THE NARAYANA SEQUENCE IN FINITE GROUPS

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Abstract. In this paper, the Narayana sequence modulo m is studied. The paper outlines the definition of Narayana numbers and some of their combinatorial links with Eulerian, Catalan and Delannoy numbers and other special functions. From the definition, the Narayana orbit of a 2-generator group for a generating pair $(x, y) \in G$ is defined, so that the lengths of the period of the Narayana orbit can be examined. These yield in turn the Narayana lengths of the polyhedral group and the binary polyhedral group for the generating pair (x, y) and associated properties.

1. Introduction

The Narayana numbers and their properties were studied by Özkan, Ramirez, Petersen *et al.* [18, 19, 20, 21]. Petersen [20] especially placed them in the Euler-Macmahon-Carlitz/Riordan combinatorial spectrum.

The Narayana sequence is defined by the third order linear, homogeneous recurrence relation

$$N_{n+3}= N_{n+2} + N_n \tag{1.1}$$

with the initial values $N_0 = 0, N_1 = 1$ and $N_2 = 1$. That is, the Narayana sequence is {0,1,1,1,2,3,4,6,9,13,19,28,41,60, ...} A078012. We can also obtain Narayana numbers with a matrix just like Fibonacci numbers as follows.

$$T^n = \begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix} \tag{1.2}$$

where $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. [26]

The Narayana numbers $N_{n,k}$ were discovered by Tadepalli Venkata Narayana, a Canadian mathematician.

$N_{n,k}$ are usually expressed by the Narayana Triangle which summarizes their symmetry $N_{n,k} = N_{n,n-k+1}$, as in the Pascal triangle. We know that

$$N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k} \tag{1.3}$$

which can also be expressed as

$$N_{n,k} = \binom{n-1}{k}\binom{n+1}{k+1} - \binom{n}{k}\binom{n}{k+1} \tag{1.4}$$

which has echoes of the Catalan result

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}. \tag{1.5}$$

We give the first few of them in Table 1.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	1								
3	1	3	1							
4	1	6	6	1						
5	1	10	20	10	1					
6	1	15	50	50	15	1				
7	1	21	105	175	105	21	1			
8	1	28	196	490	490	196	28	1		
9	1	36	336	1176	1764	1176	336	36	1	
10	1	45	540	2520	5292	5292	2520	540	45	1

TABLE 1. Some of the Narayana numbers

From Table 1, we see that $N_{n,1}$ is given as sequences A000217 and $N_{10,k}$ is a part of sequence A001263 in the On-Line Encyclopedia of Integer Sequences [25]. Also, the sums along the leading diagonals yield $\{1, 1, 2, 4, 8, 17, 37, 82, 185, \ldots]$, the generalized Catalan sequence, A004148 [25]. Many of the relevant number theory features were elaborated by Carlitz and Riordan [4] and Deveci and Shannon [10].

The first study on Fibonacci sequences in groups started with Wall [28]. He investigated the ordinary Fibonacci sequences in cyclic groups. The concept was later extended to some special linear recurrence sequences by some authors [6, 7, 9, 11, 15, 16, 17]. The theory was expanded to 3-step Fibonacci sequence by Özkan, Aydın and Dikici [14] and extended to k-step Fibonacci sequences by Lü and Wang [13].

Simple groups of order less than a million were considered by Campbell, Doostie and Robertson [2] and the binary polyhedral and binary polyhedral groups were studied by Taş and Karaduman [27] and C.M. and P.P Campbell [3]. They defined the Fibonacci length of the basic Fibonacci orbit in a 2-generator group. Knox [12] expanded the theory to k-nacci sequences in a finite group. Deveci and Karaduman defined the basic k-nacci sequences and the basic periods of these sequences in finite groups [8] and also extended the concept to Pell, Pell-Padovan, Jacobsthal-Padovan sequences in finite groups [5].

We next extend the concept to the Narayana sequences. In Section 2, it is explained with some theorems and examples that the Narayana sequence forms a periodic sequence according to $\bmod\, m$. In Section 3, for a generating pair $(x, y) \in G$, we define the Narayana orbit. In Section 4, we have obtained the Narayana lengths of the polyhedral and the binary polyhedral groups with specific examples.

2. The Narayana Sequences

A sequence is simply periodic with period k if the first different k elements in the sequence form a repeating subsequence. For example, $x_1, x_2, x_3, x_4, x_5, x_1, x_2, x_3, x_4, x_5 \ldots$ is simply periodic with period 5. Let us denote $N_i(\bmod\, m)$ with $\left\{N_i^{(m)}\right\}$. That is,

$$\left\{N_i^{(m)}\right\} = \left\{N_0^{(m)}, N_1^{(m)}, N_2^{(m)}, \ldots, N_n^{(m)}, \ldots\right\}.$$

which has the same recurrence relation as in (1.1). These recurrences belong to a family of third order lacunary-type Padovan, Perrin and Plastic sequences which have been extensively explored by Anderson, Horadam and Shannon [1, 22, 23, 24].

Theorem 2.1. $\left\{N_n^{(m)}\right\}$ *forms a simply periodic sequence.*

Proof. Since there are only a finite number m^3 of possible term triplets, the sequence repeats, and repeating the triple results in iteration of all subsequent terms.

From definition of the Narayana sequence, we have

$$N_{n+2}= N_{n+3} - N_n$$

so, if

$$N_{i+2}^{(m)} = N_{j+2}^{(m)},$$
$$N_{i+1}^{(m)} = N_{j+1}^{(m)},$$
$$N_{i}^{(m)} = N_{j}^{(m)}$$

then

$$N_{i-j+2}^{(m)} = N_2^{(m)}, N_{i-j+1}^{(m)} = N_1^{(m)} \quad \text{and} \quad N_{i-j}^{(m)} = N_0^{(m)},$$

which implies that the sequence $\left\{N_n^{(m)}\right\}$ is simply periodic, as required. □

Now, we let $\text{Per}(N^{(m)})$ denote the smallest period of the sequence $\left\{N_n^{(m)}\right\}$, the period of the Narayana sequence modulo m. Let p_i be distinct primes. If $m = \prod_{i=1}^{t} {p_i}^{e_i} (t \geq 1\ )$ then we get

$$\text{Per}(N^{(m)}) = \text{lcm}[\text{Per}(N^{({p_i}^{e_i})})],$$

the least common multiple of the $\text{Per}(N^{({p_i}^{e_i})})$.

Example 2.2. $\left\{N_n^{(3)}\right\} = \{0, 1, 1, 1, 2, 0, 1, 0, 0, 1, 1, 1, 2, 0, 1, 0, 0, 1, \dots\} \Longrightarrow \text{Per}(N^{(3)})\ = 8.$

As a sequence of the Theorem 2.1, we give the following Conjecture.

Conjecture 2.3. *For $m > 1$, we have*

$$\left\{N_{\text{Per}(N^{(m)})\ -7}^{(m)}, N_{\text{Per}(N^{(m)})\ -6}^{(m)}, \dots, N_{\text{Per}\left(N^{(m)}\right)-1}^{(m)}, N_{\text{Per}\left(N^{(m)}\right)}^{(m)}, N_{\text{Per}\left(N^{(m)}\right)+1}^{(m)}, N_{\text{Per}\left(N^{(m)}\right)+2}^{(m)}\right\}$$
$$= \{m-2, 1, 1, m-1, \dots, 0, 1, 1\}.$$

Example 2.4. *For* $\left\{N_n^{(3)}\right\} = \{0, 1, 1, 1, 2, 0, 1, 0, 0, 1, 1, 1, 2, 0, 1, 0, 0, \dots\} \Longrightarrow \text{Per}(N^{(3)})\ = 8.$

$$\left\{N_{\text{Per}(N^{(3)})\ -7}^{(3)}, N_{\text{Per}(N^{(3)})\ -6}^{(3)}, \dots, N_{\text{Per}\left(N^{(3)}\right)-1}^{(3)}, N_{\text{Per}\left(N^{(3)}\right)}^{(3)}, N_{\text{Per}\left(N^{(3)}\right)+1}^{(3)}, N_{\text{Per}\left(N^{(3)}\right)+2}^{(3)}\right\}$$
$$\left\{N_{8\ -7}^{(3)}, N_{8\ -6}^{(3)}, \dots, N_{8-1}^{(3)}, N_{8}^{(3)}, N_{8+1}^{(3)}, N_{8+2}^{(3)}\right\} = \{1, 1, 1, 2, \dots, 0, 0, 1, 1\}$$
$$= \{m-2, 1, 1, m-1, \dots, 0, 1, 1\}.$$

For the matrix $A = [a_{ij}]_{(k+1)\times(k+1)}$ with a_{ij} integers, $A(\text{mod}\, m)$ means that all entries of A are reduced modulo m, that is, $A\,(\text{mod}\, m) = (a_{ij}\ (\text{mod}\, m))$. Let $\langle N\rangle_{p^\alpha} = \left\{T^i\ (\text{mod}\, p^\alpha) \mid i \geq 0\right\}$ be a cyclic group and $\left|\langle N\rangle_{p^\alpha}\right|$ denote the order of $\langle N\rangle_{p^\alpha}$. From 1.2, we have that $\text{Per}(N^{(p^\alpha)}) = \left|\langle N\rangle_{p^\alpha}\right|$.

Theorem 2.5. *Let t be the positive integer such that* $\text{Per}(N^{(p)}) = \text{Per}(N^{(p^t)})$. *Then we have* $\text{Per}(N^{(p^\alpha)}) = p^{\alpha-t}\text{Per}(N^{(p)})$, $\alpha \geq t$. *In particular, if* $\text{Per}(N^{(p)}) \neq \text{Per}(N^{(p^2)})$ *then we have* $\text{Per}(N^{(p^\alpha)}) = p^{\alpha-1}\text{Per}(N^{(p)})$, $\alpha > 1$.

Proof. Let q be a positive integer. Since $T^{\mathrm{Per}(N^{(p^{q+1})})} \equiv I(\bmod\, p^{q+1})$ and $T^{\mathrm{Per}(N^{\left(p^{q+1}\right)})} \equiv I(\bmod\, p^{q})$, we get that $\mathrm{Per}(N^{(p^q)})$ divides $\mathrm{Per}(N^{\left(p^{q+1}\right)})$ where I is an identity matrix. On the other hand, we know

$$T^{\mathrm{Per}(N^{(p^q)})} = I + ({a_{ij}}^{(q)} p^q).$$

So, we have

$$T^{\mathrm{Per}(N^{(p^q)})p} = (I + {a_{ij}}^{(q)} p^q)^p = \sum_{i=0}^{p} \binom{p}{i} ({a_{ij}}^{(q)} p^q)^i \equiv I(\bmod\, p^{q+1})$$

which yields the result that $\mathrm{Per}(N^{(p^{q+1})}$ divides $\mathrm{Per}(N^{(p^q)}p$. Therefore, we get

$$\mathrm{Per}(N^{\left(p^{q+1}\right)}) = \mathrm{Per}(N^{(p^q)}) \quad \text{or} \quad \mathrm{Per}(N^{\left(p^{q+1}\right)} = \mathrm{Per}(N^{(p^q)}p)$$

and $\mathrm{Per}(N^{\left(p^{q+1}\right)} = \mathrm{Per}(N^{(p^q)}p)$ if and only if there is an ${a_{ij}}^{(q)}$ which is not divisible by p. Since $\mathrm{Per}(N^{\left(p^t\right)}) \neq \mathrm{Per}(N^{\left(p^{t+1}\right)})$, there is an ${a_{ij}}^{(t+1)}$ which is not divisible by p. Thus, we get $\mathrm{Per}(N^{\left(p^{t+1}\right)}) \neq \mathrm{Per}(N^{\left(p^{t+2}\right)})$. The proof is finished by induction on t. □

Let us explain this theorem with an example.

Example 2.6. *For* $p = 3$ and $q = 1$*,* $T^{\mathrm{Per}(N^{(9)})} \equiv T^{24} \equiv I(\bmod 9)$*, so that,*

$$T^{24} = \begin{pmatrix} N_{25} & N_{23} & N_{24} \\ N_{24} & N_{22} & N_{23} \\ N_{23} & N_{21} & N_{22} \end{pmatrix} = \begin{pmatrix} 5896 & 2745 & 4023 \\ 4023 & 1873 & 2745 \\ 2745 & 1278 & 1873 \end{pmatrix}_{\bmod 9} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I,$$

$$T^{3\mathrm{Per}\left(N^{(3)}\right)} = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 3 \begin{pmatrix} 1965 & 915 & 1341 \\ 1341 & 624 & 915 \\ 915 & 426 & 624 \end{pmatrix} \right)^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\bmod 3^2).$$

In this example, for $t = 1$ and $\alpha = 2$*, we have*

$$\mathrm{Per}\left(N^{(p^\alpha)}\right) = p^{\alpha - t}\mathrm{Per}\left(N^{(p)}\right) = \mathrm{Per}\left(N^{(3^2)}\right) = 3^1 \mathrm{Per}\left(N^{(3)}\right)$$

where $\mathrm{Per}\left(N^{(3)}\right) = 8$ *and* $\mathrm{Per}\left(N^{(3^2)}\right) = 24$ *from Example 2.2. Also, in this example for* $\mathrm{Per}\left(N^{(p)}\right) \neq \mathrm{Per}\left(N^{(p^2)}\right)$, $\mathrm{Per}(N^{(p^\alpha)}) = p^{\alpha-1}\mathrm{Per}(N^{(p)})$ *is provided.*

Theorem 2.7. *If* $m = \prod_{i=1}^{t} {p_i}^{e_i} (t \geq 1\ )$ *where the* p_i *are distinct primes, then* $\mathrm{Per}(N^{(m)}) = \mathrm{lcm}[\mathrm{Per}(N^{({p_i}^{e_i})})]$*, the least common multiple of the* $\mathrm{Per}(N^{({p_i}^{e_i})})$*.*

Proof. The statement $\mathrm{Per}(N^{({p_i}^{e_i})})$ is the length of the period of $\left\{N_{\mathrm{n}}^{({p_i}^{e_i})}\right\}$ which implies that the sequence $\left\{N_{\mathrm{n}}^{({p_i}^{e_i})}\right\}$ repeats only after blocks of length $u\mathrm{Per}(N^{({p_i}^{e_i})}), u \in \mathbb{N}$ and the statement $\mathrm{Per}(N^{(m)})$ is the length of the period $\left\{N_{\mathrm{n}}^{(m)}\right\}$ which implies that $\left\{N_{\mathrm{n}}^{({p_i}^{e_i})}\right\}$ repeats after $\mathrm{Per}(N^{(m)})$for all values i.

Thus, $\mathrm{Per}(N^{(m)})$ is of the form $u.\mathrm{Per}(N^{({p_i}^{e_i})})$ for all values of i, and any such number gives a period of $\left\{N_{\mathrm{n}}^{(m)}\right\}$. Then we get that $\mathrm{Per}(N^{(m)}) = \mathrm{lcm}\left[\mathrm{Per}(N^{({p_i}^{e_i})})\right]$, as required. □

3. The Narayana length of generating pairs in groups

Let G be a group and $x, y \in G$. If every element of G can be written as a word

$$x^{u_1} y^{u_2} x^{u_3} y^{u_4} \dots x^{u_{m-1}} y^{u_m}$$

where $1 \leq i \leq m$, $u_i \in \mathbb{Z}$, then we say that x and y generate G,

$$G = \langle x_0 = x,\ x_1 = y :\ x_{i+3} = x_{i+2}.x_i,\ i \geq 0 \rangle \,,$$

and that G is a 2-generator group. Let G be a finite 2-generator group and X be the subset of $G \times G$ such that $(x, y) \in X$ if and only if G is generated by x and y. We call (x, y) a generating pair for G.

Definition 3.1. *For a generating pair $(x, y) \in G$, we define the Narayana orbit as*

$$N_{x,y,y}(G) = \{x_i\}$$

with $x_0 = x,\ x_1 = y,\ x_2 = y,\ x_{i+3} = x_{i+2}.x_i,\ i \geq 0$.

Theorem 3.2. *A Narayana orbit of a finite group is simply periodic.*

Proof. Let n be the order of G. Since there are n^3 distinct 3-tuples of elements of G, at least one of the 3-tuples appears twice in a Narayana orbit of G. Thus, the subsequence follows the 3-tuples. Because of the repetitions, the Narayana orbit is periodic.

Since the Narayana orbit is periodic, there exist natural numbers i and j, with $i > j$, such that

$$x_{i+1} = x_{j+1}, \quad x_{i+2} = x_{j+2}, \quad x_{i+3} = x_{j+3}.$$

By the definition of the Narayana orbit, we know that

$$x_i = (x_{i+2})^{-1}.x_{i+3} \quad \text{and} \quad x_j = (x_{j+2})^{-1}.x_{j+3} \,.$$

Hence, $x_i = x_j$ and it then follows that

$$x_{i-j} = x_{j-j} = x_0, \quad x_{i-j+1} = x_{j-j+1} = x_1, \quad x_{i-j+2} = x_{j-j+2} = x_2.$$

Thus, the Narayana orbit is simply periodic. □

We denote the period of the Narayana orbit $N_{x,y,y}(G)$ by $LN_{x,y,y}(G)$ and we call the Narayana length of G with respect to the generating pair (x, y).

Theorem 3.3. *The Narayana length of $\mathbb{Z}_n \times \mathbb{Z}_m$, where $\mathbb{Z}_n = \langle x \rangle$ and $\mathbb{Z}_m = \langle y \rangle$, equals*

$$\operatorname{lcm}[\operatorname{Per}(N^{(n)}), \operatorname{Per}(N^{(m)})].$$

Proof. We know that $\mathbb{Z}_n \times \mathbb{Z}_m$ have the presentation

$$\langle x, y :\ x^n = y^m = e, xy = yx \rangle \,.$$

The Narayana orbit is

$$x_0 = x,\ x_1 = y,\ x_2 = y,\ x_3 = xy,\ x_4 = xy^2,\ x_5 = xy^3,$$
$$x_6 = x^2 y^4,\ x_7 = x^3 y^6\, x_8 = x^4 y^9,\ x_9 = x^6 y^{13},\ \dots$$

If we get $x_i = x,\ x_{i+1} = y,\ x_{i+2} = y$ then the proof is finished. Examining this statement in more detail, it yields

$$x^{N_{i-2}} y^{N_i} = x = x_0, \quad x^{N_{i-1}} y^{N_{i+1}} = y = x_1, \quad x^{N_i} y^{N_{i+2}} = y = x_2,$$

and the least non-trivial integer satisfying the above conditions occurs when

$$i = \operatorname{lcm}[\operatorname{Per}(N^{(n)}), \operatorname{Per}(N^{(m)})].$$

□

4. Applications

Definition 4.1. *The polyhedral group* $(\ell, m, n\ )$, *for* $\ell, m, n > 1$, *is defined by the presentation*

$$\Big\langle x, y, z:\ x^\ell = y^m = z^n = xyz = e\Big\rangle \quad or \quad \Big\langle x, y:\ x^\ell = y^m = (xy)^n = e\Big\rangle.$$

The polyhedral group $(\ell, m, n\ )$ *is finite if and only if the number*

$$k = \ell mn\left(\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} - 1\right) = mn + n\ell + \ell m - \ell mn$$

is positive. Its order is $\frac{2\ell mn}{k}$.

Definition 4.2. *The binary polyhedral group* $\langle \ell, m, n\rangle$, *for* $\ell, m, n > 1$, *is defined by the representation*

$$\Big\langle x, y, z:\ x^\ell = y^m = z^n = xyz\Big\rangle, \quad or \quad \Big\langle x, y:\ x^\ell = y^m = (xy)^n\Big\rangle.$$

The binary polyhedral group $\langle \ell, m, n\rangle$ *is finite if and only if the number*

$$k = \ell mn\left(\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} - 1\right) = mn + n\ell + \ell m - \ell mn$$

is positive. Its order is $\frac{4\ell mn}{k}$.

Now we obtain the Narayana lengths of the polyhedral groups $(2,2,2)$, $(n,2,2)$, $(2,n,2)$, $(2,2,n)$ and the binary polyhedral groups $\langle 2,2,2\rangle$, $\langle n,2,2\rangle$, $\langle 2,n,2\rangle$, $\langle 2,2,n\rangle$ for the generating pair (x,y).

Theorem 4.3. *The Narayana length of the polyhedral group* $(2,2,2)$ *is* 7.

Proof. From Theorem 3.3, we can see that $LN_{x,y,y}((2,2,2)) = 7$. Since $(2,2,2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we have

$$x_0 = x,\ x_1 = y,\ x_2 = y,\ x_3 = xy,\ x_4 = x,\ x_5 = xy,\ x_6 = e,\ x_7 = x,\ x_8 = y, \ldots$$

and $LN_{x,y,y}((2,2,2)) = 7$. □

Theorem 4.4. *The Narayana length of the polyhedral group* $(2,n,2)$ *is as follows:*

$$LN_{x,y,y}((2,n,2)) = \begin{cases} \frac{7n}{2}, & n \equiv 0 \pmod 4, \\ 7n & n \equiv 2 \pmod 4,\ n > 2, \\ 14n, & otherwise. \end{cases}$$

Proof. If $\langle x, y:\ x^2 = y^n = (xy)^2 = e\rangle$, $|x| = 2$, then $|y| = n$ and $|xy| = 2$. The Narayana orbit is

$$x, y, y, xy, x, yx, y^{-2}, xy^{-2}, y^{-1}, y^{-3}, xy^{-5}, xy^{-4}, xy^{-1}, y^4, x, y, y^5, xy^5, xy^4, xy^{-1},$$
$$y^{-6}, xy^{-2}, y^{-1}, y^{-7}, xy^{-9}, x^{-1}, xy^{-1}, y^8, x, y, y^9, xy^9, \ldots$$

Thus, the Narayana orbit becomes:

$$x_0 = x,\ x_1 = y,\ x_2 = y, \ldots$$
$$x_7 = xy^{-2},\ x_8 = y^{-1},\ x_9 = y^{-3}, \ldots$$
$$x_{14} = x,\ x_{15} = y,\ x_{16} = y^5, \ldots$$
$$x_{28=14i} = x,\ x_{29=14i+1} = y,\ x_{30=14i+2} = y^{9=4i+1}, \ldots$$
$$x_{42} = x,\ x_{43} = y,\ x_{44} = y^{13}, \ldots$$

so, we get $4i = un$ for $u \in \mathbb{N}$ and $i \in \mathbb{N}$. If $n \equiv 0 \pmod 4$, there are two subcases:

First case: If $\frac{n}{2} \equiv 0 \pmod 4$, then $i = \frac{n}{4}$. So, we get $LN_{x,y,y}((2,n,2)) = \frac{7n}{2}$.

$n = 8k$, $4i = u8k \Longrightarrow i = 2uk$ and $k = \frac{i}{2u}$ where $n = \frac{4i}{u}$ and $i = \frac{un}{4}$. Since the smallest number u satisfying the equation is 2, we get $i = \frac{n}{2}$. From Theorem 3.3, it is easy to see that $LN_{x,y,y}((2,n,2)) = \frac{7n}{2}$, $n \equiv 0 \pmod 4$.

Second case: If $\frac{n}{2} \equiv 2 \pmod 4$, then $i = \frac{n}{4}$. So, we get $LN_{x,y,y}((2,n,2)) = \frac{7n}{2}$. Similarly, from Theorem 3.3, it is easy to see that $LN_{x,y,y}((2,n,2)) = \frac{7n}{2}$, $n \equiv 0 \pmod 4$. If $n \equiv 2 \pmod 4$ then $i = \frac{n}{2}$. So, fromTheorem 3.3, it is easy to see that $LN_{x,y,y}((2,n,2)) = 7n$. If n is odd, then $i = n$. So, from Theorem 3.3, it is easy to see that $LN_{x,y,y}((2,n,2)) = 14n$. □

Theorem 4.5. *Let G be any one of the polyhedral groups $(n,2,2)$ and $(2,2,n)$. Then*

$$LN_{x,y,y}(G) = \begin{cases} \frac{7n}{2}, & n \equiv 0 \pmod 4, \\ 7n & n \equiv 2 \pmod 4,\ n > 2, \\ 14n, & \text{otherwise.} \end{cases}$$

Proof. Firstly, let us consider the polyhedral group $(n,2,2)$. We know $\langle x, y : x^n = y^2 = (xy)^2 = e \rangle$, $|x| = n$, $|y| = 2$ and $|xy| = 2$. The Narayana orbit is

$$x, y, y, xy, x^{-1}, xy, e, x^{-1}, x^2y, x^2y, xy, x, xy, e, x, y, y, xy, x^{-1}, xy, e, \ldots$$

for polyhedral groups $(n,2,2)$. So, $LN_{x,y,y}((n,2,2)) = 14$.

Secondly, let us consider the polyhedral group $(2,2,n)$. Since $\langle x, y : x^2 = y^2 = (xy)^n = e \rangle$, $|x| = 2$, $|y| = 2$ and $|xy| = n$, the Narayana orbit is

$$x, y, y, xy, yxy, xy, (xy)^2, y(xy)^3, y(xy)^2, y, \ldots.$$

So, we have

$$\begin{gathered} x_0 = x,\ x_1 = y,\ x_2 = y, \ldots \\ x_7 = y(xy)^3,\ x_8 = (xy)^3 x,\ x_9 = y, \ldots \\ x_{14} = (xy)^4 x,\ x_{15} = (xy)^3 x,\ x_{16} = y, \ldots \\ x_{28=14i} = (xy)^{8=4i} x,\ x_{29=14i+1} = (xy)^{7=4i-1} x,\ x_{30=14i+2} = y, \ldots \\ x_{42} = (xy)^{12} x,\ x_{43} = (xy)^{11} x,\ x_{44} = y, \ldots \end{gathered}$$

If $n \equiv 0 \pmod 4$, $LN_{x,y,y}((2,2,n)) = \frac{7n}{2}$.

If $n \equiv 2 \pmod 4$, $LN_{x,y,y}((2,2,n)) = 7n$.

If n is odd, $LN_{x,y,y}((2,2,n)) = 14n$.

The proof is like that of Theorem 4.2. □

The Theorem 4.4 and 4.5 are supported by the following examples to make it more understandable.

Example 4.6. *For $n = 4$, the Narayana lenght of polyhedral groups $(4,2,2)$ is 14.*

$$\left\langle x, y : x^4 = y^2 = (xy)^2 = e \right\rangle, \ |x| = 4,\ |y| = 2,\ |xy| = 2.$$

So, the Narayana sequence in the $(4,2,2)$ is

$$x,\ y,\ y,\ xy,\ x^3,\ xy,\ e, x^3,\ x^2y,\ x^2y,\ xy,\ x,\ xy,\ e,\ x,\ y,\ xy, \ldots$$

and the Narayana length of the polyhedral groups $(4,2,2)$ is 14.

Theorem 4.7. *The Narayana length of the binary polyhedral group $\langle 2,2,2 \rangle$ is 14.*

Proof. Since $\langle x, y : x^2 = y^2 = (xy)^2 \rangle,\ |x| = 4$, the Narayana orbit is

$$x, y, y, xy, x, yx, e, x, y^3, y^3, xy^3, x, y^3x, e, x, y, y, \ldots.$$

So, we get $LN_{x,y,y}\left((2,2,2)\right) = 14$. □

Theorem 4.8. *For $n \geq 1$, the Narayana length of the binary polyhedral group $\langle 2, n, 2 \rangle$ is as follows:*

$$LN_{x,y,y} = \begin{cases} \frac{7n}{2}, & n \equiv 0 \pmod 4, \\ 7n & n \equiv 2 \pmod 4, \\ 14n, & \textit{otherwise.} \end{cases}$$

Proof. Since $\langle x, y : x^2 = y^n = (xy)^2 \rangle,\ |x| = 4,\ |y| = 2n,\ |xy| = 4$, the Narayana orbit is

$$x, y, y, xy, x, yx, x^2y^{-2}, x^3y^{-2}, y^{-1}, x^2y^{-3}, xy^{-1}, x, x^3y^3, y^4, xy^4, y, y^5, xy^9, xy^8,$$
$$xy^3, x^2y^{-6}, x^3y^{-14}, y^{-17}, x^2y^{-23}, xy^{-9}, xy^8, x^3y^{31}, \ldots$$

If $n \equiv 0 \,(\mathrm{mod}\, 4)$, $LN_{x,y,y}\left(\langle 2, n, 2 \rangle\right) = \frac{7n}{2}$.
If $n \equiv 2 \,(\mathrm{mod}\, 4)$, $LN_{x,y,y}\left(\langle 2, n, 2 \rangle\right) = 7n$.
If n is odd, $LN_{x,y,y}\left(\langle 2, n, 2 \rangle\right) = 14n$.
The proof is like that of Theorem 4.4 and 4.5. □

Theorem 4.9. *Let G_n be any one of the binary polyhedral groups $\langle n, 2, 2 \rangle$ and $\langle 2, 2, n \rangle$. Then we get*

$$LN_{x,y,y} = \begin{cases} \frac{7n}{2}, & n \equiv 0 \pmod 4, \\ 7n & n \equiv 2 \pmod 4,\ n \geq 1, \\ 14n, & \textit{otherwise.} \end{cases}$$

Proof. Firstly, let us consider the binary polyhedral group $\langle n, 2, 2 \rangle$. The group is defined by $\langle x, y : x^n = y^2 = (xy)^2 \rangle,\ |x| = 2n,\ |y| = 4,\ |xy| = 4$. So,
If $n \equiv 0 \,(\mathrm{mod}\, 4)$, $LN_{x,y,y}\left(\langle n, 2, 2 \rangle\right) = \frac{7n}{2}$.
If $n \equiv 2 \,(\mathrm{mod}\, 4)$, $LN_{x,y,y}\left(\langle n, 2, 2 \rangle\right) = 7n$.
If n is odd, $LN_{x,y,y}\left(\langle n, 2, 2 \rangle\right) = 14n$. □

Example: For $n = 2$, the Narayana length of the binary polyhedral group $\langle 2, 2, 2 \rangle$ is 14. Since $\langle x, y : x^2 = y^2 = (xy)^2 \rangle,\ |x| = 4,\ |y| = 4,\ |xy| = 4$, the Narayana sequence in the $\langle 2, 2, 2 \rangle$ is

$$x, y, y, xy, x, yx, e, x, y^3, y^3, xy^3, x, y^3x, e, x, y, y, xy, \ldots$$

For $n = 4$, the Narayana length of the binary polyhedral group $\langle 4, 2, 2 \rangle$ is 14. Since $\langle x, y : x^4 = y^2 = (xy)^2 \rangle,\ |x| = 8,\ |y| = 4,\ |xy| = 4$, the Narayana sequence in the $\langle 4, 2, 2 \rangle$ is

$$x, y, y, xy, x^3, yx^3, e, x^3, yx^6, yx^6, yx^3, yx^3y, yx^7, e, x, y, y, xy, \ldots.$$

For $n = 1$, the Narayana length of the binary polyhedral group $\langle 1, 2, 2 \rangle$ is 14.

We next note that in the group defined by

$$\left\langle x, y : x = y^2 = (xy)^2 \right\rangle,\ |x| = 2,\ |y| = 4,\ |xy| = 4.$$

So, the Narayana sequence in the $\langle 1, 2, 2 \rangle$ is

$$x, y, y, xy, e, y, e, e, y, y, y, x, yx, e, x, y, y, xy, \ldots$$

Secondly, let us consider the binary polyhedral group $\langle 2, 2, n \rangle$. We first note that in the group defined by

$$\left\langle x, y : x^2 = y^2 = (xy)^n \right\rangle,\ |x| = 4,\ |y| = 4,\ |xy| = 2n.$$

If $n \equiv 0 \pmod 4$, $LN_{x,y,y}(\langle 2,2,n\rangle) = 7n$.
If $n \equiv 2 \pmod 4$, $LN_{x,y,y}(\langle 2,2,n\rangle) = 7n$.
If n is odd, $LN_{x,y,y}(\langle 2,2,n\rangle) = 14n$.
The proofs are like those of Theorem 4.4 and 4.5.

Example 4.10. *For $n = 3$, the Narayana length of the binary polyhedral group $\langle 2,2,3\rangle$ is 42. We first note that in the group defined by*

$$\left\langle x, y:\ x^2 = y^2 = (xy)^3 \right\rangle,\ |x| = 4,\ |y| = 4,\ |xy| = 6.$$

So, the Narayana sequence in the $\langle 2,2,3\rangle$ is

$$x, y, y, xy, yxy, (xy)^4, (xy)^5, y, (xy)^4y, (xy)^3y, e, (xy)^4y, yx^3, yx^3, (xy)^4x, x, y, (xy)^5,$$
$$x(xy)^5, (xy)^4, (xy)^3, x(xy)^2, (xy)^2x, (xy)^5x, y^3x, x, (xy)^2,$$
$$x^3y, y, (xy)^2y, y, y^2, xyx, xy^3, xy, x, (yx)^2x, yx^2, (yx)^2, x, yx^3, e, x, y, y, xy, yxy, \ldots$$

5. CONCLUSION

We have recalled the essential features of the Narayana sequence and the main properties so that we could examine the Narayana sequences modulo m. So, we have defined the Narayana orbits of 2-generator finite groups. Finally, we have obtained the Narayana lengths of the polyhedral and the binary polyhedral groups with specific examples.

ACKNOWLEDGMENTS

We thank the editor and the referees and for all their comments and suggestions to improve the paper.

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MSC2020: 11B39, 11B50, 11B75

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