GLOBAL SERIES FOR ZETA FUNCTIONS

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Abstract. We provide two general families of everywhere-convergent series expansions for Barnes multiple zeta functions involving Bernoulli polynomials of the second kind and weighted Stirling numbers. These contain the classical results of Ser and Hasse and several recent generalizations as special cases. We also show how these series have good p-adic analogues.

1. Introduction

The values of the Riemann zeta function $\zeta(s)$ at positive integers k may be interpreted as the (reciprocals of) probabilities that a set of k "randomly chosen" positive integers is relatively prime; this is a consequence of the celebrated Euler product

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{\text{primes } p} (1-p^{-s})^{-1} \qquad (\Re(s) > 1) \tag{1.1}$$

observed for real values of s by Euler as an expression of the Fundamental Theorem of Arithmetic. Upon analytic continuation, the values of $\zeta(s)$ at the negative integers are rational numbers whose numerators are related to the class numbers of cyclotomic fields. And of course the behavior of $\zeta(s)$ in the critical strip $0 < \Re(s) < 1$ is intimately connected to questions concerning the distribution of prime numbers.

Since zeta functions reveal different arithmetic information depending on where they are evaluated, there is some philosophical interest in expressions for them which are valid on the entire complex plane. Series expansions of this type were given by Ser [17] and by Hasse [9] for the Hurwitz zeta function, and several generalizations of these results are also known. In this article we give two very general families (Theorems 3.1 and 3.2) of series expansions for multiple zeta functions which unify many of the known series, such as those in [3, 17, 9]. As applications we obtain series for zeta values, Stieltjes constants, and related constants in terms of Bernoulli numbers of the second kind, hyperharmonic numbers, and related sequences.

Beyond providing a comprehensive general framework for complex series of this type, a further peculiar feature of these series is that they are equally valid as p-adic series for p-adic multiple zeta functions, under suitable conditions (Theorems 6.1 and 6.2). In the last two sections we give several examples of series derived from these general theorems which converge in both real and p-adic senses to analogous zeta values.

2. Zeta functions and Bernoulli polynomials

For a positive integer r, the Barnes multiple zeta function [1, 15] of order r, denoted by $\zeta_r(s,a)$, is defined by

$$\zeta_r(s,a) := \sum_{t_1=0}^{\infty} \cdots \sum_{t_r=0}^{\infty} (a+t_1+\cdots+t_r)^{-s} \tag{2.1}$$

for $\Re(s) > r$ and $\Re(a) > 0$, and continued meromorphically to $s \in \mathbb{C}$ with simple poles at $s = 1, 2, \ldots, r$. When $r = 1$ or $a = 1$ that part of the notation is often suppressed, so that

$\zeta_1(s,1) = \zeta(s)$ denotes the Riemann zeta function. Note also that $\zeta_0(s,a) = a^{-s}$ by convention. For $\Re(s) > r$ and $\Re(a) > 0$ these functions may also equivalently be given [21, eq. (3.3)] by the single Dirichlet series

$$\zeta_r(s,a) = \sum_{m=0}^{\infty} \binom{m+r-1}{m} (m+a)^{-s}. \tag{2.2}$$

For our purposes here it is useful to use (2.2) to extend the definition to include zeta functions of negative integer order as well. For any nonnegative integer r, the definition (2.2) gives

$$\zeta_{-r}(s,a) = \sum_{j=0}^{r} \binom{r}{j} (-1)^j (a+j)^{-s}, \tag{2.3}$$

so that $(-1)^r \zeta_{-r}(s,a)$ is the r-th forward difference of the power function a^{-s} with respect to the a parameter. With these definitions we have the difference equation

$$\zeta_r(s,a) - \zeta_r(s,a+1) = \zeta_{r-1}(s,a) \tag{2.4}$$

for all integers r, and the derivative-shift identity

$$\frac{\partial}{\partial a} \zeta_r(s,a) = -s\zeta_r(s+1,a) \tag{2.5}$$

[15, eq. (3.11)] for all integers r.

The n-th Bernoulli polynomial of order z, denoted $B_n^{(z)}(x)$, is defined [4] by

$$\left(\frac{t}{e^t - 1}\right)^z e^{xt} = \sum_{n=0}^{\infty} B_n^{(z)}(x) \frac{t^n}{n!}. \tag{2.6}$$

These are polynomials of degree n in x and of degree n in the order z. They satisfy a difference equation

$$B_n^{(z)}(x+1) - B_n^{(z)}(x) = nB_{n-1}^{(z-1)}(x) \tag{2.7}$$

[4, eq. (1.5)] and derivative identity

$$\frac{\partial}{\partial x} B_n^{(z)}(x) = nB_{n-1}^{(z)}(x) \tag{2.8}$$

[4, eq. (1.6)]. Their dual companions are the order z Bernoulli polynomials of the second kind $b_n^{(z)}(x)$, which are defined [4] by the generating function

$$\left(\frac{t}{\log(1+t)}\right)^z (1+t)^x = \sum_{n=0}^{\infty} b_n^{(z)}(x) t^n. \tag{2.9}$$

These are also polynomials of degree n in x and of degree n in the order z. When $z = 1$ or $x = 0$ that part of the notation is often suppressed, so that $B_n^{(z)}$ denotes $B_n^{(z)}(0)$, $b_n(x)$ denotes $b_n^{(1)}(x)$, and B_n denotes $B_n^{(1)}(0)$. The numbers b_n are also often called *Gregory coefficients* and $n!b_n$ are sometimes called *Cauchy numbers*. The polynomials $b_n^{(z)}(x)$ satisfy a difference equation

$$b_n^{(z)}(x+1) - b_n^{(z)}(x) = b_{n-1}^{(z)}(x) \tag{2.10}$$

[4, eq. (2.4)] and derivative identity

$$\frac{\partial}{\partial x} b_n^{(z)}(x) = b_{n-1}^{(z-1)}(x) \tag{2.11}$$

[4, eq. (2.3)]. The polynomials $B_n^{(z)}(x)$ and $b_n^{(z)}(x)$ may be used interchangeably by means of Carlitz's identities

$$n!b_n^{(z)}(x) = B_n^{(n-z+1)}(x+1), \qquad B_n^{(z)}(x) = n!b_n^{(n-z+1)}(x-1) \tag{2.12}$$

[4, eq. (2.11), (2.12)].

It is well known that for positive integers r, the values of Barnes zeta functions at the negative integers

$$\zeta_r(-k,a) = \frac{(-1)^r k!}{(r+k)!} B_{r+k}^{(r)}(a) \tag{2.13}$$

[15, eq.(3.10)] are given in terms of Bernoulli polynomials, as are the residues

$$\operatorname*{Res}_{s=k} \zeta_r(s,a) = \frac{(-1)^{r-k} B_{r-k}^{(r)}(a)}{(k-1)!(r-k)!} \tag{2.14}$$

[15, eq. (3.9)] of $\zeta_r(s,a)$ at each of its r poles at $s = 1, \ldots, r$.

3. Series for zeta functions

Our first family of series, involving the Bernoulli polynomials of the second kind, give a broad generalization of the original series representation of Ser [17]. These express finite combinations of positive order zeta functions $\zeta_r(s,a)$ as series of negative order zeta functions $\zeta_{-m}(s,a) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} (j+a)^{-s}$ weighted by Bernoulli polynomials $b_m^{(k)}(x)$ of the second kind.

Theorem 3.1. *For all nonnegative integers N and k, there is an identity of analytic functions*

$$\sum_{m=0}^{\infty} (-1)^{m+N} b_{m+N}^{(k)}(x) \sum_{j=0}^{m} (-1)^j \binom{m}{j} (j+a)^{-s}$$
$$= \frac{\zeta_{N-k}(s-k, a+x)}{(s-1)_k} - \sum_{m=0}^{N-1} (-1)^m b_m^{(k)}(x) \zeta_{N-m}(s,a)$$

for all $s \in \mathbb{C}$, provided that $a > 0$ and $a + x > 0$.

Remarks. The special case where $N = 1$, $k = 1$, $a = 1$, $x = 0$ reduces to the original result

$$\sum_{m=0}^{\infty} (-1)^m b_{m+1} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (j+1)^{-s} = \zeta(s) - \frac{1}{s-1} \tag{3.1}$$

of Ser [17]. The cases where $N = 1$, $k = 1$ were recently given by Blagouchine [3, Theorem 3, eq. (28)]. The positive-order zeta functions on the right side of the above identity have simple poles at $s = 1, 2, \ldots, N$, however the residues all sum to zero, as the series on the left represents an analytic function of $s \in \mathbb{C}$. The highest order zeta which occurs on the right side is $\zeta_N(s,a)$, except when $k = 0$ in which case there is also the term $\zeta_N(s, a+x)$. In these series, the index shift N of the Bernoulli polynomial series coefficients on the left determines the order of the zeta functions on the right, and the order k of the Bernoulli polynomial series coefficients on the left determines a shift, in order and in the s variable, of one zeta function term on the right.

Examples. Taking $N = 1$ in this theorem gives

$$\sum_{m=0}^{\infty} (-1)^m b_{m+1}^{(k)}(x) \zeta_{-m}(s,a) = \zeta(s,a) - \frac{\zeta_{1-k}(s-k, a+x)}{(s-1)_k}, \tag{3.2}$$

where the two terms on the right side both have simple poles at $s = 1$ with residues summing to $1 - 1 = 0$, for any k. Taking $N = 2$ yields

$$\sum_{m=0}^{\infty} (-1)^m b_{m+2}^{(k)}(x)\zeta_{-m}(s,a) = \frac{\zeta_{2-k}(s-k, a+x)}{(s-1)_k} - \zeta_2(s,a) + \big(x + \frac{k}{2}\big)\zeta(s,a), \tag{3.3}$$

whose $k = 1$ case was given by Blagouchine [3, eq. (77)]. At $s = 1$, the residues of the three zeta function terms on the right side of (3.3) sum to

$$\left(-a - x + \frac{2-k}{2}\right) + (a-1) + \left(x + \frac{k}{2}\right) = 0 \tag{3.4}$$

and at $s = 2$ the residues sum to $1 - 1 + 0 = 0$. For $N = 3$ we have

$$\begin{aligned} \sum_{m=0}^{\infty} (-1)^m b_{m+3}^{(k)}(x)\zeta_{-m}(s,a) \quad &= \zeta_3(s,a) - (x + \frac{k}{2})\zeta_2(s,a) \\ &\quad + b_2^{(k)}(x)\zeta(s,a) - \frac{\zeta_{3-k}(s-k, a+x)}{(s-1)_k}, \end{aligned} \tag{3.5}$$

and once again all residues on the right at $s = 1, 2, 3$ sum to zero.

Proof of Theorem 3.1. We first prove the theorem in the case $k = 0$. We begin by fixing a nonnegative integer N, a complex number s with $\Re(s) > N + 1$, and a positive real number a. We construct the Newton series for the analytic function $f(x) = \zeta_N(s, a + x)$ of the complex variable x in the right half plane described by $\Re(a + x) > 0$. Under these conditions the Newton series is

$$\begin{aligned} \zeta_N(s, a+x) \quad &= \sum_{m=0}^{\infty} \binom{x}{m} \Delta^m \left[\zeta_N(s, a+x)\right]_{x=0} \\ &= \sum_{m=0}^{N-1} (-1)^m \binom{x}{m} \zeta_{N-m}(s,a) \\ &\qquad + \sum_{m=N}^{\infty} \binom{x}{m} \Delta^{m-N} \left(\Delta^N \left[\zeta_N(s, a+x)\right]_{x=0}\right) \\ &= \sum_{m=0}^{N-1} (-1)^m \binom{x}{m} \zeta_{N-m}(s,a) \\ &\qquad + \sum_{m=0}^{\infty} (-1)^{m+N} \binom{x}{m+N} \zeta_{-m}(s,a), \end{aligned} \tag{3.6}$$

where Δ is the difference operator defined by $\Delta f(x) = f(x+1) - f(x)$. In order to justify the existence and convergence of this Newton series, we first appeal to the Mellin transform integral representation

$$\zeta_N(s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^s e^{-at}}{(1 - e^{-t})^N} \frac{dt}{t} \tag{3.7}$$

[15] which is valid for $\Re(s) > N$. Since we assume that $s \in \mathbb{C}$ is fixed with $\Re(s) > N + 1$, we can verify that the function $f(t) = t^{s-1}e^{-at}(1 - e^{-t})^{-N}$ satisfies $\lim_{t\to 0} f(t) = 0$ and

$\lim_{t\to\infty} f(t) = 0$, and therefore is bounded on $(0, \infty)$. It follows that for fixed s, a as described we have

$$|\zeta_N(s, a + x)| \leqslant C \int_0^\infty |e^{-xt}|\, dt \tag{3.8}$$

for some positive constant C, which shows that $\zeta_N(s, a + x)$ has exponential order less than π, as an analytic function of x on the half-plane $\Re(a + x) > 0$. Thus the hypotheses of Carlson's uniqueness theorem [6] are satisfied for the function $\zeta_N(s, a+x)$, so that it is uniquely determined by its Newton series (3.6), provided that the series converges.

For $s, a \in \mathbb{C}$, $a \notin \mathbb{Z}^+$, we have the estimate

$$\left|\frac{B_n^{(n+s)}(a)}{n!}\right| \sim \left|\frac{1}{(\log n)^s n^a \Gamma(1-a)}\right| \tag{3.9}$$

as $n \to \infty$ [20, Lemma 2], and for $a > 0$ we have

$$\zeta_{-m}(s, a) \sim \frac{(\log m)^{s-1}\Gamma(a)}{m^a \Gamma(s)} \tag{3.10}$$

as $m \to \infty$ [14, Theorem 1.7]. Since $B_n^{(n+1)}(a) = n!\binom{a-1}{n}$, for any s, a, x, the m-th term in the series (3.6) satisfies

$$\left|\binom{x}{m+N}\zeta_{-m}(s, a)\right| \sim \left|\frac{(\log m)^{s-2}\Gamma(a)}{m^{a+x+1}\Gamma(-x)\Gamma(s)}\right| \tag{3.11}$$

as $m \to \infty$ when x is not a nonnegative integer. Therefore for any positive real number a and $x \in \mathbb{C}$ with $\Re(a + x) > 0$, the series (3.6) converges absolutely and uniformly on compact subsets of $s \in \mathbb{C}$ to an entire function. Since the positive-order zeta functions have meromorphic continuations to the entire complex plane, it follows that the identity

$$\zeta_N(s, a + x) - \sum_{m=0}^{N-1} (-1)^m \binom{x}{m} \zeta_{N-m}(s, a) = \sum_{m=0}^{\infty} (-1)^{m+N} \binom{x}{m+N} \zeta_{-m}(s, a) \tag{3.12}$$

is an identity of entire functions of $s \in \mathbb{C}$. Taking x to be real then proves the theorem in the case $k = 0$, since it may be written as

$$\zeta_N(s, a + x) - \sum_{m=0}^{N-1} (-1)^m b_m^{(0)}(x) \zeta_{N-m}(s, a) = \sum_{m=0}^{\infty} (-1)^{m+N} b_{m+N}^{(0)}(x) \zeta_{-m}(s, a). \tag{3.13}$$

Now we consider x to be a real variable with $a + x > 0$. Observing that

$$\int_x^{x+1} b_n^{(k)}(t)\, dt = b_{n+1}^{(k+1)}(x + 1) - b_{n+1}^{(k+1)}(x) = b_n^{(k+1)}(x) \tag{3.14}$$

and

$$\int_x^{x+1} \zeta_N(s, a + t)\, dt = \frac{\zeta_N(s-1, a+x) - \zeta_N(s-1, a+x+1)}{s-1} = \frac{\zeta_{N-1}(s-1, a+x)}{s-1}, \tag{3.15}$$

we obtain the general statement of the theorem by k-fold application of the operator $\int_x^{x+1} dx$ to both sides of (3.13). For any positive integer k, estimates (3.9), (3.10) show that the series thus obtained also converge absolutely and uniformly on compact subsets of $s \in \mathbb{C}$, so that interchanging integration and summation is justified. The theorem is therefore proven. □

Our second family of series includes original results of Ser [17] and Hasse [9] and shows how they are interrelated. It involves the *weighted Stirling numbers of the first kind* [5], which may be defined by the vertical generating function

$$(1+t)^{-r}(\log(1+t))^k = k!\sum_{m=k}^{\infty} s(m,k|r)\frac{t^m}{m!} \tag{3.16}$$

or by the horizontal generating function

$$(x)_m := x(x-1)\cdots(x-m+1) = \sum_{k=0}^{m} s(m,k|r)(x+r)^k. \tag{3.17}$$

For purposes of derivation we regard $s(m,k|r)$ as a polynomial of degree $m-k$ in the variable r, although the important combinatorial applications occur when r is a nonnegative integer. In this case, $(-1)^{m-k}s(m,k|r) := \left[{m+r \atop k+r}\right]_r$ is a positive integer known as an *r-Stirling number* [2], which counts the number of permutations of $\{1,2,\dots,m+r\}$, having exactly $k+r$ cycles, in which the elements $1,2,\dots,r$ are restricted to appear in different cycles.

Theorem 3.2. *For all positive integers k and nonnegative integers N, there is an identity of analytic functions*

$$\begin{aligned}\sum_{m=0}^{\infty}(-1)^{m+N}\frac{s(m+N,k|r)}{(m+N)!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(j+a)^{-s}\\ = (-1)^k\binom{s+k-1}{k}\zeta_N(s+k,a-r) - \sum_{m=k}^{N-1}(-1)^m\frac{s(m,k|r)}{m!}\zeta_{N-m}(s,a)\end{aligned}$$

for all $s\in\mathbb{C}$, provided that $a>0$ and $a-r>0$.

Remark. Again, the zeta functions on the right side have simple poles at $s=1,2,\dots,N$, but the residues sum to zero. As soon as $k\geqslant N$ the latter sum on the right vanishes to leave only a single term involving ζ_N.

Proof of Theorem 3.2. Observing that

$$\binom{-x}{m} = \frac{s(m,0|x)}{m!}, \tag{3.18}$$

we see from (3.12) with $r=-x$ that the statement of the theorem holds if $k=0$. We put $r=-x$ and apply the differentiation operator d/dr to both sides of (3.12) k times, using (2.5) and the relation [22, eq. (2.5)]

$$s'(n,k|r) = -(k+1)s(n,k+1|r). \tag{3.19}$$

Comparing the generating functions (2.9) and (3.16) shows that

$$s(m+k,k|r) = \frac{(m+k)!}{k!}b_m^{(-k)}(-r) = \binom{m+k}{k}B_m^{(m+k+1)}(1-r). \tag{3.20}$$

Therefore it follows from estimates (3.9), (3.10) that the given series converges absolutely and uniformly on compact subsets of $s\in\mathbb{C}$ under the given conditions on a and r, justifying the interchanging of summation and differentiation. This completes the proof. □

When we take $k = 1$ in this theorem, the series may be written in terms of *hyperharmonic numbers* $H_m^{[r]}$ which are defined by $H_m^{[0]} = \frac{1}{m}$ for $m > 0$, $H_0^{[r]} = 0$, and

$$H_m^{[r]} = \sum_{i=1}^{m} H_i^{[r-1]} \tag{3.21}$$

for positive integers r (cf. [2, 12]). Thus $H_n = H_n^{[1]}$ denotes the usual harmonic number. The hyperharmonic numbers are connected to weighted Stirling numbers by means of the identity $H_m^{[r]} = (-1)^{m+1} s(m, 1|r)/m!$.

Corollary 3.3. *For all nonnegative integers r and N we have*

$$\sum_{m=0}^{\infty} H_{m+N}^{[r]} \zeta_{-m}(s, a) = s\zeta_N(s+1, a-r) - \sum_{m=1}^{N-1} H_m^{[r]} \zeta_{N-m}(s, a)$$

for all $s \in \mathbb{C}$ and $a > r$.

Examples. Taking $N = 1$ in this corollary gives

$$\sum_{m=0}^{\infty} H_{m+1}^{[r]} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (j+a)^{-s} = s\zeta(s+1, a-r). \tag{3.22}$$

The $r = 0$ case of this formula is the original series of Hasse [9], whereas the $r = 1$ case was recently given by Blagouchine [3, eq. (133)]. If we take $r = 0$, $N = 2$, and $a = 1$ in the corollary we obtain a series for $\zeta_2(s)$ which reduces to Ser's original series

$$\sum_{m=0}^{\infty} \frac{1}{m+2} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (j+1)^{-s} = (s-1)\zeta(s) \tag{3.23}$$

[17] by means of the identity $\zeta_2(s) = \zeta(s-1)$, which follows directly from (2.2). So these classical series are members of the same hyperharmonic family, which is valid for all positive integer orders r.

4. Values at negative integers

Since the values of the Barnes zeta function at negative integers are given by Bernoulli polynomials as in (2.13), evaluating our global series at a negative integer will yield finite sum identities for Bernoulli polynomials, which may be of combinatorial interest. When evaluated at negative integers $s = -n$, the negative-order zeta functions $\zeta_{-m}(s, a)$ satisfy

$$\zeta_{-m}(-n, a) = (-1)^m m! S(n, m|a) \tag{4.1}$$

in terms of the *weighted Stirling numbers of the second kind* [5] which are defined by the generating function

$$e^{at}(e^t - 1)^m = m! \sum_{n=m}^{\infty} S(n, m|a) \frac{t^n}{n!}. \tag{4.2}$$

The weighted Stirling number $S(n, m|a)$ is a polynomial of degree $n - m$ in the variable a.

Corollary 4.1. *For all nonnegative integers N and k, we have the polynomial identity*

$$\sum_{m=0}^{n} m! b_{m+N}^{(k)}(x) S(n, m|a) = n! \left(\frac{B_{n+N}^{(N-k)}(a+x)}{(n+N)!} - \sum_{m=0}^{N-1} b_m^{(k)}(x) \frac{B_{N+n-m}^{(N-m)}(a)}{(N+n-m)!} \right)$$

for all nonnegative integers n.

Remark. This corollary may be viewed as a decomposition of the Bernoulli polynomial $B^{(N-k)}_{n+N}(a+x)$ of degree $n+N$ into a sum of $n+N+1$ "separable" polynomial terms of the form $f(x)g(a)$, each of total degree $n+N$. In the case $N=1$ it simplifies to

$$\sum_{m=0}^{n} m!b^{(k)}_{m+1}(x)S(n,m|a) = \frac{B^{(1-k)}_{n+1}(a+x) - B_{n+1}(a)}{n+1}. \tag{4.3}$$

Examples. For order $k=0$ the above says

$$\sum_{m=0}^{n} m!\binom{x}{m+1}S(n,m|a) = \frac{B_{n+1}(a+x) - B_{n+1}(a)}{n+1}, \tag{4.4}$$

and for $k=1$ it says

$$\sum_{m=0}^{n} m!b_{m+1}(x)S(n,m|a) = \frac{(a+x)^{n+1} - B_{n+1}(a)}{n+1}. \tag{4.5}$$

Corollary 4.2. *For all nonnegative integers N, n, and k such that $N+n \geqslant k$, we have the polynomial identity*

$$\sum_{m=0}^{n} \frac{s(m+N,k|r)S(n,m|a)}{(m+1)_N} = \frac{n!B^{(N)}_{N+n-k}(a-r)}{k!(N+n-k)!} - \sum_{m=k}^{N-1} \frac{n!s(m,k|r)B^{(N-m)}_{N+n-m}(a)}{m!(N+n-m)!}.$$

Examples. Taking $k=1$ in this corollary gives the polynomial identity for hyperharmonic numbers

$$\sum_{m=0}^{n}(-1)^{m+N}m!H^{[r]}_{m+N}S(n,m|a) = -\frac{n!B^{(N)}_{N+n-1}(a-r)}{(N+n-1)!} + \sum_{m=1}^{N-1}\frac{(-1)^m n!H^{[r]}_m B^{(N-m)}_{N+n-m}(a)}{m!(N+n-m)!} \tag{4.6}$$

for all nonnegative integers n. Taking $N=k$ in this corollary yields

$$\sum_{m=0}^{n} \frac{s(m+k,k|r)S(n,m|a)}{(m+1)_k} = \frac{B^{(k)}_n(a-r)}{k!}. \tag{4.7}$$

5. Values at positive integers

There are two important principles we often make use of when evaluating these series at positive integers. The first is the identity

$$\zeta_{-m}(n+1,a) := \sum_{j=0}^{m}(-1)^j\binom{m}{j}(j+a)^{-n-1} = \frac{m!P_n(h^{(1)}_m(a),\ldots,h^{(n)}_m(a))}{a(a+1)\cdots(a+m)} \tag{5.1}$$

[24, eq. (2.28)], where $P_n(x_1,\ldots,x_n)$ denotes the modified Bell polynomial defined by

$$\exp\left(\sum_{n=1}^{\infty} x_n\frac{t^n}{n}\right) = \sum_{n=0}^{\infty} P_n(x_1,\ldots,x_n)t^n \tag{5.2}$$

which is evaluated at generalized harmonic numbers

$$h^{(n)}_m(a) = \sum_{j=0}^{m}\frac{1}{(a+j)^n}. \tag{5.3}$$

For $s=n+1\in\mathbb{N}$, this identity may be used to rewrite the inner sum of the infinite series on the left hand side in terms of generalized harmonic numbers. The second observation is that the individual positive-order zeta functions $\zeta_N(s,a)$ on the right hand side have poles at positive integers $s \in \{1, 2, \dots, N\}$ and are regular if $s > N$. For example, the case $N = 1$, $k = 1$ case of Theorem 3.1 reads

$$\sum_{m=0}^{\infty} (-1)^m b_{m+1}(x)\zeta_{-m}(s,a) = \zeta(s,a) - \frac{(a+x)^{1-s}}{(s-1)}, \tag{5.4}$$

and the evaluation of the right side at $s = 1$ will be

$$\lim_{s\to 1} \left(\zeta(s,a) - \frac{1}{s-1}\right) - \left(\frac{(a+x)^{1-s} - 1}{s-1}\right) = -\psi(a) + \log(a+x), \tag{5.5}$$

where $\psi(a) = \frac{\partial}{\partial a}\frac{\partial}{\partial s}\zeta(s,a)\big|_{s=0}$ is the digamma function. For $s = 2, 3, \dots$ the evaluation is obtained by direct substitution, without requiring the use of limits. By means of these two principles, we present here a general statement for the case $N = 1$. (For general N one may also derive similar results.)

Theorem 5.1. *For $a > 0$ and $a + x > 0$ we have*

$$\sum_{m=0}^{\infty} \frac{(-1)^m m! b_{m+1}^{(k)}(x) P_n(h_m^{(1)}(a), \dots, h_m^{(n)}(a))}{a(a+1)\cdots(a+m)}$$

$$= \begin{cases} \psi(a+x) - \psi(a), & \text{if } k = n = 0, \\ \zeta(n+1,a) - \zeta(n+1,a+x), & \text{if } k = 0, n > 0, \\ \frac{(-1)^k}{(k-1)!}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}(j+a+x)^{k-1}\ln(j+a+x) - \psi(a), & \text{if } k > 0, n = 0, \\ \zeta(n+1,a) + \frac{(-1)^{k-n}}{n!(k-n-1)!}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}(j+a+x)^{k-n-1}\ln(j+a+x), & \text{if } 0 < n < k, \\ \zeta(n+1,a) - \frac{1}{(n)_k}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}(j+a+x)^{k-n-1}, & \text{if } 0 < k \leqslant n. \end{cases}$$

Examples. We observe some special cases of the above theorem. Taking $n = 0$ and $k = 1$, and observing that $P_0(x) = 1$, for $a > 0$ and $a + x > 0$, we have

$$\sum_{m=0}^{\infty} \frac{(-1)^m m! b_{m+1}(x)}{a(a+1)\cdots(a+m)} = \log(a+x) - \psi(a), \tag{5.6}$$

generalizing the classical Mascheroni series

$$\sum_{m=0}^{\infty} \frac{(-1)^m b_{m+1}}{m+1} = \gamma \tag{5.7}$$

[3, eq. (100)] to arbitrary a and x. Taking $n = 1$, $k = 1$, $a = 1$, and observing that $P_1(x) = x$ and $h_m^{(1)}(1) = H_{m+1}$, the harmonic number, for $x > -1$ we have

$$\sum_{m=0}^{\infty} \frac{(-1)^m b_{m+1}(x) H_{m+1}}{m+1} = \zeta(2) - \frac{1}{1+x}. \tag{5.8}$$

Taking $x=-1$ and $a=2$, and observing that $b_m(-1)=B_m^{(m)}/m!$ gives

$$\sum_{m=0}^{\infty}\frac{(-1)^m B_{m+1}^{(m+1)}}{(m+1)(m+2)!}=\gamma-1 \tag{5.9}$$

and

$$\sum_{m=0}^{\infty}\frac{(-1)^m B_{m+1}^{(m+1)}H_{m+2}}{(m+1)(m+2)!}=\zeta(2)+\gamma-3. \tag{5.10}$$

Although we do not give a general statement for all cases with $N>1$, we mention a few isolated identities of this type. In a previous article [21, §5] we derived identities

$$\sum_{m=0}^{\infty}\frac{(-1)^m b_{m+2}H_{m+1}}{m+1}=\frac{\zeta(2)}{2}-1 \tag{5.11}$$

and

$$\sum_{m=0}^{\infty}\frac{(-1)^m b_{m+3}^{(2)}H_{m+1}}{m+1}=\frac{\pi^2}{72}+\frac{\log(2\pi)}{2}-\frac{\gamma}{2}-\frac{3}{4}. \tag{5.12}$$

These series may also be evaluated by the principles of this section. For the first one, taking $k=1$, $N=2$, $a=1$, $x=0$ in Theorem 3.1 gives

$$\sum_{m=0}^{\infty}\frac{(-1)^m b_{m+2}H_{m+1}}{m+1}=\lim_{s\to 2}\left(\frac{\zeta(s-1)}{s-1}-\zeta_2(s)\right)+\frac{\zeta(2)}{2}. \tag{5.13}$$

Agreement between (5.11) and (5.13) may be obtained by means of the identity $\zeta_2(s)=\zeta(s-1)$ which follows from (2.2). For the second one, take $k=2$, $N=3$, $a=1$, $x=0$ in Theorem 3.1 to give

$$-\sum_{m=0}^{\infty}\frac{(-1)^m b_{m+3}^{(2)}H_{m+1}}{m+1}=\lim_{s\to 2}\left(\frac{\zeta(s-2)}{(s-1)(s-2)}-\zeta_3(s)+\zeta_2(s)\right)-\frac{\zeta(2)}{12}. \tag{5.14}$$

Agreement between (5.12) and (5.14) may be obtained by means of $\zeta(0)=-1/2$, $\zeta'(0)=-\log(2\pi)/2$, and the identity $\zeta_3(s)=(\zeta(s-1)+\zeta(s-2))/2$ which also follows from (2.2).

The algebraic nature of such constants related to zeta is a topic of considerable interest. A *period* [11] is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, in finitely many variables, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients. The set of periods forms a countable ring which contains all algebraic numbers, but also some transcendental numbers such as powers of π and logarithms of positive algebraic numbers. We make the following observation concerning the nature of Bernoulli polynomial series at positive integers.

Theorem 5.2. *Suppose that a and x are algebraic numbers such that $a>0$ and $a+x>0$. Then for any positive integer k the sum of the series*

$$\sum_{m=0}^{\infty}\frac{(-1)^m m!\, b_{m+N}^{(k)}(x)P_n(h_m^{(1)}(a),\ldots,h_m^{(n)}(a))}{a(a+1)\cdots(a+m)}$$

is a period when $n\geqslant N$.

Proof. Evaluating the series of Theorem 3.1 at $s=n+1\geqslant N+1$, all of the zeta functions on the right side are of the form $\zeta_r(s,a)$ where $s\geqslant r+1$ and $a\in\bar{\mathbb{Q}}$. The change of variables $u=1-e^{-t}$ in the integral representation (3.7) shows that any such value is a period, using the fact that $-\log(1-u)=\int_0^u dx/(1-x)$ for $0\leqslant u<1$. □

Remark. Although it is not even known to be irrational, Euler's constant γ is believed to not be a period. Certain other constants, such as e and $1/\pi$, are also believed to not be periods. When $n < N$ the evaluation of the series in Theorem 5.2 typically involves γ, as in (5.12), and in such cases is presumably not a period. The summands in the stated series are polynomials in x and rational functions of a, and the above theorem implies that as long as the Bernoulli polynomial index shift is not greater than n, the value of the given series is guaranteed to be a period.

Similar principles hold for the Stirling number series. Although we do not state the most general result here, we do give a fairly general simple result obtained by assuming $k \geqslant N$.

Theorem 5.3. *For $k \geqslant N$, $n \geqslant 0$, $a > 0$, and $a - r > 0$, we have*

$$\sum_{m=0}^{\infty} (-1)^{m+N} \frac{s(m+N,k|r) P_n(h_m^{(1)}(a), \ldots, h_m^{(n)}(a))}{(m+1)_N a(a+1)\cdots(a+m)} = (-1)^k \binom{n+k}{k} \zeta_N(n+k+1, a-r).$$

Examples. Taking $n = 0$ gives

$$\sum_{m=0}^{\infty} \frac{(-1)^{m+N} s(m+N,k|r)}{(m+1)_N a(a+1)\cdots(a+m)} = (-1)^k \zeta_N(k+1, a-r), \tag{5.15}$$

and in the special case where $N = k = 1$ and $a = r+1$ we obtain the hyperharmonic series

$$\sum_{m=0}^{\infty} \frac{m! H_{m+1}^{[r]}}{(r+1)\cdots(r+m+1)} = \zeta(2), \tag{5.16}$$

valid for any nonnegative integer order r. Similarly, taking $n = 1$ gives

$$\sum_{m=0}^{\infty} \frac{(-1)^{m+N} s(m+N,k|r)}{(m+1)_N a(a+1)\cdots(a+m)} \sum_{j=0}^{m} \frac{1}{j+a} = (-1)^k (k+1) \zeta_N(k+2, a-r), \tag{5.17}$$

and in the special case where $N = k = 1$ and $a = r+1$ we obtain the hyperharmonic series

$$\sum_{m=0}^{\infty} \frac{m! H_{m+1}^{[r]} (H_{m+r+1} - H_r)}{(r+1)\cdots(r+m+1)} = 2\zeta(3), \tag{5.18}$$

valid for any nonnegative integer order r.

We also have the following theorem concerning the algebraic nature of these Stirling number series, analogous to Theorem 5.2.

Theorem 5.4. *Suppose that a and r are algebraic numbers such that $a > 0$ and $a - r > 0$. Then the sum of the series*

$$\sum_{m=0}^{\infty} (-1)^m \frac{s(m+N,k|r) P_n(h_m^{(1)}(a), \ldots, h_m^{(n)}(a))}{(m+1)_N a(a+1)\cdots(a+m)}$$

is a period when $n + k \geqslant N$.

6. p-adic versions of the series

One of our primary motivations for considering these series is the fact that they may be also interpreted p-adically. Although as p-adic series they do not give *entire* functions of s, under fairly mild hypotheses on a they give analytic functions of s on p-adic disks which contain the ring of integers. Roughly speaking, the condition that $a > 0$ and $a + x > 0$ for complex series is replaced by the condition that a has p-adic absolute value larger than 1 and x is a p-adic integer.

For a prime number p we use $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ to denote the ring of p-adic integers, the field of p-adic numbers, and the completion of an algebraic closure of $\mathbb{Q}_p$, respectively. Let $|\cdot|_p$ denote the unique absolute value defined on $\mathbb{C}_p$ normalized by $|p|_p = p^{-1}$. Given $a \in \mathbb{C}_p^\times$, we define the p-adic valuation $\nu_p(a) \in \mathbb{Q}$ to be the unique exponent such that $|a|_p = p^{-\nu_p(a)}$. By convention we set $\nu_p(0) = \infty$.

We choose an embedding of the algebraic closure $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$ and fix it once and for all. Let $p^{\mathbb{Q}}$ denote the image in $\mathbb{C}_p^\times$ of the set of positive real rational powers of p under this embedding. Let μ denote the group of roots of unity in $\mathbb{C}_p^\times$ of order not divisible by p. If $a \in \mathbb{C}_p$, $|a|_p = 1$ then there is a unique element $\hat{a} \in \mu$ such that $|a - \hat{a}|_p < 1$ (called the *Teichmüller representative* of a); it may also be defined analytically by $\hat{a} = \lim_{n\to\infty} a^{p^{n!}}$. We extend this definition to $a \in \mathbb{C}_p^\times$ by

$$\hat{a} = (\widehat{a/p^{\nu_p(a)}}), \tag{6.1}$$

that is, we define $\hat{a} = \hat{u}$ if $a = p^r u$ with $p^r \in p^{\mathbb{Q}}$ and $|u|_p = 1$. We then define the function $\langle\cdot\rangle$ on $\mathbb{C}_p^\times$ by $\langle a\rangle = p^{-\nu_p(a)} a/\hat{a}$. This yields an internal direct product decomposition of multiplicative groups

$$\mathbb{C}_p^\times \simeq p^{\mathbb{Q}} \times \mu \times D \tag{6.2}$$

where $D = \{a \in \mathbb{C}_p : |a - 1|_p < 1\}$, given by

$$a = p^{\nu_p(a)} \cdot \hat{a} \cdot \langle a\rangle \mapsto (p^{\nu_p(a)}, \hat{a}, \langle a\rangle). \tag{6.3}$$

Since the projection $\langle a\rangle \in D$ is p-adically close to 1 for any $a \in \mathbb{C}_p^\times$, we may define useful p-adic power functions $\langle a\rangle^s$. First, for any $a \in \mathbb{C}_p^\times$, the power function may be defined using the binomial expansion

$$\langle a\rangle^s := \sum_{n=0}^{\infty} \binom{s}{n} (\langle a\rangle - 1)^n \tag{6.4}$$

which converges to (at least) a C^∞ function of $s \in \mathbb{Z}_p$ [16]. Beyond this, if we assume that a lies in some finite extension K of $\mathbb{Q}_p$ whose ramification index over $\mathbb{Q}_p$ is less than $p - 1$, then the composition of power series

$$\langle a\rangle^s := \exp(s \log\langle a\rangle) \tag{6.5}$$

converges to an *analytic* function of s on a disk in $\mathbb{C}_p$ which contains $\mathbb{Z}_p$, and the definitions (6.4) and (6.5) agree on this disk [18, Proposition 2.1].

In [18] we defined p-adic multiple zeta functions $\zeta_{p,r}(s, a)$ for $r \in \mathbb{Z}^+$ and $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$ by the r-fold Volkenborn integral

$$\zeta_{p,r}(s, a) = \frac{1}{(s-1)\cdots(s-r)} \int_{\mathbb{Z}_p^r} \frac{(a + t_1 + \cdots + t_r)^r}{\langle a + t_1 + \cdots + t_r\rangle^s} \, dt_1 \cdots dt_r. \tag{6.6}$$

For fixed a, $\zeta_{p,r}(s, a)$ is a C^∞ function of s on $\mathbb{Z}_p \setminus \{1, \ldots, r\}$, and is an analytic function of s on a disc of positive radius about zero, on which it is independent of the choice made to define the $\langle\cdot\rangle$ function. If a lies in a finite extension K of $\mathbb{Q}_p$ whose ramification index over $\mathbb{Q}_p$

is less than $p-1$ then $\zeta_{p,r}(s,a)$ is analytic for $s \in \mathbb{C}_p$ such that $|s|_p < |\pi|_p^{-1} p^{-1/(p-1)}$, except for simple poles at $s = 1, \ldots, r$. If $s \notin \{1, \ldots, r\}$, the function $\zeta_{p,r}(s,a)$ is locally analytic as a function of a on $\mathbb{C}_p \setminus \mathbb{Z}_p$ [18, Theorem 3.1].

At the negative integers the values of the p-adic zeta function $\zeta_{p,r}$ are given by

$$\zeta_{p,r}(-n,a) = \frac{(-1)^r r!}{(n+r)!} \left(\frac{\langle a \rangle}{a} \right)^n B_{n+r}^{(r)}(a) \tag{6.7}$$

for $|a|_p > 1$ ([18], Theorem 3.2(v)), in agreement with (2.3) up to a power of the locally constant, algebraic factor $\langle a \rangle / a$.

For a nonnegative integer r we define, as in [23], the p-adic Barnes zeta function $\zeta_{p,-r}(s,a)$ of order $-r$ by

$$\zeta_{p,-r}(s,a) = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \langle a+j \rangle^{-s}, \tag{6.8}$$

for $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$ and $s \in \mathbb{C}_p$ such that the series (6.4) or (6.5) converges, so that $(-1)^r \zeta_{p,-r}(s,a)$ is the r-th forward difference of the power function $\langle a \rangle^{-s}$ with respect to the a parameter. With these definitions we have the difference equation

$$\zeta_{p,r}(s,a) - \zeta_{p,r}(s,a+1) = \zeta_{p,r-1}(s,a) \tag{6.9}$$

for all integers r, and the derivative-shift identity

$$\frac{\partial}{\partial a} \zeta_{p,r}(s,a) = -s \frac{\langle a \rangle}{a} \zeta_{p,r}(s+1,a) \tag{6.10}$$

for all integers r [18, 23].

We now present the p-adic analogues of Theorem 3.1 and Theorem 3.2.

Theorem 6.1. *For all nonnegative integers N and k, there is an identity*

$$\sum_{m=0}^{\infty} (-1)^{m+N} b_{m+N}^{(k)}(x) \sum_{j=0}^{m} (-1)^j \binom{m}{j} \langle j+a \rangle^{-s}$$

$$= \left(\frac{a}{\langle a \rangle} \right)^k \frac{\zeta_{p,N-k}(s-k, a+x)}{(s-1)_k} - \sum_{m=0}^{N-1} (-1)^m b_m^{(k)}(x) \zeta_{p,N-m}(s,a)$$

for $s, x \in \mathbb{Z}_p$, provided that $|a|_p > 1$. These are analytic functions of $x \in \mathbb{Z}_p$ and (at least) C^∞ functions of $s \in \mathbb{Z}_p$ which are analytic functions of s on some disk in $\mathbb{C}_p$ containing $s = 0$. If in addition a lies in a finite extension K of $\mathbb{Q}_p$ whose ramification index over $\mathbb{Q}_p$ is less than $p-1$, then these are analytic functions of s on a disk in $\mathbb{C}_p$ containing $\mathbb{Z}_p$.

Proof. Concerning convergence of the series, we proved in [23, Theorem 3] that $|\zeta_{-m}(s,a)|_p \leqslant |m!|_p |a|_p^{-m}$ for $s \in \mathbb{Z}_p$ and $|a|_p > 1$. Howard [10] proved that $\nu_p(b_m) \geqslant -[\![m/(p-1)]\!]$, so it follows from the generating function (2.9) that $\nu_p(b_m^{(k)}(x)) \geqslant -[\![m/(p-1)]\!]$ for all $k \in \mathbb{Z}^+$ and $x \in \mathbb{Z}_p$. Therefore the above series converges absolutely and uniformly in $s \in \mathbb{Z}_p$ when $|a|_p > 1$ and $s \in \mathbb{Z}_p$. Comparison of (4.1) and Corollary 4.1 shows that the stated equality holds when $s = -n$ is a negative integer. Since both sides are (at least) C^∞ functions of $s \in \mathbb{Z}_p$ and they agree at the negative integers, they agree for all indicated values of s. □

Theorem 6.2. *For all positive integers k and nonnegative integers N, there is an identity*

$$\sum_{m=0}^{\infty}(-1)^{m+N}\frac{s(m+N,k|r)}{(m+N)!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\langle j+a\rangle^{-s}$$
$$=\left(\frac{\langle a\rangle}{a}\right)^k(-1)^k\binom{s+k-1}{k}\zeta_{p,N}(s+k,a-r)-\sum_{m=k}^{N-1}(-1)^m\frac{s(m,k|r)}{m!}\zeta_{p,N-m}(s,a)$$

for $s,r\in\mathbb{Z}_p$, provided that $|a|_p>1$. These are analytic functions of $r\in\mathbb{Z}_p$ and (at least) C^∞ functions of $s\in\mathbb{Z}_p$ which are analytic functions of s on some disk in $\mathbb{C}_p$ containing $s=0$. If in addition a lies in a finite extension K of $\mathbb{Q}_p$ whose ramification index over $\mathbb{Q}_p$ is less than $p-1$, then these are analytic functions of s on a disk in $\mathbb{C}_p$ containing $\mathbb{Z}_p$.

It will be observed that these series are identical to those of Theorems 3.1 and 3.2 up to replacing $\zeta_r(s,a)$ with $\zeta_{p,r}(s,a)$, and the inclusion of a power of $\langle a\rangle/a$ in one term on the right. The same principles which were used to evaluate those series at positive integers $s=n+1$ may also be applied to the p-adic series, namely we have

$$\zeta_{p,-m}(n+1,a):=\sum_{j=0}^{m}(-1)^j\binom{m}{j}\langle j+a\rangle^{-n-1}=\left(\frac{a}{\langle a\rangle}\right)^{n+1}\frac{m!P_n(h_m^{(1)}(a),\ldots,h_m^{(n)}(a))}{a(a+1)\cdots(a+m)} \quad (6.11)$$

for $|a|_p>1$ [24, eq. (2.28)], in direct analogy to (5.1).

Examples. Taking $N=1$, $k=1$, $s=1$, $n=0$ in Theorem 6.1 and equation (5.6) yields

$$\sum_{m=0}^{\infty}\frac{(-1)^m m!b_{m+1}(x)}{a(a+1)\cdots(a+m)}=\begin{cases}\log(a+x)-\psi(a)\text{ in }\mathbb{R}, & \text{if } a>0 \text{ and } a+x>0,\\ \log_p(a+x)-\psi_p(a)\text{ in }\mathbb{C}_p, & \text{if } |a|_p>1 \text{ and } x\in\mathbb{Z}_p,\end{cases} \quad (6.12)$$

where $\log_p$ is the p-adic Iwasawa logarithm, defined by its usual power series on D and extended to all of $\mathbb{C}_p^\times$ by $\log_p a:=\log_p\langle a\rangle$, and $\psi_p(a)=\frac{\partial}{\partial a}\frac{\partial}{\partial s}\zeta_p(s,a)|_{s=0}$ is the p-adic digamma function. This gives a generalization to all $x\in\mathbb{Z}_p$ of a result of [23, Theorem 5], so that the exact same series of polynomials in x and rational functions of a converges, in both real and p-adic metrics, to analogous transcendental functions, under appropriate conditions.

The remaining examples we present here all involve the prime $p=2$, which we choose for simplicity while remarking that similar identities hold for any prime p. Taking $N=1$, $k=1$, $s=2$, $n=1$, $a=1/2$ in Theorem 5.1 and Theorem 6.1 gives

$$\sum_{m=0}^{\infty}\frac{(-1)^m4^{m+1}b_{m+1}(x)O_{m+1}}{(2m+1)\binom{2m}{m}}=\begin{cases}\frac{2}{1+2x}-3\zeta(2)\text{ in }\mathbb{R}, & \text{if } x>-1/2,\\ \frac{2}{1+2x}-4\zeta_{2,1}(2,\frac{1}{2})\text{ in }\mathbb{C}_2, & \text{if } x\in\mathbb{Z}_2,\end{cases} \quad (6.13)$$

where $O_m:=\sum_{j=1}^m\frac{1}{2j-1}$ is the m-th "odd harmonic" number, generalizing [23, eq. (5.18)]. (Note that $\zeta_{2,1}(2,\frac{1}{2})=0$ by the reflection formula [18, Theorem 3.2(iii)].) Taking $N=1$, $k=1$, $n=0$, $s=1$, $x=-1$, $a=3/2$ gives

$$\sum_{m=0}^{\infty}\frac{(-1)^m4^{m+1}B_{m+1}^{(m+1)}}{(m+1)(2m+3)(m+1)!\binom{2m+2}{m+1}}=\begin{cases}-\log 2-\psi(3/2) & \text{in }\mathbb{R},\\ -\log_2 2-\psi_2(3/2) & \text{in }\mathbb{Q}_2,\end{cases} \quad (6.14)$$

which is similar to [23, eq. (5.17)]; note also that $\log_2 2 = 0$. Taking $N = 1$, $k = 1$, $n = 0$, $s = 1$, $a = r + \frac{1}{2}$ in Theorem 5.3 and Theorem 6.2 yields the hyperharmonic series

$$\sum_{m=0}^{\infty} \frac{4^{m+1}\binom{2r}{r} H_{m+1}^{[r]}}{(2r+2m+1)\binom{2m+2r}{m+r}\binom{m+r}{r}} = \begin{cases} 6\zeta(2) = \pi^2 & \text{in } \mathbb{R}, \\ 8\zeta_{2,1}(2, \frac{1}{2}) = 0 & \text{in } \mathbb{Q}_2 \end{cases} \tag{6.15}$$

which is valid both in $\mathbb{R}$ and in $\mathbb{Q}_2$ for any nonnegative integer order r. Taking $N = 1$, $k = 1$, $n = 1$, $s = 2$, $a = r + \frac{1}{2}$ yields

$$\sum_{m=0}^{\infty} \frac{4^{m+1}\binom{2r}{r} H_{m+1}^{[r]}(O_{r+m+1} - O_r)}{(2r+2m+1)\binom{2m+2r}{m+r}\binom{m+r}{r}} = \begin{cases} 7\zeta(3) & \text{in } \mathbb{R}, \\ 16\zeta_{2,1}(3, \frac{1}{2}) & \text{in } \mathbb{Q}_2, \end{cases} \tag{6.16}$$

again valid in both metrics for any nonnegative integer order r.

7. Stieltjes constants

The Stieltjes constants $\gamma_n(a)$ are the coefficients in the Laurent expansion of $\zeta(s, a)$ at $s = 1$, defined, together with their p-adic counterparts $\gamma_{p,n}(a)$, by

$$\zeta(s, a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(a)(s-1)^n \qquad (a > 0), \tag{7.1}$$

$$\zeta_{p,1}(s, a) = \frac{a/\langle a \rangle}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_{p,n}(a)(s-1)^n \qquad (|a|_p > 1). \tag{7.2}$$

We remark that under the assumption that $|a|_p > 1$, the p-adic zeta function $\zeta_{p,1}(s, a)$ is always meromorphic on some disk in $\mathbb{C}_p$ containing $s = 1$, with a simple pole at $s = 1$ with residue $a/\langle a \rangle$, so the above series expansion is valid on some disk in $\mathbb{C}_p$ containing $s = 1$. The following theorem is obtained by applying $\frac{\partial^n}{\partial s^n}\big|_{s=0}$ to the series of Theorems 3.1, 3.2, 6.1, and 6.2 for $\zeta(s, a)$ and $\zeta_{p,1}(s, a)$.

Theorem 7.1. *For the n-th Stieltjes constant, we have the Bernoulli polynomial formula*

$$\gamma_n(a) = \sum_{m=0}^{\infty} (-1)^m b_{m+1}(x) \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{\log(j+a)^n}{j+a} - \frac{\log(a+x)^{n+1}}{n+1},$$

$$\gamma_{p,n}(a) = \sum_{m=0}^{\infty} (-1)^m b_{m+1}(x) \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{\log_p(j+a)^n}{j+a} - \frac{\log_p(a+x)^{n+1}}{n+1},$$

and the weighted Stirling number formula

$$\gamma_n(a-r) = \frac{-1}{n+1} \sum_{m=0}^{\infty} H_{m+1}^{[r]} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \log(j+a)^{n+1},$$

$$\gamma_{p,n}(a-r) = \frac{-1}{n+1} \sum_{m=0}^{\infty} H_{m+1}^{[r]} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \log_p(j+a)^{n+1}.$$

As real series, these are valid when $a > 0$, $a + x > 0$, $a - r > 0$. As p-adic series, they are valid when $|a|_p > 1$ and $x, r \in \mathbb{Z}_p$.

We remark that the above (real) Bernoulli formula for $\gamma_n(a)$ was recently given by Blagouchine [3, eq. (89)], who also gave the $r = 0$ and $r = 1$ cases of the real hyperharmonic formula [3, eq. (123), eq. (138)]. We emphasize that all these formulas are of the same general family, and they have perfectly natural p-adic analogues.

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