SOME COMBINATORIAL ASPECTS OF BI-PERIODIC INCOMPLETE HORADAM SEQUENCES

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Abstract. We have recently introduced the bi-periodic incomplete Horadam numbers as a generalization of incomplete Horadam numbers, and studied their properties. In this paper, we provide some combinatorial interpretations of bi-periodic incomplete Horadam numbers by using the weighted tilings approach. We also define bi-periodic hyper Horadam numbers and show that each bi-periodic hyper Horadam number can be written as the difference of a bi-periodic Horadam number and a bi-periodic incomplete Horadam number.

1. Introduction

The bi-periodic Horadam sequence $\{w_n\}$ with arbitrary initial values w_0 and w_1 is defined for $n \geq 2$ by the recurrence relation

$$w_n = a^{\xi(n+1)} b^{\xi(n)} w_{n-1} + c w_{n-2}, \tag{1.1}$$

where a, b and c are nonzero real numbers. Here, $\xi(n) = [1 - (-1)^n]/2$. Note that $\xi(n) = 0$ when n is even, and $\xi(n) = 1$ when n is odd. This sequence is a natural generalization of the classical Horadam sequence $\{W_n\}$, obtained when $a = b = p$ and $c = q$. In particular, $\{w_n\}$ reduces to the generalized bi-periodic Fibonacci sequence $\{u_n\}$ for the initial values $u_0 = 0$ and $u_1 = 1$. If in addition $a = b = c = 1$, we obtain the Fibonacci sequence $\{F_n\}$. We refer to [4, 6, 9, 13, 15, 16, 17] for basic properties of the generalized bi-periodic Fibonacci sequence and bi-periodic Horadam sequence.

An n-board is a board of dimensions $1 \times n$ with 1×1 cells labeled $1, 2, \ldots, n$. Suppose that the board is covered with 1×1 squares and 1×2 dominoes, where a square covers a single cell and a domino covers two cells. It is well-known that the $(n+1)$th Fibonacci number F_{n+1} counts the number of distinct tilings of an n-board using squares and dominoes [3]. It can be expressed as

$$F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i},$$

where i can be thought of as the number of dominoes on the n-board. This expression gives rise to an interesting class of integers, called incomplete Fibonacci numbers. They were introduced by Filipponi [7] as

$$F_{n+1}(k) = \sum_{i=0}^{k} \binom{n-i}{i}, \quad 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Combinatorially, $F_{n+1}(k)$ counts the number of ways to tile an n-board with at most k dominoes [1]. Incomplete Fibonacci numbers generalize the Fibonacci numbers. In fact, they reduce to the Fibonacci numbers for $k = \lfloor \frac{n}{2} \rfloor$. Similar constructions exist for Lucas numbers [7].

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Hyper-Fibonacci numbers were introduced by Dil and Mezö [5], and they satisfy the following explicit formula

$$F_{n+1}^{(k)} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+k-i}{i+k}.$$

These numbers also have a combinatorial interpretation: $F_{n+1}^{(k)}$ counts the number of distinct ways to tile an $(n+2k)$-board with at least k dominoes [1].

Since Horadam numbers generalize Fibonacci and Lucas numbers, Belbachir and Belkhir [2] introduced a generalization of the incomplete Fibonacci and the incomplete Lucas numbers. These numbers are called the incomplete Horadam numbers and they are defined by

$$W_n(k) = \sum_{i=0}^{k} \frac{(n-2i)W_1 + piW_0}{n-i} \binom{n-i}{i} p^{n-2i-1} q^i, \quad 0 \le k \le \left\lfloor \frac{n}{2} \right\rfloor,$$

where $\{W_n\}$ is the Horadam sequence. In the spirit of [2], we have recently introduced and studied [14] the bi-periodic incomplete Horadam numbers $w_n(k; a, b, c)$, or $w_n(k)$ for short, given by

$$w_n(k) = a^{\xi(n-1)} \sum_{i=0}^{k} \frac{(n-2i)w_1 + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i} c^i \tag{1.2}$$

for positive integers n and k with $0 \le k \le \lfloor \frac{n}{2} \rfloor$. Note that the bi-periodic incomplete Horadam numbers $w_n(k)$ reduce to the bi-periodic Horadam numbers w_n when $k = \lfloor \frac{n}{2} \rfloor$. Moreover, they reduce to the generalized bi-periodic incomplete Fibonacci numbers $u_n(k)$ with the initial values 0 and 1, given explicitly by

$$u_n(k) = a^{\xi(n-1)} \sum_{i=0}^{k} \binom{n-i-1}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i} c^i. \tag{1.3}$$

The incomplete Horadam numbers are also a special case of these numbers since they are obtained for $a = b = p$ and $c = q$.

As in the case of incomplete Fibonacci numbers and their various generalizations, the incomplete Horadam numbers also have combinatorial aspects. Our goal in this paper is to study some of such aspects by means of weighted tilings. In particular, we will provide combinatorial interpretation and proofs for some of the statements of [14]. We will also define the bi-periodic hyper Horadam numbers and provide a few other results.

2. Main Results

We define a weighted n-tiling as a tiling of an n-board by squares and dominoes as follows:

(1) if the first tile is a square, it has a weight of w_1;
(2) if the first tile is a domino, it has a weight of cw_0;
(3) for the remaining tiles,
   (a) if a square is in an even position, it has a weight of a;
   (b) if a square is in an odd position, it has a weight of b;
   (c) each domino must have a weight of c.

Let WT_n be the set of all weighted n-tilings. Given a weighted tiling $t \in WT_n$, we define the weight $\omega(t)$ of t as the product of the weights of its tiles. The total weight of the set WT_n

is the sum of the weights of all weighted tilings:

$$\mathcal{S}_n = \sum_{t \in WT_n} \omega(t).$$

For example, the 4-board has five distinct weighted tilings as shown in Figure 1. Therefore, the total weight of the set WT_4 is $\mathcal{S}_4 = a^2bw_1 + 2acw_1 + abcw_0 + c^2w_0$.

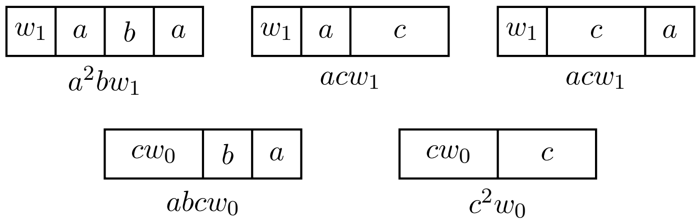

FIGURE 1. All five weighted tilings of the 4-board

In the following lemma, we obtain an expression for the total weight of weighted n-tilings with exactly i dominoes.

Lemma 2.1. *Let n and i be positive integers with $0 \le i \le \lfloor \frac{n}{2} \rfloor$, and let $\mathcal{S}_{n,i}$ be the total weight of weighted n-tilings with exactly i dominoes. Then,*

$$\mathcal{S}_{n,i} = a^{\xi(n-1)} \frac{(n-2i)w_1 + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i} c^i. \tag{2.1}$$

Proof. Let $WT_{n,i}$ be the set of all weighted tilings of length n with exactly i dominoes. Since the first tile is either a square or a domino, we can partition $WT_{n,i}$ into two subsets, the tilings beginning with a square and the tilings beginning with a domino. Let $\mathcal{S}_{n,i}^{square}$ and $\mathcal{S}_{n,i}^{domino}$ be the total weight of such subsets, respectively. Then $\mathcal{S}_{n,i} = \mathcal{S}_{n,i}^{square} + \mathcal{S}_{n,i}^{domino}$. Next, we count the total weight of each subset.

If the first tile is a square, then its weight is w_1. Now consider the tilings of the remaining $n-1$ cells with exactly i dominoes. There are $\binom{n-1-i}{i}$ ways to select such dominoes. Once the dominoes are selected, the remaining cells must be covered by squares. If $n-1$ is even, the squares must be evenly distributed in the odd and even positions. So the weight of such an $(n-1)$-tiling is

$$\binom{n-1-i}{i} c^i (ab)^{\lfloor \frac{n-1}{2} \rfloor - i}.$$

If $n-1$ is odd, on the other hand, the number of squares in even positions is one more than the number of squares in odd positions. So the weight of such an $(n-1)$-tiling is

$$\binom{n-1-i}{i} c^i a (ab)^{\lfloor \frac{n-1}{2} \rfloor - i}.$$

Hence if the first tile is a square, we obtain the following expression for the weight of the corresponding n-tiling with exactly i dominoes:

$$\mathcal{S}_{n,i}^{square} = a^{\xi(n-1)} \binom{n-1-i}{i} w_1 c^i (ab)^{\lfloor \frac{n-1}{2} \rfloor - i}. \tag{2.2}$$

Now suppose that the first tile is a domino. Then its weight is cw_0. There are $\binom{n-i-1}{i-1}$ ways to select exactly $i-1$ dominoes from the tilings of the remaining $n-2$ cells where $1 \le i \le \lfloor \frac{n}{2} \rfloor$. If $n-2$ is even, the weight of such an $(n-2)$-tiling is

$$\binom{n-1-i}{i-1} c^{i-1}(ab)^{\lfloor \frac{n}{2} \rfloor - i} = \binom{n-1-i}{i-1} c^{i-1} ab(ab)^{\lfloor \frac{n-1}{2} \rfloor - i}.$$

If $n-2$ is odd, the weight of such an $(n-2)$-tiling is

$$\binom{n-1-i}{i-1} c^{i-1} b(ab)^{\lfloor \frac{n}{2} \rfloor - i} = \binom{n-1-i}{i-1} c^{i-1} b(ab)^{\lfloor \frac{n-1}{2} \rfloor - i}.$$

Hence if the first tile is a domino, the weight of the corresponding n-tiling can be expressed as

$$\mathcal{S}_{n,i}^{domino} = a^{\xi(n-1)} \binom{n-1-i}{i-1} w_0 c^i b(ab)^{\lfloor \frac{n-1}{2} \rfloor - i}. \tag{2.3}$$

Then, (2.1) follows from adding equations (2.2) and (2.3). □

Theorem 2.2. *Let n and k be positive integers with $0 \le k \le \lfloor \frac{n}{2} \rfloor$. If $\mathcal{S}_n(k)$ is the total weight of weighted n-tilings with at most k dominoes, then*

$$\mathcal{S}_n(k) = a^{\xi(n-1)} \sum_{i=0}^{k} \frac{(n-2i)w_1 + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i} c^i.$$

Proof. This follows immediately from Lemma 2.1 by summing (2.1) over i. □

Theorem 2.3. *Let n and k be positive integers with $0 \le k \le \lfloor \frac{n}{2} \rfloor$. Then the bi-periodic incomplete Horadam number $w_n(k)$ counts the total weight of n-tilings with at most k dominoes. Specifically, $w_n(k) = \mathcal{S}_n(k)$.*

Proof. Comparing (1.2) with Theorem 2.2, we get the desired result. □

Consider two positive integers n and k with $0 \le k \le \lfloor \frac{n-1}{2} \rfloor$. Let a, b and c be nonzero real numbers. Using similar arguments as in Lemma 2.1, the generalized bi-periodic incomplete Fibonacci number $u_{n+1}(k; a, b, c)$ can be thought of as the total weight of the set of n-tilings with at most k dominoes under the following three conditions. Each domino has a weight of c. If a square is in an odd position then it has a weight of a. Otherwise, it has a weight of b.

In the following proposition, we give an expression involving the bi-periodic incomplete Horadam numbers $w_n(k; a, b, c)$ and the generalized bi-periodic incomplete Fibonacci numbers $u_n(k; a, b, c)$.

Proposition 2.4. *Let n and k be positive integers with $0 \le k \le \lfloor \frac{n}{2} \rfloor$. Then,*

$$w_n(k; a, b, c) = w_1 u_n(k; a, b, c) + cw_0 u_{n-1}(k-1; a^{\xi(n-1)} b^{\xi(n)}, a^{\xi(n)} b^{\xi(n-1)}, c). \tag{2.4}$$

Combinatorial proof. **Question:** What is the total weight of n-tilings with at most k dominoes?

Answer 1: The total weight of such tilings is $\mathcal{S}_n(k)$ which equals $w_n(k)$ by Theorem 2.3.

Answer 2: Condition on the first tile. If it is a square, then it follows from (2.2) and (1.3) that the total weight of such tilings is

$$w_1 \sum_{i=0}^{k} \binom{n-1-i}{i} c^i (ab)^{\lfloor \frac{n-1}{2} \rfloor - i} a^{\xi(n-1)} = w_1 u_n(k; a, b, c).$$

Otherwise, the first tile is a domino and, using (2.3) with i replaced by $i+1$ and (1.3), the total weight is

$$\begin{cases} cw_0 \sum_{i=0}^{k-1} \binom{n-2-i}{i} c^i (ab)^{\lfloor \frac{n-2}{2} \rfloor - i} = cw_0 u_{n-1}(k-1; a, b, c), & \text{if } n \text{ is even;} \\ cw_0 \sum_{i=0}^{k-1} \binom{n-2-i}{i} c^i b (ab)^{\lfloor \frac{n-2}{2} \rfloor - i} = cw_0 u_{n-1}(k-1; b, a, c), & \text{if } n \text{ is odd.} \end{cases}$$

Considering both cases we obtain the desired result. □

In the following, we provide a combinatorial proof for Proposition 2.5 of [14].

Proposition 2.5. *For $0 \le k \le \frac{n-r-1}{2}$, we have*

$$\sum_{i=0}^{r} \binom{r}{i} w_{n+i}(k+i) a^{\left\lfloor \frac{i+\xi(n+1)}{2} \right\rfloor} b^{\left\lfloor \frac{i+\xi(n)}{2} \right\rfloor} c^{r-i} = w_{n+2r}(k+r). \tag{2.5}$$

Combinatorial proof. **Question:** What is the total weight of the $(n+2r)$-tilings with at most $k+r$ dominoes?

Answer 1: The total weight of such tilings is $\mathcal{S}_{n+2r}(k+r)$ which equals $w_{n+2r}(k+r)$ by Theorem 2.3.

Answer 2: Consider the number of dominoes appearing among the last r tiles. There are $\binom{r}{i}$ ways to select positions for i dominoes among the last r tiles where $0 \le i \le r$. Now, if $r-i$ is odd, then the weight of the remaining $r-i$ squares is given by

$$\begin{cases} ba^{\lfloor \frac{r-i}{2} \rfloor} b^{\lfloor \frac{r-i}{2} \rfloor}, & \text{if } n+1+r-i \text{ is odd;} \\ aa^{\lfloor \frac{r-i}{2} \rfloor} b^{\lfloor \frac{r-i}{2} \rfloor}, & \text{if } n+1+r-i \text{ is even;} \end{cases} \iff \begin{cases} ba^{\lfloor \frac{r-i}{2} \rfloor} b^{\lfloor \frac{r-i}{2} \rfloor}, & \text{if } n \text{ is odd;} \\ aa^{\lfloor \frac{r-i}{2} \rfloor} b^{\lfloor \frac{r-i}{2} \rfloor}, & \text{if } n \text{ is even;} \end{cases}$$

and hence $a^{\left\lfloor \frac{r-i+\xi(n+1)}{2} \right\rfloor} b^{\left\lfloor \frac{r-i+\xi(n)}{2} \right\rfloor}$. See Figure 2.

Similarly, if $r-i$ is even, we have

$$a^{\left\lfloor \frac{r-i}{2} \right\rfloor} b^{\left\lfloor \frac{r-i}{2} \right\rfloor} = a^{\left\lfloor \frac{r-i+\xi(n+1)}{2} \right\rfloor} b^{\left\lfloor \frac{r-i+\xi(n)}{2} \right\rfloor}.$$

Note that the weight of the remaining $(n+r-i)$-tilings with at most $k+r-i$ dominoes is given by $w_{n+r-i}(k+r-i)$. Altogether, the weight of the $(n+2r)$-tilings with exactly i dominoes among the last r tiles is given by

$$\binom{r}{i} a^{\left\lfloor \frac{r-i+\xi(n+1)}{2} \right\rfloor} b^{\left\lfloor \frac{r-i+\xi(n)}{2} \right\rfloor} c^i w_{n+r-i}(k+r-i).$$

Summing the above quantity yields the desired result.

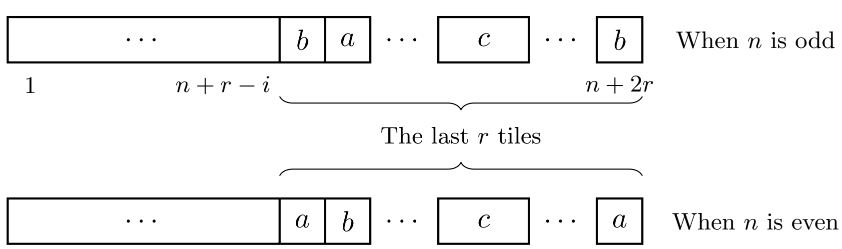

FIGURE 2. An $(n+2r)$-tiling with i dominoes and $r-i$ squares from cells $n+r-i+1$ to $n+2r$

□

Finally, we provide a combinatorial proof for Proposition 2.6 of [14].

Proposition 2.6. *For $r \geq 2k+2$, we have*

$$\sum_{i=0}^{r-1} a^{\left\lfloor \frac{r-\xi(n+1)}{2} \right\rfloor - \left\lfloor \frac{i+\xi(n)}{2} \right\rfloor} b^{\left\lfloor \frac{r-\xi(n)}{2} \right\rfloor - \left\lfloor \frac{i+\xi(n+1)}{2} \right\rfloor} c w_{n+i}(k) = w_{n+r+1}(k+1) - a^{\left\lfloor \frac{r+\xi(n+1)}{2} \right\rfloor} b^{\left\lfloor \frac{r+\xi(n)}{2} \right\rfloor} w_{n+1}(k+1). \tag{2.6}$$

Combinatorial proof. **Question:** What is the total weight of the $(n+r+1)$-tilings with at most $k+1$ dominoes under the condition that at least one of the cells $n+1, \ldots, n+r$ is occupied by the left half of a domino?

Answer 1: The total weight of such tilings corresponds to the total weight of the set of $(n+r+1)$-tilings with at most $k+1$ dominoes excluding the weight of the subset of $(n+r+1)$-tilings with only squares from cells $n+2$ to $n+r+1$. In fact, the total weight of the set of $(n+r+1)$-tilings with at most $k+1$ dominoes is $w_{n+r+1}(k+1)$. On the other hand, the total weight of the $(n+r+1)$-tilings with only squares from cells $n+2$ to $n+r+1$ is given as follows:

(1) If r is odd, the weight is

$$\begin{cases} w_{n+1}(k+1) a^{\lfloor \frac{r}{2} \rfloor} b^{\lfloor \frac{r}{2} \rfloor} b, & \text{if } n \text{ is odd;} \\ w_{n+1}(k+1) a a^{\lfloor \frac{r}{2} \rfloor} b^{\lfloor \frac{r}{2} \rfloor}, & \text{if } n \text{ is even;} \end{cases}$$

and hence $w_{n+1}(k+1) a^{\left\lfloor \frac{r+\xi(n+1)}{2} \right\rfloor} b^{\left\lfloor \frac{r+\xi(n)}{2} \right\rfloor}$.

(2) If r is even, the weight is

$$w_{n+1}(k+1) a^{\lfloor \frac{r}{2} \rfloor} b^{\lfloor \frac{r}{2} \rfloor} = w_{n+1}(k+1) a^{\left\lfloor \frac{r+\xi(n+1)}{2} \right\rfloor} b^{\left\lfloor \frac{r+\xi(n)}{2} \right\rfloor}.$$

Answer 2: Suppose that the last domino from the right hand side occupies the cells $i+1$ and $i+2$ where $n \leq i \leq n+r-1$. Then, for any i, the weight to tile cells from 1 to i with at most k dominoes is $w_i(k)$; the weight of the domino that occupies cells $i+1$, $i+2$ is c; and the weight of the remaining cells is $a^{\left\lfloor \frac{r-\xi(n+1)}{2} \right\rfloor - \left\lfloor \frac{i-n+\xi(n)}{2} \right\rfloor} b^{\left\lfloor \frac{r-\xi(n)}{2} \right\rfloor - \left\lfloor \frac{i-n+\xi(n+1)}{2} \right\rfloor}$. Hence, the weight of the $(n+r+1)$-tilings with at least one domino from cells $n+1$ to $n+r+1$ is

$$a^{\left\lfloor \frac{r-\xi(n+1)}{2} \right\rfloor - \left\lfloor \frac{i-n+\xi(n)}{2} \right\rfloor} b^{\left\lfloor \frac{r-\xi(n)}{2} \right\rfloor - \left\lfloor \frac{i-n+\xi(n+1)}{2} \right\rfloor} c w_{n+i}(k).$$

Summing yields the desired result. □

In the following, we give a connection between the bi-periodic incomplete Horadam numbers and the bi-periodic Horadam numbers. To this purpose, we define the bi-periodic hyper Horadam numbers.

Definition 2.7. *Let n and k be positive integers. The bi-periodic hyper Horadam numbers $w_n^{(k)}$ are defined by*

$$w_n^{(k)} = a^{\xi(n-1)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-2i)w_1 + (k+i)bw_0}{n+k-i} \binom{n+k-i}{i+k} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i} c^{k+i}. \tag{2.7}$$

Theorem 2.8. *Let n and k be positive integers. Then the bi-periodic hyper Horadam number $w_n^{(k)}$ counts the total weight of the weighted $(n+2k)$-tilings with at least k dominoes.*

Proof. From Lemma 2.1 and (2.7), we get the desired result. □

Proposition 2.9. *Let n and k be positive integers. The bi-periodic hyper Horadam sequence satisfies the recurrence relation*

$$w_n^{(k)} = a^{\xi(n+1)} b^{\xi(n)} w_{n-1}^{(k)} + c w_n^{(k-1)} \tag{2.8}$$

with initial values $w_n^{(0)} = w_n$ and $w_0^{(k)} = c^k w_0$.

Proof. Let t be a weighted n-tiling with at least k dominoes. If t ends with a square and n is even (resp. n is odd) then the weight of t is $aw_{n-1}^{(k)}$ (resp. $bw_{n-1}^{(k)}$). If t ends with a domino then its weight is $cw_n^{(k-1)}$. □

Using (2.7) and (2.8), we obtain the following non-homogeneous recurrence relation:

$$\begin{aligned} w_n^{(k)} &= a^{\xi(n+1)} b^{\xi(n)} w_{n-1}^{(k)} + c w_{n-2}^{(k)} \\ &\quad + a^{\xi(n-1)} \frac{n w_1 + b(k-1) w_0}{n+k-1} \binom{n+k-1}{k-1} (ab)^{\lfloor \frac{n-1}{2} \rfloor} c^k. \end{aligned} \tag{2.9}$$

Finally, we relate the bi-periodic Horadam numbers, the bi-periodic incomplete Horadam numbers, and the bi-periodic hyper Horadam numbers to one another. This is stated in the next proposition.

Proposition 2.10. *Let n and k be integers with $k > 0$. Then we have*

$$w_n^{(k)} = w_{n+2k} - w_{n+2k}(k-1). \tag{2.10}$$

Proof. Note that the bi-periodic Horadam number w_{n+2k} is given by

$$w_{n+2k} = a^{\xi(n-1)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor + k} \frac{(n+2k-2i)w_1 + biw_0}{n+2k-i} \binom{n+2k-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor + k - i} c^i. \tag{2.11}$$

Replacing the index i by $i-k$ in (2.7), we obtain

$$w_n^{(k)} = a^{\xi(n-1)} \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor + k} \frac{(n+2k-2i)w_1 + biw_0}{n+2k-i} \binom{n+2k-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor + k - i} c^i. \tag{2.12}$$

Then, subtracting equation (2.12) from (2.11) gives $w_{n+2k}(k-1)$. This completes the proof. □

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