max_{2^r \leq n < 2^{r+1}} (a(n) - s_2(n)) = F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor,$$

where F_r is the rth Fibonacci number.

2. Proof of (1)

Proof. First, we note that $a(0) = 0 = s_2(0)$. We will prove that for nonnegative integers r and $2^r \leq n < 2^{r+1}$, $a(n) \geq s_2(n)$. The proof is by induction on r.

Base Step. $r = 0$.

$$a(1) = 1 = s_2(1).$$

Therefore, for all n such that $2^0 \leq n < 2^1$, $a(n) \geq s_2(n)$. Thus, the base step is true.

Induction Step. Assume that for some integer $r \geq 0$, if $2^r \leq n < 2^{r+1}$, then $a(n) \geq s_2(n)$. We will prove this statement for $r+1$. Let n be an integer such that $2^{r+1} \leq n < 2^{r+2}$.

n is even. That is, $n = 2m$ for some $2^r \leq m < 2^{r+1}$. Then, by the induction hypothesis,

$$a(2m) = a(m) \geq s_2(m) = s_2(2m).$$

Therefore, the result is true for even n.

n is odd. That is, $n = 2m+1$ for some $2^r \le m < 2^{r+1}$. Then, by the induction hypothesis and the fact that if $m \ge 1$, $a(m+1) \ge 1$,

$$a(2m+1) = a(m) + a(m+1) \ge s_2(m) + 1 = s_2(2m) + 1 = s_2(2m+1).$$

Hence the result is true for odd n. Therefore, for all n such that $2^{r+1} \le n < 2^{r+2}$, we have $a(n) \ge s_2(n)$. Thus, the induction step is true.

Therefore, by mathematical induction, (1) is true for all nonnegative integers n. □

3. Proof of (2)

Proof. ($\Longleftarrow$) First, if $i = j \ge 0$, then $n = 2^i - 2^j = 2^i - 2^i = 0$, so $a(0) = 0 = s_2(0)$. Now, let $n = 2^i - 2^j$, where $i > j \ge 0$. Then, $n = 2^j(2^{i-j} - 1)$, so that the base 2 representation of n is

$$\underbrace{11\cdots1}_{i-j}\underbrace{00\cdots0}_{j}.$$

Therefore, $s_2(n) = i - j$.

Also, $a(n) = a(2^{i-j} - 1) = a(2^{i-j-1}) + a(2^{i-j-1} - 1) = 1 + a(2^{i-j-1} - 1)$. Continuing this process, in the last steps, we have

$$a(n) = i - j - 2 + a(3) = i - j - 2 + a(1) + a(2) = i - j - 2 + 1 + 1 = i - j.$$

Therefore, $a(n) = s_2(n)$.

($\Longrightarrow$) First, we handle the cases $n = 0$, $n = 1$, $n = 2$, and $n = 3$. For $n = 0$, $a(0) = 0 = s_2(0)$ and $n = 0 = 2^i - 2^i$ for all nonnegative integers i. For $n = 1$, $a(1) = 1 = s_2(1)$ and $n = 1 = 2^1 - 2^0$. For $n = 2$, $a(2) = 1 = s_2(2)$ and $n = 2 = 2^2 - 2^1$. For $n = 3$, $a(3) = 2 = s_2(3)$ and $n = 3 = 2^2 - 2^0$. Therefore, the result is true for $n = 0$, $n = 1$, $n = 2$, and $n = 3$. Now consider the following lemma.

Lemma. *Let $r \ge 2$ be an integer such that $2^r \le n < 2^{r+1}$ and $a(n) = s_2(n)$. Then $n = 2^{r+1} - 2^i$ for $0 \le i \le r$.*

n is odd. We first note that the only odd n in the interval $2^r \le n < 2^{r+1}$ of the form $2^i - 2^j$ is $2^{r+1} - 1$. And $a(2^{r+1} - 1) = r + 1$ and $s(2^{r+1} - 1) = r + 1$. Also, note that $a(2^r + 1) = r + 1$ and $s_2(2^r + 1) = 2$. This proves the lemma for the odd numbers $2^r + 1$ and $2^{r+1} - 1$ in the interval. To prove the lemma for the other odd n in the interval, we will prove the following statement.

For odd n such that $2^r + 1 < n < 2^{r+1} - 1$, we have $a(n) \ge r + 1$. Once this statement is proven, we note that for the odd n in the statement, $s_2(n) < r + 1$. So, the lemma will be true for all odd n in the interval $2^r \le n < 2^{r+1}$. The proof of the statement will be by induction on r.

Base Step. $r = 2$: The statement is vacuously true. $r = 3$: $a(11) = 5$ and $a(13) = 5$, so the statement is true.

Induction Step. Let $r \ge 3$ and assume the statement is true for r. We will prove the statement for $r + 1$.

Let n be odd such that $2^{r+1} + 1 < n < 2^{r+2} - 1$. Then $n = 2k + 1$ for some $2^r < k < 2^{r+1} - 1$. And $a(n) = a(2k+1) = a(k) + a(k+1)$. But either k or $k+1$ is odd and between $2^r + 1$ and $2^{r+1} - 1$. So the a value of either k or $k+1$ is greater than or equal to $r + 1$ and the a value of the other one is greater than or equal to 1. Therefore, $a(n) \ge r + 2$. This is what we wanted to prove, so this completes the proof of the induction step.

Therefore, by mathematical induction, the statement is true for the other odd n. Hence, this completes the proof of the odd n case in the lemma. *n is even.* Assume $a(n) = s_2(n)$ for some $2^r \le n < 2^{r+1}$ and assume that $n = 2^e \cdot m$, where $e \ge 1$ and m is odd. But $a(2^e m) = a(m)$ and $s_2(2^e m) = s_2(m)$. Since $a(n) = s_2(n)$, $a(m) = s_2(m)$. But, the only odd m such that $a(m) = s_2(m)$ are $m = 2^i - 1$, where $i \ge 1$. Therefore, $n = 2^e m = 2^e(2^i - 1) = 2^{e+i} - 2^e$. Hence the result is true for even n, and thus, the lemma is true for r.

Therefore, by the principle of mathematical induction, the lemma is true for all nonnegative integers r. This completes the proof of (2). □

4. Proof of (3)

Before we begin the proof of (3), we need the following statements and definitions. In each interval $2^r \le m \le 2^{r+1}$, the maximum value of $a(m)$ is the Fibonacci number F_{r+2}. It was shown by Lehmer [2] that this maximum occurs at $m \in \{\alpha_r, \beta_r\}$ as defined below.

Definition. *In row $r \ge 0$ for $2^r \le m \le 2^{r+1}$, maximums $a(m) = F_{r+2}$ occur at $m \in \{\alpha_r, \beta_r\}$ given explicitly by*

$$\alpha_r = \frac{2^{r+2} - (-1)^r}{3}, \qquad \beta_r = \frac{5 \cdot 2^r + (-1)^r}{3}.$$

Also, let $\alpha_1 = 3$ and $\beta_1 = 3$. Then for a positive integer r, we have the recursions

$$\alpha_{r+1} = 2 \cdot \alpha_r + (-1)^r, \qquad \beta_{r+1} = 2 \cdot \beta_r + (-1)^{r+1}.$$

Let r be a positive integer. Note that if r is odd, the base 2 representation of α_r is

$$1\,\underbrace{01\cdots01}_{\lfloor\frac{r}{2}\rfloor}\,1,$$

and if r is even, the base 2 representation of α_r is

$$1\,\underbrace{01\cdots01}_{\lfloor\frac{r}{2}\rfloor}.$$

Also, if r is odd, the base 2 representation of β_r is

$$11\,\underbrace{01\cdots01}_{\lfloor\frac{r-1}{2}\rfloor},$$

and if r is even, the base 2 representation of β_r is

$$11\,\underbrace{01\cdots01}_{\lfloor\frac{r-1}{2}\rfloor}\,1.$$

Therefore,

$$s_2(\alpha_r) = \left\lfloor \frac{r+1}{2} \right\rfloor + 1, \qquad s_2(\beta_r) = \left\lfloor \frac{r}{2} \right\rfloor + 2.$$

Then, for the Fibonacci values F_{r+2} at locations $m \in \{\alpha_r, \beta_r\}$ in $2^r \le m < 2^{r+1}$, we obtain the maximum discrepancies in part (3) such that

$$a(\alpha_r) - s_2(\alpha_r) = F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor,$$
$$a(\beta_r) - s_2(\beta_r) \le F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor,$$

with equality in the β_r case for odd r.

Next, we restate a theorem in [1] for the second largest values in rows of Stern sequences.

Theorem. *For all $r \geq 4$ in rows $2^r \leq n < 2^{r+1}$, the second largest values $M_2(r)$ are given by $F_{r+2} - F_{r-3} = 2\,F_r + F_{r-2} = F_r + L_{r-1}$ for Fibonacci numbers F_n and Lucas numbers L_n.*

Table 1 from [1] lists actual values in rows of Stern sequences, with the largest as $M_1(r)$ and second largest as $M_2(r)$. We also define $d(r)$ and $e(r)$ in Table 1 by the following identities:

$$d(r) = F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor \text{ as the discrepancy formula in (3),}$$

$$e(r) = d(r) - M_2(r) = M_1(r) - \left\lfloor \frac{r+3}{2} \right\rfloor - M_2(r) = F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor - M_2(r).$$

TABLE 1. $M_1(r)$, $d(r)$, $M_2(r)$, $e(r)$ in rows $3 \leq r \leq 12$.

row r	3	4	5	6	7	8	9	10	11	12
$M_1(r)$	5	8	13	21	34	55	89	144	233	377
$d(r)$	2	5	9	17	29	50	83	138	226	370
$M_2(r)$	4	7	12	19	31	50	81	131	212	343
$e(r)$	-2	-2	-3	-2	-2	0	2	7	14	27
$e(r)+2$	0	0	-1	0	0	2	4	9	16	29

We note the minimum $s_2(n)$ is at least 2 for n between the ends of the rows. Identity (3) is true in any row r, if $e(r) + 2 \geq 0$. Thus, the greatest $M_2(r) - s_2(n)$ possible challenge to (3) would be if $d(r) - (M_2(r) - 2) < 0$. Seen in the last row of Table 1, this possibility happens only in row $r = 5$ which had $e(r) + 2 = -1$.

However, a quick check of the actual values $a(n)$ and $s_2(n)$ in row $r = 5$ showed that when $a(n) = M_2(r) = 12$, then $s_2(n) = 4$, and by [1] when $a(n) = M_3(r) = 11$, then $s_2(n) \in \{3, 5\}$. Therefore, even in row $r = 5$ we have $12 - 4 = 8 < 9 = d(5)$ and $11 - 3 = 8 < 9 = d(5)$.

Proof of (3). For values $a(n)$ and $s_2(n)$ in $n \in [0, 2^{r+1})$ through row $r = 12$, we verified (3) as

$$\max_{2^r \leq n < 2^{r+1}} (a(n) - s_2(n)) = F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor.$$

Now assume $r > 12$. We want to prove that no other values $a(n) - s_2(n)$ exceed that maximum in (3). To show this, let m be a value between $2^r + 1$ and $2^{r+1} - 1$, where $m \neq \alpha_r, \beta_r$. Then, $a(m) - s_2(m) \leq M_2(r) - 2$. Hence,

$$\begin{aligned} F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor - (a(m) - s_2(m)) &\geq F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor - (M_2(r) - 2) \\ &= F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor - (F_{r+2} - F_{r-3} - 2) \\ &= F_{r-3} - \left\lfloor \frac{r+3}{2} \right\rfloor + 2 \geq 0. \end{aligned}$$

Thus, with the above inequality proven, we have

$$\max_{2^r \leq n < 2^{r+1}} (a(n) - s_2(n)) = F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor.$$

This completes the proof of (3). $\square$

Note: The recursion for $d(r)$ and $e(r)$ could start at $r = 8$, where $d(8) = 50$ and $e(8) = 0$ from Table 1. Then the recursions for $r \geq 9$ are

$$d(r) = d(r-1) + F_{r-5} + \begin{cases} 0 & \text{for even} \quad r, \\ 1 & \text{for odd} \quad r, \end{cases}$$

$$e(r) = e(r-1) + F_r + \begin{cases} 0 & \text{for even} \quad r, \\ -1 & \text{for odd} \quad r. \end{cases}$$

Recursion $e(r)$ also shows that subsequent $e(r)$ values are positive and become larger for $r \geq 9$.

References

[1] J. Lansing, *Largest values for the Stern sequence*, J. Integer Seq., **17** (2014), Article 14.7.5, 18 pp.
[2] D. H. Lehmer, *On Stern's diatomic series*, Amer. Math. Monthly, **36** (1929), 59–67.

4. Second Solution for Problem 7, by Tanay Wakhare

The Stern sequence is recursively defined by $a_0 = 0, a_1 = 1$ and for $n \geq 1$ as

$$a_{2n} = a_n,$$
$$a_{2n+1} = a_n + a_{n+1},$$

where we abbreviated $a_n = a(n)$ for convenience.

We also define $s_2(n)$ as the sum of the binary digits of n, abbreviated as s_n, which is completely characterized by the recurrences

$$s_{2n} = s_n,$$
$$s_{2n+1} = s_n + 1,$$

with $s_1 = 1$. The Stern sequence achieves relative maxima at the Jacobstal numbers

$$J(n) = \frac{1}{3}\left(2^n - (-1)^n\right),$$

with $a_{J(n)} = F_n$, where F_n is the n-th Fibonacci number. However, s_n has relative maxima when the underlying bit string is all 1s, so that $n = 2^N - 1$ and $s_{2^N-1} = N$.

We now define the discrepancy $\delta_n := a_n - s_n$, which satisfies the recurrences

$$\delta_{2n} = \delta_n,$$
$$\delta_{2n+1} = \delta_n + a_{n+1} - 1,$$

with initial conditions $\delta_0 = \delta_1 = 0$.

We will combine parts (1) and (2) of the problem by Larry Ericksen into the following theorem.

Theorem 4.1. *We have $a_n \geq s_n$, with equality if and only if $n = 2^N$ or $n = 2^N - 2^M$ for some integers $N > M$.*

Proof. In general, since $a_{n+1} \geq 1$, we have $\delta_n \geq 0$ and $a_n \geq s_n$.

To find examples of equality, we need to characterize indices such that $\delta_n = 0$. First, we note that $n = 2^N$ works since we successively apply the top recurrence. We can now prove that at $n = 2^N - 2^M$, $\delta_n = 0$. We induct on N, then divide out by powers of two until we have

$$\delta_n = \delta_{2^{N-M}-1} = \delta_{2^{N-M-1}-1} + a_{2^{N-M}} - 1 = \delta_{2^{N-M-1}-1} = 0.$$

If we consider any index n which is a difference of powers of 2, we successively divide out by 2 until we're left with a_n at an index of the form $n = 2^N - l$, where l is odd and $l \neq 1$. Applying the bottom recurrence gives

$$\delta_n = \delta_{2^{N-1}-\frac{l+1}{2}} + a_{2^{N-1}-\frac{l-1}{2}} - 1.$$

Since $l \neq 1$, we're left with a at an index which isn't a power of two. However, $a_n = 1$ if and only if $n = 2^N$ (else $a_{2n+1} = a_n + a_{n+1} \geq 2$). Therefore, $a_{2^{N-1}-\frac{l-1}{2}} \geq 2$ and $\delta_n \geq 1$. $\square$

5. Third Solution for Problem 7, by Karyn McLellan

1. Proof of (1) and (2)

Recall the first two statements in the problem posed by Larry Ericksen:

(1) Prove that for nonnegative integers n, $a(n) \geq s_2(n)$.
(2) Prove that for nonnegative integers n, $a(n) = s_2(n)$ if and only if $n = 2^i - 2^j$ for $i \geq j \geq 0$.

Proof. We will begin by proving that if $n = 2^i - 2^j$ then $a(n) = s_2(n)$. Note that the values of $n = 2^i - 2^j$ give OEIS sequence A023758: 0, 1, 2, 3, 4, 6, 7, 8, 12,.... According to the entry, this is also the sequence of numbers whose binary representations are nonincreasing.

Recall that the n^{th} Stern number $a(n)$ gives us the number of hyperbinary representations of $n-1$. In listing all such representations for a given n we can use the following two operations:

$$02 \leftrightarrow 10$$
$$12 \leftrightarrow 20$$

We are changing two smaller numbers into a bigger number (or vice versa). There are only two possible operations using two digits. Note that 22 at the beginning of a representation requires a carry to three digits, which can be accomplished by writing as $022 \to 102 \to 110$, for example. All possible hyperbinary representations are connected via a chain or a tree using these two operations. We are simply shifting one 2^k term into two 2^{k-1} terms via the operations, so can reach any combination of powers of 2 in this way.

Now, we are interested specifically in numbers of the form $n = 2^i - 2^j$. The values of $n-1$ give OEIS sequence A089633 and according to the entry this is also the sequence of numbers whose binary representations contain at most one 0. Our aim is to find all hyperbinary representations of numbers of this form, and we will start with a simple example to illustrate.

Let $n = 30 = 11110_2$, which is $32 - 2$. The hyperbinary representations of 29 are

$$11101 \to 11021 \to 10221 \to 2221,$$

and so $a(30) = 4 = s_2(30)$. Subtracting 1 from 30 removes one of the initial ones from the binary representation and brings us to the first hyperbinary representation of 29. The rest of the 1's in the binary representation of 30 (the first 3 in 29) each give us a new hyperbinary representation of 29, using the $10 \to 02$ operation.

We can generalize this to all n of the form $2^i - 2^j$. We know the binary representation of n is nonincreasing, so it has form $1\ldots110\ldots0$ and $s_2(n)$ 1's. Subtracting 1 gives $n - 1 = 1\ldots101\ldots1_2$, which has exactly one 0 and $s_2(n) - 1$ initial 1's. We consider this the first hyperbinary representation of $n - 1$. Each of the initial 1's uses the operation $10 \to 02$, introducing a 0 where the 1 was so the process can continue. This creates the rest of the hyperbinary representations for a total of $s_2(n)$, and hence $a(n) = s_2(n)$. We must note that at no point in the process are any other hyperbinary representations possible; from the starting point $1\dots101\dots1$ and all intermediate steps $1\dots102\dots21\dots1$ the only move to make is $10 \to 02$, and when we reach $2\dots21\dots1$ there is nowhere to go but back. Note, that if $n = 10\dots0_2$ then $n-1 = 01\dots1_2$, in which case there are no operations to perform and $a(n) = s_2(n) = 1$. This completes the proof that if $n = 2^i - 2^j$ then $a(n) = s_2(n)$.

Next we will prove that if $n \neq 2^i - 2^j$ then $a(n) > s_2(n)$. We will start by breaking the binary representation of n into non-increasing blocks of the form $1\dots10\dots0$ so that

$$n = 1\dots10\dots0 \mid 1\dots10\dots0 \mid \dots \mid 1\dots110\dots0.$$

We must have at least two blocks because the binary representation of n is not non-increasing. The final block works the same as the $n = 2^i - 2^j$ case. Subtracting 1 from n gives the binary representation (and first hyperbinary representation) of $n-1$, and the final block will have form $1\dots101\dots1$. Each of the leading 1's here gives us a new hyperbinary representation using the $10 \to 02$ operation. Similarly, for each of the previous blocks this operation gives a hyperbinary representation for each 1. This gives a total of $s_2(n)$ such representations so far, and we are left to show that there must be at least one more. We know that $n = 2^i - 2^j$ if and only if $n-1$ contains at most one 0 in its binary representation. Therefore if $n \neq 2^i - 2^j$ then $n-1$ contains more than one 0. We have two possible cases for the location of the 0's:

Case 1: We have two 0's in a row. Using our operations we have $100 \to 020 \to 012$. This gives an extra hyperbinary representation containing 12, apart from the $s_2(n)$ described above.
Case 2: We have two 10 pairs. Using the operation $10 \to 02$, $10\dots10$ can become $10\dots02$ or $02\dots10$ or we can use the operation twice to obtain $02\dots02$. This also gives an extra hyperbinary representation, apart from the $s_2(n)$ we already have. (Combinatorially we can see that there are many ways of applying this to our blocks, as well as the additional operation $20 \to 12$ operation where applicable, so for large n, $s_2(n)+1$ is a small lower bound for $a(n)$.) We now have $a(n) > s_2(n)$, completing the proof.

We have shown that if $n = 2^i - 2^j$ then $a(n) = s_2(n)$, and if $n \neq 2^i - 2^j$ then $a(n) > s_2(n)$. Because these are the only two options for n, combining these statements gives us the proof of (1), namely, $a(n) \geq s_2(n)$. Now, taking the contrapositive of the second statement proven above, we get that if $a(n) \leq s_2(n)$, then $n = 2^i - 2^j$. But $a(n) < s_2(n)$ is not possible by (1) and so we are left with the statement if $a(n) = s_2(n)$ then $n = 2^i - 2^j$. Combining this with the first statement proved above we get the proof of (2), namely, $a(n) = s_2(n)$ if and only if $n = 2^i - 2^j$. Statements (1) and (2) hold for $n \geq 0$, but of course $n = 0$ is the trivial case $a(n) = s_2(n) = 0$. □

2. Proof of (3)

Recall the third statement in the problem:

(3) Let r be a nonnegative integer. Prove that

$$\max_{2^r \leq n < 2^{r+1}} (a(n) - s_2(n)) = F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor,$$

where F_r is the rth Fibonacci number.

Proof. The expression $F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor$ for $r \geq 2$ gives OEIS sequence A129696, namely 1, 2, 5, 9, 17, 29, 50, 83, 138, ... (Here the expression is shifted to $F_{r+3} - 2 - \left\lfloor \frac{r}{2} \right\rfloor$ for $r \geq 1$.)

Let $L_i(r)$ be the i^{th} largest value of the Stern sequence $a(n)$ occurring in the interval $2^r \leq n < 2^{r+1}$. We know from [2] that $L_1(r) = F_{r+2}$, for $r \geq 0$, and this maximum occurs in two locations:

$$n_r = \frac{4 \cdot 2^r - (-1)^r}{3}, \quad n_r^* = \frac{5 \cdot 2^r + (-1)^r}{3}.$$

The first set of indices are the Jacobsthal numbers, OEIS sequence A001045: 1, 3, 5, 11, 21, 43, 85 ...; the second, OEIS sequence A048573: 2, 3, 7, 13, 27, 53, 107, ... (although $n = 2$ does not belong in the first interval for $r = 0$, so we will disregard this term). Note also, that OEIS sequence A086893: 1, 3, 5, 13, 21, 53, 85, ... gives the index of the term F_{r+2} in the ordered pair (F_{r+2}, F_{r+1}) which occurs in each interval $2^r \leq n < 2^{r+1}$, for $r \geq 0$ (although the first interval includes only $n = 1$). This F_{r+2} alternates between locations n_r and n_r^*.

Let us now consider $s_2(n)$ for those n with $a(n) = L_1(r)$. We will first break down the above three sequences into subsequences of alternating terms. The Jacobsthal sequence, A001045, splits into 1, 5, 21, 85, 341, ... (A002450) and 3, 11, 43, 171, ... (A007583). According to the entries, the binary representation of the former sequence has form $(10)^k1$ for $k \geq 0$, and so $s_2(A002450(k)) = k + 1$. Similarly, the binary representation of the latter sequence has form $(10)^k11$, for $k \geq 0$, and so $s_2(A007583(k)) = k + 2$. Combining these results, we get that $s_2(A001045(r)) = 1, 2, 2, 3, 3, 4, 4, \ldots = \left\lfloor \frac{r+3}{2} \right\rfloor$ for $r \geq 0$.

If we next split the sequence A086893 into subsequences we get A002450 and 3, 13, 53, 213, ... (A072197). According to the entry, the binary representation of the latter sequence has form $11(01)^k$, for $k \geq 0$, and so $s_2(A072197(k)) = k + 2$. Combining these results, we get that $s_2(A086893(r)) = 1, 2, 2, 3, 3, 4, 4, \ldots = \left\lfloor \frac{r+3}{2} \right\rfloor$ for $r \geq 0$.

Lastly, if we next split the sequence A048573 into subsequences we get A072197 and 7, 27, 107, 427, ... (A136412). The binary representation of the latter sequence has form $1(10)^k11$, for $k \geq 0$, and so $s_2(A072197(k)) = k + 3$. (Aside: $A007583(k) = \frac{2^{2k+3}+1}{3}$ and binary form $(10)^k11$. Adding 2^{2k+2} gives $A136412(k) = \frac{5 \cdot 4^{k+1}+1}{3}$. Therefore this sequence has the binary form $1(10)^k11$.) Combining these results, we get that $s_2(A048573(r)) = 2, 3, 3, 4, 4, \ldots = \left\lfloor \frac{r+5}{2} \right\rfloor$ for $k \geq 0$.

We have just shown that for values of n with $a(n) = L_1(n)$, i.e., in sequences A001045 and A086893 (n_r and n_r^*), $a(n) - s_2(n) = F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor$. We now must show these are the maximal values of $a(n) - s_2(n)$ on the intervals $2^r \leq n < 2^{r+1}$. For values of n in sequence A048573, we have maximal $a(n)$ but $(a(n) - s_2(n)) = F_{r+2} - \left\lfloor \frac{r+5}{2} \right\rfloor$ is not maximal.

The first part of the following table gives the observed maximum value of $(a(n) - s_2(n))$ for intervals $2^r \leq n < 2^{r+1}$ with $r \leq 10$, and the corresponding values of n. Comparing columns we can confirm that $\max(a(n) - s_2(n)) = F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor$ for these intervals.

r	$\max(a(n) - s_2(n))$	n	$L_1(r)$	$\left\lfloor \frac{r+3}{2} \right\rfloor$	$F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor$	$L_2(r)$	F_{r+2}	F_{r-3}
0	0	1	1	1	0	-	1	-
1	0	3	2	2	0	1	2	-
2	1	5	3	2	1	2	3	-
3	2	11/13	5	3	2	4	5	-
4	5	21	8	3	5	7	8	1
5	9	43/53	13	4	9	12	13	1
6	17	85	21	4	17	19	21	2
7	29	171/213	34	5	29	31	34	3
8	50	341	55	5	50	50	55	5
9	83	683/853	89	6	83	81	89	8
10	138	1365	144	6	138	131	144	13

What about $r > 10$? If we start by assuming $a(n) = L_1(n)$, we have shown above that $a(n) - s_2(n) = F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor$. To prove maximality in general we need to consider the second largest value of the Stern sequence in each interval, namely $L_2(r)$. We know from [1] that for $r \geq 4$ we have that $L_2(r) = F_{r+2} - F_{r-3}$. Since $F_{r-3} > \left\lfloor \frac{r+3}{2} \right\rfloor$ for $r \geq 9$, we have that

$$F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor > F_{r+2} - F_{r-3} = L_2(r).$$

This means that $\max(a(n) - s_2(n))$ cannot occur for $a(n) = L_2(r)$ (or any other smaller value of the Stern sequence). We must have $a(n) = L_1(r)$ and so $\max(a(n) - s_2(n)) = F_{r+2} - \left\lfloor \frac{r+3}{2} \right\rfloor$ on $2^r \leq n < 2^{r+1}$ for $r \geq 9$. Combined with the observed maximality for $r \leq 8$, we are done. $\square$

References

[1] J. Lansing, *Largest Values for the Stern Sequence*, Journal of Integer Sequences, **17** (2014), Article 14.7.5.
[2] D. H. Lehmer, *On Stern's Diatomic Sequence*, Amer. Math. Monthly, **36.2** (1929), 59–67.
[3] OEIS Foundation Inc. (2016), The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/`.

6. Notes for Problem 12

It appears that $y(x)$ is given in two pieces: $y(x) = x^2/4$ for $0 \leq x \leq 4$ and $y(x) = 2x - 4$ for $x > 4$, and that similarly defined functions have similarly simple formulas. Is there a simple proof?