PROBLEM SESSION

Problem 1: Half-companion Pell Numbers

Proposed by aBa Mbirika, University of Wisconsin-Eau Claire, mbirika@uwec.edu

It is well known that the only Fibonacci numbers F_n which are perfect powers a^b for $a, b \in \mathbb{N}$ and $b > 1$ are $F_n = 1, 8$, and 144. Likewise, it is known that the only Lucas numbers which are perfect powers are $L_n = 1$ and 4. Similarly, the only Pell numbers $P_0 = 0, P_1 = 1, P_{n+2} = 2P_{n+1} + P_n$ which are perfect powers are $P_n = 1$ and 169.

We may also define the half-companion (or associated) Pell numbers $Q'_0 = 1, Q'_1 = 1, Q'_{n+2} = 2Q'_{n+1} + Q'_n$. In other words, $Q'_n = Q_n/2$, where Q_n is the sequence of companion Pell numbers (otherwise referred to as the Pell-Lucas numbers).

We thus ask for a classification of the half-companion Pell numbers Q'_n which are perfect powers a^b for $a, b \in \mathbb{N}$ and $b > 1$.

Problem 2: Generalizing Continued Fractions

Proposed by Giuliano Romeo, Politecnico di Torino, giuliano.romeo@polito.it

A continued fraction can be defined as

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

where $a_i \in \mathbb{Z}^+$.

The following two results hold in the field of real numbers.

(1) The continued fraction expansion is finite if and only if the number is a rational.
(2) The continued fraction expansion is eventually periodic if and only if the number is a quadratic irrational.

It is natural to generalize continued fractions over the field of p-adic numbers $\mathbb{Q}_p$. While there exist algorithms for generating continued fractions, in the p-adic case there don't exist any satisfying (2). For example, the p-adic continued fraction expansion $[b_0, b_1, \ldots]$ of $\alpha_0 \in \mathbb{Q}_p$ provided by Browkin is obtained by iterating the following steps for all $n \geq 0$:

$$\begin{aligned} b_n &= s(\alpha_n) \\ \alpha_{n+1} &= \frac{1}{\alpha_n - b_n} \end{aligned}$$

where $s : \mathbb{Q}_p \to \mathbb{Q}$ is the function that replaces the role of the floor function in the classical continued fractions over $\mathbb{R}$. This algorithm satisfies (1), but not (2).

Is there an algorithm for generating p-adic continued fractions which satisfies both (1) and (2)?

Problem 3: Partition Related Functions

Proposed by Faye Jackson, University of Michigan, alephnil@umich.edu

A *partition* of a natural number n is an increasing sequence of natural numbers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $n = \sum_{i=1}^{k} \lambda_i$. Let

$$T(r,t,n) \;=\; \sum_{\lambda \vdash n} \#\{\lambda_j \;:\; \lambda_j \equiv r \pmod{t}\}.$$

As a matter of convenience we always take the representative r to satisfy $1 \le r \le t$. Beckwith and Mertens proved that as $r, s \to \infty$,

$$\frac{T(r,t,n)}{T(s,t,n)} \;\to\; 1.$$

Furthermore, for n sufficiently large, if $1 \le r < s \le t$ then $T(r,t,n) \ge T(s,t,n)$.

What can be said about the functions

$$D_k^{\times}(r,t,n) \;=\; \sum_{\substack{\lambda \vdash n \\ \forall \lambda_j,\, k \nmid \lambda_j}} \#\{\lambda_j \;:\; \lambda_j \equiv r \pmod{t}\},$$

and is there a combinatorial proof for the biases? Is there a combinatorial proof of the inequality $T(r,t,n) \ge T(s,t,n)$ when $1 \le r < s \le t$?

Problem 4: Fibonacci, Lucas and Primes

Proposed by Rigoberto Florez, The Citadel, rflorez1@citadel.edu

Are there infinitely many prime numbers of the form $F_r + L_{r\pm1}$? Or equivalently, $F_k + L_{k+1}$ or $F_k + L_{k-1}$?

Acknowledgments

Many thanks to Faye Jackson, David Luo and Alicia Smith Reina for assistance in typing these up, Steven J. Miller for moderating the session, and the participants for the discussions on the problems.